CHEVALLEY GROUPS OVER LOCAL RINGS

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Dedicated to Professor Tadao Tannaka on his 60th birthday

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Introduction.

0.1. Let $G_c$ be a connected complex semi-simple Lie group. Following Chevalley (cf. [2] and [3]), we have a uniquely determined affine group scheme (i.e. a representable covariant functor $G$ from the category of commutative rings with a unit into the category of groups) such that

1. $G(C)$ is a connected complex semi-simple Lie group isomorphic to $G_c$, where $C$ is the field of complex numbers.

2. For any algebraically closed field $k$, $G(k)$ is a connected semi-simple algebraic group defined and split over the prime field of $k$ and its type is the same with that of $G_c$.

We call $G$ the Chevalley-Demazure group scheme associated with $G_c$ and we shall say that $G$ is simple, of rank $r$ or simply connected if the Lie group $G_C$ is so. In Section 1, we shall introduce briefly the definition of $G$.

0.2. Let $R$ be a commutative ring with a unit, $\alpha$ be an ideal of $R$, $f: R \to R/\alpha$ be the natural homomorphism. Then, there is a group homomorphism $G(f): G(R) \to G(R/\alpha)$. Denote by $G(R, \alpha)$ (resp. $G^*(R, \alpha)$) the kernel (resp. the inverse image of the center of $G(R/\alpha)$) of $G(f)$ and we call it the special (resp. general) congruence subgroup modulo $\alpha$ of $G(R)$. Any subgroup $N$ of $G(R)$ such that $G^*(R, \alpha) \supseteq N \supseteq G(R, \alpha)$ for an ideal $\alpha$ of $R$ is a normal subgroup of $G(R)$. Such a normal subgroup of $G(R)$ we shall call a congruence subgroup of $G(R)$.

0.3. Now, let $R$ be a local ring, $\mathfrak{m}$ be the maximal ideal and $k$ be the residue class field $R/\mathfrak{m}$, $p$ be the characteristic of $k$. W. Klingenberg has proved (cf. [5], [6]) that if $G=SL_{n+1}$ or $Sp_{2n}$, the only normal subgroups of $G(R)$ are the congruence subgroups provided that the characteristic of $k$ is $\neq 2$.
and $k \neq F_3$ for the groups $G = SL_2$ and $G = Sp_{2n}$. In this note, for a simple Chevalley-Demazure group scheme and a local ring $R$, we shall reduce the determination of the normal subgroups of $G(R)$ to the determination of certain submodules of $R$, except the following cases:

(a) $G$ is of type $A$, and $p = 2$ or $k = F_3$

(b) $G$ is of type $B$, or $G_2$, and $k = F_2$,

where $F_q$ is the finite field with $q$ elements. In particular, if $G$ is simply connected, we have that the only normal subgroups are the congruence subgroups provided that the characteristic of $k$ is $\neq 2$ (resp. $\neq 3$) if $G$ is of type $B_n$, $C_n$ or $F_4$ (resp. of type $G_2$). The main theorem is stated in Section 1 with the preliminary definitions. In Section 2, we give some basic properties of certain subgroups of $G(R)$ for our later use and, in Section 3, we prove a key proposition (2.17) and then prove our main theorem (1.9).

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1. Chevalley-Demazure group scheme, Statement of the main theorem.

In this section, we shall introduce the Chevalley-Demazure group scheme associated with a connected complex semi-simple Lie group (cf. [2], [3]) and then state our main theorem.

1.1. Let $G_c$ be a connected complex semi-simple Lie group, $T_c$ a maximal torus of $G_c$. Denote by $g_c$, $t_c$ the Lie algebras of $G_c$ and $T_c$ respectively. Let $\Delta$ be the system of roots of $g_c$ with respect to $t_c$, $\Pi = \{a_1, \cdots, a_i\}$ be a fundamental root system of $\Delta$, $g_z$ be a Chevalley lattice of $g_c$ generated by $[H_{a_1}, \cdots, H_{a_i}, X_{\alpha}, \alpha \in \Delta]$. For each $\alpha \in \Delta$, the element $H_{\alpha}$ is contained in the submodule $t_z = g_z \cap t_c$. We have

(1) $\alpha(H_{\alpha}) = 2$,

(2) if $\alpha, \beta$ are roots, then $\beta(H_{\alpha}) = v - \mu$, where $v, \mu$ are non-negative integers such that $\beta + i\alpha$ is a root for each integer $-v \leq i \leq \mu$, or

(3) if $\alpha, \beta$ and $\alpha + \beta$ are roots, $[X_{\alpha}, X_{\beta}] = N_{\alpha\beta}X_{\alpha + \beta}$, where $N_{\alpha\beta} = \pm 0 + 1$.

1.2. Let $\rho$ be a faithful representation of $G_c$ in an $n$-dimensional vector space $V$ over $C$, $dp$ the differential of $\rho$ which is a representation of $g_c$ in $V$. Then, there exists a $Z$-free module $V_z$ generated by $[v_1, \cdots, v_n]$ in $V$ such that

(4) $(m!)^{-1}dp(X_{\alpha})^mV_z \subset V_z$ for all integers $m \geq 0$ and all roots $\alpha \in \Delta$,

(5) $dp(H_{\alpha})v_i = \Lambda_i(H_{\alpha})v_i, \Lambda_i(H_{\alpha}) \in Z$, for all roots $\alpha \in \Delta$ and all $i (1 \leq i \leq n)$. 

Such a module $V_x$ is called to be admissible (cf. [2] and [7]). The base $\{v_1, \ldots, v_n\}$ of $V_x$ determines the coordinates $x_{ij}$ ($1 \leq i \leq j \leq n$) on $GL(V)$ and the restrictions of $x_{ij}$ to $G_c$ generate a subring $Z[G]$ of the affine algebra $C[G]$ of $G_c$. The ring $Z[G]$ has a structure of a Hopf algebra and defines a group scheme $G$ over $Z$. Namely,

$$ R \longrightarrow G(R) = \text{Hom}(Z[G], R) $$

is a covariant functor from the category of commutative rings with 1 into the category of groups. We shall call $G$ the Chevalley-Demazure group scheme associated with $G_c$. In particular, if $G_c$ is simply connected of type $A_n$ (resp. of type $C_n$), then $G$ is isomorphic to the functor $SL_{n+1}$ (resp. $SO_{2n+1}$).

1.3. For any $t \in C$, $x_a(t) = \exp t \, d\rho(X_a)$ is an element of $G_c$ and the coordinates of $x_a(t)$ are polynomial functions on $t$ with coefficients in $Z$. Let $Z[\xi]$ be the algebra over $Z$ generated by one variable $\xi$. Then we have a homomorphism of $Z[G]$ onto $Z[\xi]$ which assigns to each $x_{ij}$ the $(i, j)$-coordinate of $x_a(\xi)$. The homomorphism induces an injective homomorphism of groups

$$ G_a(R) = \text{Hom}(Z[\xi], R) \longrightarrow G(R) = \text{Hom}(Z[G], R) . $$

We denote also by $x_a(t)$, $t \in R$, the element of $G(R)$ corresponding to an element of $G_a(R)$ such that $\xi \rightarrow t$.

1.4. Let $P$ (resp. $X$, $P_r$) the additive group generated by the weights of all representations of $G$ (resp. the weights of $\rho$, the roots of $g_c$). Then, these are free abelian groups of rank $l$ such that $P \supseteq X \supseteq P_r$; $X$ is generated by $\Delta_1, \ldots, \Delta_n$ over $Z$; if $G$ is simply connected, then $P = X$. For any $\chi \in \text{Hom}(X, C^*)$, $h(\chi) = \text{diag}(\chi(\Delta_1), \ldots, \chi(\Delta_n))$ is an element of $G_c$. Let $Z[T]$ be the algebra generated by $\Lambda_1, \Lambda_1^{-1}, \ldots, \Lambda_n, \Lambda_n^{-1}$ over $Z$. Then, we have a homomorphism of $Z[G]$ onto $Z[T]$ which assigns to each $x_{ij}$ the $(i, j)$-coordinate of $h(\chi)$. The homomorphism induces an injective homomorphism of groups

$$ T(R) = \text{Hom}(Z[T], R) \longrightarrow G(R) = \text{Hom}(Z[G], R) . $$

We denote by $h(\chi)$ the element of $G(R)$ corresponding to an element $\chi \in \text{Hom}(Z[T], R)$.

1.5. DEFINITION. Let $R$ be a commutative ring with 1 and $G$ be a Chevalley-Demazure group scheme. We denote by $G_a(R)$ the subgroup of $G(R)$ generated by $x_a(t)$ for all $t \in R$ and all $\alpha \in \Delta$ and by $h(\chi)$ for all $\chi \in \text{Hom}(Z[T], R)$, and denote by $E(R)$ the subgroup of $G(R)$ generated by $x_a(t)$
for all $t \in \mathbb{R}$ and all $\alpha \in \Delta$. We know that if $\mathbb{R}$ is a field or the ring of integers of a field with a non-archimedean discrete valuation, then $G(\mathbb{R}) = G_0(\mathbb{R})$.

Further, if $G$ is simply connected of rank $\geq 1$ and if $\mathbb{R}$ is a semi-local ring, then $G(\mathbb{R}) = E(\mathbb{R})$ (cf. [8]). However, we don't know whether, in general, $G(\mathbb{R}) = G_0(\mathbb{R})$ for a group scheme $G$ (not necessarily simply connected) and a semi-local ring $\mathbb{R}$. We shall show in Section 3 the following.

1.6. PROPOSITION. Let $G$ be a Chevalley-Demazure group scheme and $\mathbb{R}$ be a local ring, then $G(\mathbb{R}) = G_0(\mathbb{R})$. In particular, if $G$ is simply connected, then $G(\mathbb{R}) = E(\mathbb{R})$.

1.7. For a root $\alpha \in \Delta$, let $g(\alpha, \alpha) = \sum_{\gamma \in \Delta} \gamma(H_\alpha)^{\alpha}$. The length $\lambda(\alpha)$ of $\alpha$ is defined to be 1 if $(\alpha, \alpha) \leq (\beta, \beta)$ for any root $\beta \in \Delta$, and is defined to be $\lambda$ if $(\alpha, \alpha)/(\beta, \beta) = \lambda$ for some root $\beta$ of length 1. If $G$ is of type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$) or $E_n$ ($n = 6, 7, 8$), then $\lambda(\alpha) = 1$ for all roots $\alpha$; if $G$ is of type $B_n$ ($n \geq 2$), $C_n$ ($n \geq 2$) or $F_4$ (resp. of type $G_2$), there are roots of lengths 1 and 2 (resp. 1 and 3).

1.8. DEFINITION. Let $G$ be a simple Chevalley-Demazure group scheme. We call $G$ is of symplectic type if $G$ is of type $C_n$ ($n \geq 2$) and simply connected. Let $\mathbb{R}$ be a commutative ring with 1, $\mathfrak{a}$ be an ideal of $\mathbb{R}$ and for a positive integer $\lambda$, $\mathfrak{a}(\lambda)$ be the ideal of $\mathbb{R}$ generated by $\lambda \mathfrak{a}$, $\mathfrak{a}^\lambda$ for all $\mathfrak{a} \in \mathfrak{a}$. We shall call a special submodule associated with $(G, \mathfrak{a})$ a submodule $\mathfrak{b}$ of $\mathbb{R}$ such that

(a) $\mathfrak{a} \supseteq \mathfrak{b} \supseteq \mathfrak{a}(\lambda)$, where $\lambda$ is the length of the long root in $\Delta$,

(b) if $G$ is of symplectic type, $r^2 \mathfrak{b} \subseteq \mathfrak{b}$ for any $r \in \mathbb{R}$ and $\mathfrak{b} \in \mathfrak{b}$,

(b') if $G$ is not of symplectic type, $\mathfrak{b}$ is an ideal of $\mathbb{R}$.

For convenience, we shall denote $\mathfrak{a}$ (resp. $\mathfrak{b}$) by $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$). Thus, by our notation, for an element $x_\mathfrak{a}(t)$ of $G(\mathbb{R})$, $t \in \mathfrak{a}_1(\mathfrak{a})$ means that $t \in \mathfrak{a}_1$ or $\mathfrak{b}$ according as $\lambda(\mathfrak{a}) = 1$ or $\lambda$. Now, we shall define certain subgroups of $G(\mathbb{R})$. $E(\mathbb{R}, \mathfrak{a}_1, \mathfrak{a}_2)$ is the normal subgroup of $E(\mathbb{R})$ generated by $x_\mathfrak{a}(t)$ for all roots $\alpha$ and $t \in \mathfrak{a}_1(\mathfrak{a})$; $E^*(\mathbb{R}, \mathfrak{a}_1, \mathfrak{a}_2)$ is the normal subgroup of $G(\mathbb{R})$ consisting of the elements $x$ of $G(\mathbb{R})$ such that $(x, G(\mathbb{R})) \subseteq E(\mathbb{R}, \mathfrak{a}_1, \mathfrak{a}_2)$, where for any subsets $A, B$ of $G(\mathbb{R})$, $(A, B)$ is the subgroup of $G(\mathbb{R})$ generated by $a^{-1}b^{-1}ab$ for $a \in A, b \in B$.

In particular, if $\mathfrak{a}_1 = \mathfrak{a}_2$, we denote $E(\mathbb{R}, \mathfrak{a}_1, \mathfrak{a}_2)$ (resp. $E^*(\mathbb{R}, \mathfrak{a}_1, \mathfrak{a}_2)$) by $E(\mathbb{R}, \mathfrak{a}_1)$ (resp. $E^*(\mathbb{R}, \mathfrak{a}_1)$) and if $\mathfrak{a}_1 = \mathfrak{a}_2 = \mathbb{R}$, by definition $E(\mathbb{R}, \mathfrak{a}_1) = E(\mathbb{R})$. Then, our main theorem is the following which is proved in Section 3.
1.9. Theorem. Let $G$ be a simple Chevalley-Demazure group scheme. Let $R$ be a local ring, $\mathfrak{m}$ be the maximal ideal of $R$, $k = R/\mathfrak{m}$ be the residue class field, $p$ be the characteristic of $k$. Assume that if $G$ is of type $A_1$ then $p \neq 2$ and $k \neq \mathbb{F}_2$ and if $G$ is of type $B_2$ or $G_2$ then $k \neq \mathbb{F}_2$.

Let $N$ be a subgroup of $G(R)$ normalized by $E(R)$. Then $N$ is normal and there exist uniquely determined ideal $a$ of $R$ and a special submodule $b$ associated with $(G, a)$ such that

$$E^s(R, a, b) \supseteq N \supseteq E(R, a, b).$$

1.10. Corollary. Under the same conditions as (1.9), if, in particular, $G$ is simply connected, then $G(R, a) = E(R, a)$ for any ideal $a$ of $R$.

1.11. Corollary. Under the same conditions as (1.9), if, in particular, $G$ is simply connected and the characteristic $p$ of $k$ is different from the length $\lambda$ of the long root, then, for any normal subgroup $N$ of $G(R)$, there exists an ideal $a$ of $R$ such that

$$G^s(R, a) \supseteq N \supseteq G(R, a).$$

2. Certain subgroups of $G(R)$. In this section, we shall deal with the structure of certain subgroups of $G(R)$. We assume that $R$ is a local ring and $G$ is simple. Notations and definitions are the same as those in the previous sections.

2.1. Definition. $U(R, a_1, a_2)$ (resp. $V(R, a_1, a_2)$) is the subgroup of $G(R)$ generated by $x_a(t)$, $t \in a_{i(a)}$ for all positive (resp. negative) roots $a \in \Delta$. In particular, if $a_i = a_0$, we denote it by $U(R, a_i)$ (resp. $V(R, a_i)$), and if $a_i = a_0 = R$, we denote it by $U(R)$ (resp. $V(R)$). Note that $U$ and $V$ are subgroup schemes of $G$. $T(R)$ is the subgroup of $G(R)$ consisting of all $h(\chi)$ for all $\chi \in \text{Hom}(\mathbb{Z}[T], R)$ which is isomorphic to $\text{Hom}(\mathbb{Z}[T], R)$ the direct product of $l$ copies of $G_{\text{a}}(R)$. $T'(R)$ is the subgroup of $T(R)$ generated by $h(\chi_{\bar{a}, a})$ for all roots $a \in \Delta$ and $u \in R^*$ (the group of units of $R$) where $\chi_{\bar{a}, a}(\Lambda_i) = u^a(\Lambda_i)$ ($1 \leq i \leq n$). $T(R, a)$ is the subgroup of $T(R)$ generated by all $h(\chi)$ such that $\chi(\alpha) \equiv 1 \pmod{a}$ for all root $\alpha$. Now, we denote by $T'(R, a, a_i)$ the subgroup of $T'(R)$ generated by $h(\chi_{\bar{a}, a})$ for all pairs $(\alpha, u)$ of $\alpha \in \Delta$ and $u \in R^*$ such that $u = 1 + st$ for $s \in R$ and $t \in a_{i(a)}$.

2.2. As for the relations of generators for $G(R)$, we know the following (cf. [1], [3]).

\begin{equation}
(1) \quad h(\chi_{\bar{a}, a}) = x_{-a}(u^{-1} - 1) x_a(1) x_{-a}(u - 1) x_a(1)^{-1} x_a(1 - u^{-1}), \quad u \in R^*.
\end{equation}
Let $\omega_a = x_a(1) x_{-a}(-1) x_a(1)$, then

$$\omega_a x_\beta(t) \omega_a^{-1} = x_{w_a(t)}(\pm t), \quad t \in R,$$

where $w_a$ is the reflection in the hyperplane orthogonal to $\alpha$ and it is an element of the Weyl group.

Let $\Delta^+$ be the set of the positive roots. If $\Delta$ is of type $A_2$,

$$\Delta^+ = \{\alpha, \beta, \alpha + \beta\}; \quad \lambda(\alpha) = \lambda(\beta) = \lambda(\alpha + \beta) = 1$$

and we have

$$\langle x_a(t), x_\beta(u) \rangle = x_{a+\beta}(\pm tu) \quad \text{for any} \quad t, u \in R.$$

If $\Delta$ is of type $B_2$,

$$\Delta^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}; \quad \lambda(\alpha) = \lambda(\alpha + \beta) = 1, \quad \lambda(\beta) = \lambda(2\alpha + \beta) = 2$$

and we have

$$\langle x_a(t), x_\beta(u) \rangle = x_{2a+\beta}(\pm tu), \quad t, u \in R.$$

If $\Delta$ is of type $G_2$,

$$\Delta^+ = \{\alpha, \beta, \alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}; \quad \lambda(\alpha) = \lambda(\beta) = \lambda(\alpha + \beta) = 1, \quad \lambda(\beta) = \lambda(3\alpha + \beta) = \lambda(3\alpha + 2\beta) = 3$$

and we have

$$\langle x_a(t), x_\beta(u) \rangle = x_{3a+2\beta}(\pm tu), \quad t, u \in R.$$
α such that \( \lambda(\alpha) = 1 \) resp. \( \lambda(\alpha) = \lambda \). Then

\[
E(R, \alpha, b) = E_i(R, \alpha)
\]

for any special submodule \( b \) associated with \( (G, \alpha) \).

**PROOF.** Since the Weyl group \( W \) is generated by \( \omega_\alpha, \alpha \in \Delta \) and \( W \) is transitive on the set of roots of the same length, from (3), it is sufficient to show that \( x_\beta(t) \in E_i(R, \alpha) \) for some root \( \beta \) of length \( \lambda \) and for all \( t \in \alpha \). Therefore, no loss of generality, we may assume that \( G \) is of type \( B_n \) or \( G_2 \). First, let \( G \) be of type \( B_n \), and let \( \Delta^+ \) be the roots (6). From (7) and (8), we have that \( x_{2a+\beta}(\pm t^2u) \) and \( x_{2a+\beta}(\pm 2tu) \) are in \( E_i(R, \alpha) \) for all \( t \in \alpha \) and \( u \in R \). Thus, by definition, we have \( E(R, \alpha, b) = E_i(R, \alpha) \). Secondly, let \( G \) be of type \( G_2 \) and let \( \Delta^+ \) be the roots (9).

From (10) and (11) we have \( z = x_{3a+\beta}(\pm t^2u)x_{3a+2\beta}(\pm t^2u^2) \) and \( x_{3a+2\beta}(\pm 3tu) \) are in \( E_i(R, \alpha) \) for all \( t \in \alpha \) and \( u \in R \). Further, \((x_{3\beta}(1), z) = x_{3a+2\beta}(\pm t^2u) \in E_i(R, \alpha) \) for all \( t \in \alpha \), and \( u \in R \). Thus by definition, we have \( E(R, \alpha, b) = E_i(R, \alpha) \). \( \text{q.e.d.} \)

2.4. PROPOSITION. Under the same notation as in (2.3),

(i) If \( \mu \neq \lambda \), then \( E_i(R, \alpha) = E(R, \alpha) = E(R, \alpha, b) \).

(ii) \( E_\lambda(R, \alpha) = E(R, \alpha) \) provided that, if \( G \) is of type \( G_2 \), \( k \neq F_2 \).

**PROOF.** It suffices to prove for the groups of type \( B_n \) and \( G_2 \).

(i) Let \( \Delta^+ \) be the positive roots (6) of type \( B_n \). Since \( \mu \neq 2,2 \) is a unit.

(8) for \( t = 2^{-1} \) and \( u \in \alpha \) shows that \( x_{2a+\beta}(\pm u) \in E_i(R, \alpha) \). Now, let \( \Delta^+ \) be the positive roots (9) of type \( G_2 \). Since \( \mu \neq 3,3 \), is a unit. (12) for \( t = 3^{-1} \) and \( u \in \alpha \) shows that \( x_{3a+2\beta}(\pm u) \in E_i(R, \alpha) \).

(ii) Let \( \Delta^+ \) be the positive roots (6) of type \( B_n \). Then from (8) for \( t = 1 \) and \( u \in \alpha \), we have \( x_{a+\beta}(u) \in E_i(R, \alpha) \). Now, let \( \Delta^+ \) be the positive roots (9) of type \( G_2 \). Then from (10) for \( t = 1 \) and \( u \in \alpha \), we have \( z = x_{a+\beta}(\pm u)x_{2a+\beta}(\pm u^2) \in E_i(R, \alpha) \) and \( z = a_{\beta}z \omega_{\alpha}^{-1} = x_{\alpha}(\pm u)x_{a+2\beta}(\pm u^2) \in E_i(R, \alpha) \). Since \( k \neq F_2 \), there exists an element \( \chi \) of Hom(Z[\( T \)], R) such that \( \chi(\alpha) = 1 \) and \( \chi(\beta) = v \) where \( v \) and \( v^-1 \) are units of \( R \). Then \( h(\chi)z \cdot h(\chi)^{-1} = x^\alpha(\pm u)x_{2a+\beta}(\pm v^2u^2) \in E_i(R, \alpha) \). Therefore, \( z^\alpha h(\chi)z^\alpha h(\chi)^{-1} = x_{3a+2\beta}(\pm (v-1)u^2) \in E_i(R, \alpha) \). This shows \( x_{3a+2\beta}(u^2) \in E_i(R, \alpha) \) and we have also \( x_\alpha(u) \in E_i(R, \alpha) \). \( \text{q.e.d.} \)

2.5. PROPOSITION. Each element of \( U(R, \alpha, a) \) is expressible in the form

\[
x_{\beta_1}(s_1)x_{\beta_2}(s_2)x_{\beta_3}(s_3) \cdots \cdots x_{\beta_N}(s_N)
\]

where \( s_i \in a_{\lambda(\alpha)}(1 \leq i \leq N) \) and \( \beta_1, \beta_2, \cdots, \beta_N \) are the positive roots of \( \Delta \), the
ordering of the roots is arbitrary chosen and fixed once for all.

Let \( U' \) be the set of elements expressible in the form as stated in the proposition. We call the order of the positive roots (or the negative roots) is regular if the heigt \( h(\alpha) = \sum_{i=1}^{i} m_i \) of \( \alpha = \sum_{i} m_i d_i \) is an increasing function of \( \alpha \). First, we prove the following lemma.

2.6. **Lemma.** Let \( \alpha, \beta \) be two positive roots. For any elements \( x_\alpha(t) \in E(R) \) and \( x_\beta(u) \in U(R, a_1, a_3) \), the commutator \( (x_\alpha(t), x_\beta(u)) \) is an element of \( U' \) which is expressible by the product of \( x_\gamma(s) \) for roots \( \gamma > \alpha, \beta \), by a regular order.

**Proof.** If \( \alpha + \beta \not\in \Delta \), then \( (x_\alpha(t), x_\beta(u)) = 1 \) and the lemma is trivial. We assume that \( \alpha + \beta \in \Delta \). Let \( \Delta_2 \) be a subsystem of roots in \( \Delta \) of rank 2 consisting of the roots \( \iota \alpha + \jmath \beta \), \( i, j \in \mathbb{Z} \).

(i) If \( \alpha - \beta \in \Delta, \{ \alpha, \beta \} \) is a fundamental system of roots of \( \Delta_2 \). When \( \Delta_2 \) is of type \( A_2 \), we have \( (x_\alpha(t), x_\beta(u)) = x_{\alpha + \beta}(\pm tu) \). If \( u \in a_1(\beta) \) then \( tu \) is also an element of \( a_1(\beta) \). When \( \Delta_2 \) is of type \( B_2 \) or \( G_2 \), we have \( (x_\alpha(t), x_\beta(u)) = x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) \) or \( x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) \) according as \( \lambda(\alpha) = 2 \) or 3. If \( \lambda(\alpha) = 1 \) (resp. = 2), then \( tu \in a_1 \) and \( t^2 u \in a_2 \) (resp. \( tu^2 \in a_2 \)). Finally, when \( \Delta_2 \) is of type \( C_2 \), we have \( (x_\alpha(t), x_\beta(u)) = x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) x_{\alpha + 3\beta}(\pm t^3 u) \) according as \( \lambda(\alpha) = 1 \) or 3. If \( \lambda(\beta) = 1 \) (resp. = 3), then \( tu, t^2 u \in a_1 \) and \( t^3 u \in a_2 \) (resp. \( tu^2, tu^3 \in a_3 \)).

(ii) If \( \alpha - \beta = \gamma \in \Delta \) and \( \alpha - 2\beta \not\in \Delta \), then \( \{ \beta, \gamma \} \) is a fundamental root system of \( \Delta_2 \) which is of type \( B_2 \) or \( G_2 \). When \( \Delta_2 \) is of type \( B_2 \), we have \( x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) x_{\alpha + 3\beta}(\pm t^3 u) \) or \( x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) x_{\alpha + 3\beta}(\pm t^3 u) \) according as \( \lambda(\alpha) = 1 \) or 3. If \( \lambda(\beta) = 1 \) (resp. = 3), then \( tu, t^2 u \in a_1 \) and \( t^3 u \in a_2 \) (resp. \( tu^2, tu^3 \in a_3 \)).

(iii) If \( \alpha - 2\beta = \gamma \in \Delta \) and \( \alpha - 3\beta \not\in \Delta \), then \( \{ \beta, \gamma \} \) is a fundamental root system of \( \Delta_2 \) which is of type \( G_2 \). We have \( x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) x_{\alpha + 3\beta}(\pm t^3 u) \) or \( x_{\alpha + \beta}(\pm tu) x_{\alpha + 2\beta}(\pm t^2 u) x_{\alpha + 3\beta}(\pm t^3 u) \) according as \( \lambda(\alpha) = 1 \) or 3. Thus \( (x_\alpha(t), x_\beta(u)) = x_{\alpha + \beta}(\pm tu) \). If \( u \in a_1 \), then \( 2tu \in a_1 \) and \( 3tu \in a_2 \). (q. e. d.)

2.7. **Proof of (2.5).** We shall show that \( U' \) is a subgroup of \( G(R) \). This proves that \( U' = U(R, a_1, a_3) \). It suffices to prove that \( x_\alpha(t) x \in U' \) for any \( x_\alpha(t) \in U \) and \( x \in U' \). We claim this by induction on a regular order of the roots \( \alpha \). If \( \alpha \) is the highest root then \( x_\alpha(t) = xx_\alpha(t) \) and the assertion is trivial. Assume that \( x_\alpha(t) x \in U' \) for any \( x_\alpha(t) \) and \( x \in U' \) such that \( \alpha > \beta \). We must show that \( x_\beta(t) x \in U' \) for any \( t \in a_1(\beta) \) and \( x \in U' \). Let \( x_\alpha = x_{\beta}(s_1) x_{\alpha}(s_{1+1}) \cdots \).
let $x_{β_ι}(s_ι)$ be an element of $U'$. Then $x_{β_ι}(t)x_ι \in U'$ is trivial by (2.6). Now assume $x_{β_ι}(t)u_κ \in U'$ for any $u_κ (κ > i)$ and we show that $x_{β_ι}(t)x_ι \in U'$. If $β = β_ι$ for some $i < j \leq N$, then this is trivial. Therefore, we may assume that $β = β_ι$ for some $j > i$. From (2.6), we have

$$x_{β_ι}(t)x_ι = x_{β_ι}(t)x_{β_ι}(s_ι)x_{ι+1} = x_{β_ι}(s_ι)x_{β_ι}(s_j)2x_{ι+1}$$

where $z$ is an element of $U'$ expressible by a product of $x_ι(t) \in U'$ for $α > β$. Further, by our assumption, $x_{β_ι}(s_j)x_ι \in U'$. Thus, we have proved $x_{β_ι}(t)x_ι \in U'$. q. e. d.

**2.8. PROPOSITION.** If $a_ι$ is a proper ideal of $R$ and $a_ι$ is a special submodule associated with $(G, a_ι)$, then

$$E(R, a_ι, a_ι) = U(R, a_ι, a_ι)T'\left(R, a_ι, a_ι\right)V(R, a_ι, a_ι).$$

First, we prove some lemmas.

**2.9. LEMMA.** For any root $α$ and a unit element $u$ of $R$, there exists $h(α) \in T'(R)$ such that $χ(α) = u^2$. Further, let $Δ$ be of rank $> 1$, then there exists $h(α) \in T'(R)$ such that $χ(α) = u$ if and only if $G$ is not of symplectic type or $λ(α) = 1$.

**PROOF.** Since $χ(α), u(α) = u^2$, the first assertion is trivial. If $X = P_ι$, the second assertion is also trivial. We may assume that $α$ is in $II = \{α_ι, \ldots , α_ι\}$, say $α = α_ι$, and let $α_{ι+1}$ be not orthogonal to $α_ι$. If $Δ_ι = \{α_ι, α_{ι+1}\}$ is of type $G_ι$, then $Δ = Δ_ι$ and the lemma holds from $P = X = P_ι$. If $Δ_ι$ is of type $A_ι$ (resp. type $B_ι$ and $λ(α) = 1$), then $χ = χ_{α, a_ι} (resp. = χ_{α, a_ι}χ_{α, a_ι})$ has the value $u$ at $α$. Thus, we can find $h(α) \in T'(R)$ such that $χ(α) = u$ except the case $G$ is of symplectic type and $λ(α) = 2$. q. e. d.

**2.10. COROLLARY.** If $a_ι$ is a proper ideal and $x_ι(t) \in E(R, a_ι, a_ι)$, then

$$h(χ)x_ι(x_ι(t))h(χ)^{-1} \in E(R, a_ι, a_ι)$$

for any $h(χ) \in T(R)$.

**PROOF.** This follows from (2) and the above lemma.

**2.11. LEMMA.** Let $Δ$ be of rank $> 1$ and $a_ι$ be proper. If $u = 1 + st$ where $s \in R$ and $t \in a_ι$, then $χ_{a_ι}(β) \equiv 1(\mod a_ι)$ for any root $β$ such that $λ(β) = 1$ and $χ_{a_ι}(β) \equiv 1(\mod a_ι)$ for any root $β$ such that $λ(β) = λ$.

**PROOF.** Note that $χ_{a_ι}(β) = (1 + st)^{λ(β)}$ where $t \in a_ι(β)$. If $λ(β) = 1$, then $a_ι(β) = a_ι$ is an ideal such that $≥ a_ι$. Therefore, the assertion is trivial.
\(\lambda(\beta) = \lambda\) and \(\lambda(\alpha) = 1\), then we have \(\chi_{a,u}(\beta) = (1 + st)^{-1} \equiv 1 \pmod{\alpha_{1}}\). Finally, let \(\lambda(\beta) = \lambda(\alpha) = \lambda\). If \(G\) is not of symplectic type, then the assertion follows from the fact that \(\alpha_{1} = \alpha_{1}(\beta)\) is an ideal of \(R\). If \(G\) is of symplectic type, then we have \(\beta(H_{a}) = 2\) or 0 according as \(\alpha = \beta\) or \(\alpha \neq \beta\). Therefore, we have also \(\chi_{a,u}(\beta) \equiv 1 \pmod{\alpha_{1}}\). q. e. d.

2.12. COROLLARY. If \(a_{1}\) is a proper ideal and \(h(\chi) \in T(R, a_{1}, a_{3})\), then \(x_{a}(s)h(\chi)x_{a}(s)^{-1} \in E(R, a_{1}, a_{3})\) for any \(x_{a}(s) \in E(R)\).

PROOF. This follows from the relation \(x_{a}(s)h(\chi)x_{a}(s)^{-1} = x_{a}((1 - \chi(\alpha))s)h(\chi)\) (cf. (2)) and the above lemma.

2.13. LEMMA. If \(a_{1}\) is a proper ideal and \(x_{a}(t) \in E(R, a_{1}, a_{3})\), then

\[
(14) \quad x_{-a}(s)x_{a}(t)x_{-a}(s)^{-1} = x_{a}(v)h(\chi_{a,u})x_{-a}(w)
\]

for any \(x_{-a}(s)\), where \(x_{a}(v)\) and \(x_{-a}(w)\) are elements of \(E(R, a_{1}, a_{3})\) and \(h(\chi_{a,u})\) is an element of \(T'(R, a_{1}, a_{3})\).

PROOF. Since \(t \in \mathfrak{m}, 1 + st\) is a unit in \(R\). Therefore, the equation

\[
\begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
u & 0 \\
0 & u^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
w & 1
\end{pmatrix}
\]

has a solution, i. e., we have \(u = (1 + st)^{-1}, v = t(1 + st)^{-1}\) and \(w = -s t(1 + st)^{-1}\). Thus, we have (14) where \(h(\chi_{a,u}) \in T'(R, a_{1}, a_{3})\) by definition. Further, if \(G\) is not of symplectic type, \(x_{a}(v), x_{-a}(w) \in E(R, a_{1}, a_{3})\) for \(a_{1}\) and \(a_{3}\) are ideals. If \(G\) is of symplectic type, since \((1 + st)^{-1} \equiv 1 - st (\text{mod }\alpha_{1}), v \equiv t(1 - st) \equiv 0, w \equiv -s t(1 - st) \equiv 0 (\text{mod }\alpha_{1})\) (cf. 1. 8. (b)). Therefore, we have also \(x_{a}(v), x_{-a}(w) \in E(R, a_{1}, a_{3})\). q. e. d.

2.14. LEMMA. If \(a_{1}\) is a proper ideal, \(x_{a}(t) \in E(R, a_{1}, a_{3})\) and \(\beta\) is a positive root \(\neq \alpha\), then

\[
(15) \quad x_{-\beta}(s)x_{a}(t)x_{-\beta}(s)^{-1} = xy \text{ for any } x_{-\beta}(s) \in E(R),
\]

where \(x \in U(R, a_{1}, a_{3})\) and \(y\) is a product of \(x_{-\gamma}(u)\)'s in \(V(R, a_{1}, a_{3})\) such that \(-\gamma > -\beta\).

PROOF. Since \(\alpha\) and \(-\beta\) are linearly independent, there exists an element \(w\) which is a product of \(\omega_{\gamma}\) for some roots \(\gamma \in \Delta\), such that \(\omega x_{a}(t)w^{-1}\) and \(\omega x_{-\beta}(s)w^{-1}\) are in \(U(R, a_{1}, a_{3})\). Therefore, \(x_{-\beta}(s)x_{a}(t)x_{-\beta}(s)^{-1} \in w^{-1}U(R, a_{1}, a_{3})w\).
From (2.5), any element of $U(R, a_1, a)$ can be expressed by the form

$$x_{\beta_1}(s_1)x_{\beta_2}(s_2)\cdots x_{\beta_N}(s_N),$$

where $s_i \in a_i(\mathfrak{g})$ and $\beta_1, \cdots, \beta_N$ are the positive roots. If we arrange the order of the roots in such a way that $w(\beta_i) > 0$ for $1 \leq i \leq j$ and $w(\beta_i) < 0$ for $j + 1 \leq i \leq N$, then we have $w^{-1}U(R, a_i, a_j)w \subseteq U(R, a_i, a_j)V(R, a_i, a_j)$. Since $x$ and $y$ are products of $x_{\alpha}(u)$'s where $\gamma$ are linear combinations of $\alpha$ and $-\beta$, we have our assertion. q. e. d.

**2.15. PROOF OF (2.8).** For convenience, denote by $UTV$ the set in the right side of the equation (13). First, we claim that $UTV$ is a subgroup of $E(R)$. It suffices to prove that $zUTV \subset UTV$ for any element $z$ of $UTV$ of the form $x_{\beta}(t)$, $h(\chi_{\beta, u})$ and $x_{-\beta}(t)$. If $z = x_{\beta}(t)$, then by (2.5), we have $x_{\beta}(t) U \subset U$. If $z = h(\chi_{\beta, u}) \in T'$, then from (2.10), we have $h(\chi_{\beta, u})U \subset U'$. Finally, if $z = x_{-\beta}(t)$, we show by induction on a regular order of the roots that

\[(16) \quad x_{-\beta}(t)U \subset UTV \quad \text{for any} \quad x_{-\beta}(t) \in V.\]

If $-\beta$ is the largest negative root, from (2.13) and (2.14), (16) is true. Assume that (16) holds for any negative root larger than $-\beta$. We must show that $x_{-\alpha}(t)x \in UTV$ for any $x \in U$. If $x = x_{\beta}(t)$, it is clear from (2.14). Now, assume that it is true for $x' = x_{\alpha}(s_{i+1}) \cdots x_{\alpha}(s_N) \in U$, and let $x = x_{\beta}(s_i)x' \in U$. Then we have again by (2.14), $x_{-\beta}(t)x_{\alpha}(s_i)x' = x_{\beta}(s_i)x'yx'$ and by our assumption $yx' \in UTV$. Thus we have $x_{-\beta}(t)x \in UTV$. This completes the proof of (16).

Secondly, we claim that $UTV$ is normal in $E(R)$. It suffices to show that $x_{\pm\alpha}(t)UTVx_{\pm\alpha}(t)^{-1} \in UTV$ for any root $\alpha, z$ and any $t \in R$. We have $x_{\alpha}(t)ux_{\alpha}(t)^{-1} \subset U$ (cf. 2.6) and $x_{\alpha}(t)h(\chi_{\beta, u})x_{\alpha}(t)^{-1} \subset U'$ for any $h(\chi_{\beta, u}) \in T'$ and any $t \in R$ (cf. 2.12). The elements of $V$ is expressible by a product of $x_{\pm\alpha}(u)$ and an element of $V^{(d)}$ consisting of elements expressible by a product of $x_{\gamma}(s)$ such that $\gamma$ are negative roots different from $-\alpha_i$ and that $s \in a_i(u)$. Since $x_{\alpha}(t)x_{-\alpha}(t)x_{\alpha}(t)^{-1} \in UTV$ and $x_{\alpha}(t)V^{(d)}x_{\alpha}(t)^{-1} \in V^{(d)}$ (cf. 2.14), we have $x_{\alpha}(t)Vx_{\alpha}(t)^{-1} \subset UTV$. Therefore, we have $x_{\alpha}(t)UTVx_{\alpha}(t)^{-1} \subset UTV$. A similar calculation applies to $x_{-\alpha}(t)$. q. e. d.

**2.16. PROPOSITION.** $B(R) = U(m)T(R)V(R)$ (resp. $B(R) = U(m)T'^{(R)}V(R)$) is a subgroup of $G(R)$ (resp. $E(R)$), where $U(m)$ is the subgroup of $U(R)$ generated by $x_{\gamma}(t)$ for all $t \in m$ and all positive root $\alpha$.

**PROOF.** Iwahori-Matsumo (4), Theorem 2.5) have proved this in the case that $R$ is the ring of integers of a field with a non-trivial, non-archimedean discrete valuation and $G$ is an adjoint group. However, their proof remains valid also in our case.

The following proposition plays a fundamental role in the proof of our main theorem.
2.17. Proposition. Let $G$ be a simple Chevalley-Demazure group scheme, $R$ be a local ring and $a$ a proper ideal of $R$ and $b$ a special submodule associated with $(G, a)$. Assume that $p \neq 2$ and $k \neq F_2$ if $G$ is of type $A_1$ and that $k \neq F_2$ if $G$ is of type $B_2$ or $G_2$. Let $N$ be a subgroup of $G(R)$ normalized by $E(R)$ such that $E^*(R, a, b) \supset N \supset E(R, a, b)$. Then $N$ contains an element $x_a(t)$ not contained in $E(R, a, b)$.

The proof will be divided into several steps. We set

$$E^*(R, a, b) = U(R, a, b)T(R, a, b)V(R, a, b)$$

where $T(R, a, b) = T(R) \cap E^*(R, a, b)$. Then $E^*_a(R, a, b)$ is a subgroup of $G(R)$ normalized by $E(R)$ such that $E^*(R, a, b) \supset E^*_a(R, a, b) \supset E(R, a, b)$. We denote by $N' = N - E^*_a(R, a, b)$. Then, (2.17) follows immediately from the following which we shall prove in the next section.

2.18. Assume that $k \neq F_2, F_3$ if $G$ is of type $A_1$. If $N' \neq \emptyset$, then $N \cap B(R) \neq \emptyset$.

2.19. Assume that $p \neq 2$ and $k \neq F_3$ if $G$ is of type $A_1$ and that $k \neq F_2$ if $G$ is of type $G_2$. If $N' \cap B(R) \neq \emptyset$, then $N' \cap x_{\beta}(R)x_{\beta}(R) \neq \emptyset$, where $\beta$, $\beta$ are dominant roots of $\Delta$ (for the definition, see 3.5).

2.20. Assume that $k \neq F_3$ if $G$ is of type $B_2$ or $G_2$. If $N' \cap x_{\beta}(R)x_{\beta}(R) \neq \emptyset$, then $N' \cap x_{\beta}(R) \neq \emptyset$ for some root $\alpha$.

2.21. Assume that $p \neq 2$ and $k \neq F_3$ if $G$ is of type $A_1$ and that $k \neq F_2$ if $G$ is of type $B_2$ or $G_2$, then $E^*_a(R, a, b) = E^*_a(R, a, b)$.

3. Proof of the main theorem. In this section, we prove (1.6), (2.17) and then prove our main theorem (1.9) and its corollaries. We use notations and definitions same as those in the previous sections.

3.1. Proposition. Let $G$ be a Chevalley-Demazure group scheme. Then $\Omega(C) = U(C)T(C)V(C)$ is an affine open subset of $G(C)$ and there exists a rational representation $\phi$ of $G(C)$ into a general linear group $GL_N(C)$ such that the coordinate function $d_\phi(g) (1 \leq i, j \leq N)$ of $\phi(g)$ is in $Z[G]$ and that the affine ring of $\Omega(C)$ is $C[G][d_\phi]$. Further, the mapping

$$\theta(C) : U(C) \times T(C) \times V(C) \to G(C)$$

defined by $\theta(C)(x, h, y) = xhy$ induces a ring isomorphism.
where $Z[U]$ (resp. $Z[V]$) is the affine ring of the subgroup $U$ (resp. $V$) of $G$.

This proposition follows from a theorem in [2].

3.2. Proof of (1.6). In (3.1), we denote by $G'$ the group scheme defined by the subring $Z[G']$ of $Z[G]$ generated by $d_{i,j}(1 \leq i, j \leq N)$. The homomorphism $\phi$ defines a homomorphism of group schemes $G \to G'$ which we denote also by $\phi$. Since $\theta(R) : U(R) \times V(R) \to \Omega(R) = \text{Hom}(Z[G] [d_{i,j}], R)$ defined by $\theta(R)(x, h, y) = xhy$ is bijective, we have $\Omega(R) \subset G_0(R)$. On the other hand, if $g \in G(R, m)$, then $\phi(g) \in G'(R, m)$. This shows that $d_{1,1}(g) \equiv 1 \pmod{m}$ and $d_{1,1}(g)$ is a unit in $R$. Therefore, $g \in \Omega(R)$. Thus, we have $G(R, m) \subset \Omega(R) \subset G_0(R)$. Now, let $\varphi$ be the homomorphism of groups $G(R) \to G(R/m)$ induced by the canonical homomorphism of rings $R \to R/m$. For any element $g \in G(R)$, $\varphi(g)$ is an element of $G_0(R)$. Therefore, $g = g_1 g_2$ where $g_1 \in G(R, m)$ and $g_2$ is an element of $G_0(R)$ such that $\varphi(g) = g_2$. Thus, we have $g \in G_0(R)$. This shows that $G(R) = G_0(R)$. If $G$ is simply connected, then $T(R) = T'(R) \subset E(R)$. Therefore, we have $G(R) = E(R)$. q.e.d.

3.3. Corollary. Let $\alpha$ be a proper ideal of $R$, then

$G(R, \alpha) = U(R, \alpha) T(R, \alpha) V(R, \alpha)$

$G^*(R, \alpha) = U(R, \alpha) T^*(R, \alpha) V(R, \alpha)$,

where $T^*(R, \alpha) = G^*(R, \alpha) \cap T(R)$.

This follows easily from the above proposition.

3.4. Proof of (2.18). If $N \subset G^*(R, m)$, then $N \subset B(R)$ and the assertion is trivial. If $N \not\subset G^*(R, m)$, then $\varphi(N)$ is a subgroup of $G(k)$ normalized by $E(k)$ not contained in the center of $G(k)$. Therefore, we have $\varphi(N) \cap T'V(k) \neq 1$ (cf. [1], p. 50. We assume that if $G$ is of type $A_l, k \neq F_2, F_3$). Thus, there exists an element $g \in N$ such that $\varphi(g) = \varphi(h) \varphi(y) \in T'V(k)$ for some elements $h \in T(R)$ and $y \in V(R)$ and that $\varphi(g)$ is not contained in the center of $G(k)$. This means that $g = g'h'y$ for some $g' \in G(R, m)$. Since $g'$ is expressed by the form $x'h'y$ where $x \in U(R, m)$, $h' \in T(R, m)$ and $y' \in V(R, m)$, we have $g \in B(R)$ and $g \not\in G^*(R, m)$. This shows that $N \cap B(R) \neq \emptyset$.

3.5. Now, we proceed to prove (2.19). First, we give some preliminary lemmas on irreducible root systems. Let $\Delta$ be an irreducible root system and
\( \Pi = \{ \alpha_1, \cdots, \alpha_l \} \) be a fundamental system of roots. A root \( \beta \in \Delta \) is called to be dominant if \( \beta(H_\alpha) \geq 0 \) for all \( \alpha \in \Pi \). By definition, the highest root is dominant. Further, if \( \lambda(\alpha) = 1 \) for all root \( \alpha \in \Delta \), then the highest root is the only dominant root. On the other hand, if \( \Delta \) has a root of length 2 or 3, there exist exactly two dominant roots and the length of these two roots are different each other (cf. [1], Lemma 13, p. 60).

3.6. **Lemma.** Let \( \Delta \) be not of type \( G_2 \), then

(i) For any positive root \( \alpha \in \Delta \) which is not in \( \Pi \), there exists a root \( \alpha_i \in \Pi \) such that \( \alpha - \alpha_i \in \Delta \) and \( \alpha + \alpha_i \in \Delta \).

(ii) For any positive root \( \alpha \in \Delta \) which is not dominant, there exists a root \( \alpha_i \in \Pi \), such that \( \alpha + \alpha_i \in \Delta \) and \( \alpha - \alpha_i \notin \Delta \).

**Proof.** We claim that for any positive root \( \alpha \) which is not in \( \Pi \), there exists a root \( \alpha_i \in \Pi \) such that \( \alpha(H_{\alpha_i}) > 0 \). We see \( \lambda(\beta)(\alpha(H_\beta)) = \lambda(\alpha)(\beta(H_\beta)) \) for any root \( \alpha, \beta \in \Delta \). If \( \alpha = \sum_{i=1}^{l} m_i \alpha_i \), then \( 2\lambda(\alpha) = \lambda(\alpha)(\alpha(H_\alpha)) = \lambda(\alpha) \sum_{i=1}^{l} m_i \lambda(\alpha_i)(\alpha(H_{\alpha_i})) > 0 \). Since \( \lambda(\alpha) > 0 \), \( m_i \geq 0 \) and \( \lambda(\alpha_i) > 0 \), \( \alpha(H_{\alpha_i}) > 0 \) for some \( \alpha_i \). Thus \( \alpha - \alpha_i \) is a root. As for a positive root which is not dominant, by definition, there exists a root \( \alpha_i \in \Pi \), such that \( \alpha(H_{\alpha_i}) < 0 \). Thus \( \alpha + \alpha_i \) is a root. Now, let \( \Delta \) be not of type \( G_2 \). Assume \( \alpha \pm \alpha_i \) are roots. Then \( \pm \alpha, \pm \alpha_i, \pm (\alpha + \alpha_i) \) and \( \pm (\alpha - \alpha_i) \) are the only linear combinations of \( \alpha \) and \( \alpha_i \) which are roots (cf. [1], Lemma 2, p. 20). This contradicts to \( \alpha(H_{\alpha_i}) \neq 0 \). Thus we have our lemma.

q. e. d.

3.7. **Lemma.** Let \( \alpha = \sum_{i=1}^{l} m_i \alpha_i \) be the highest root. We know that if \( \Delta \) is of type \( A_n, B_n, C_n, D_n, E_6 \) or \( E_7 \), then Min \( m_i = 1 \) and if \( \Delta \) is of type \( E_8, F_4 \) or \( G_2 \), then Min \( m_i = 2 \). In the former case, we set \( \alpha_i \) one of the roots \( \alpha_i \in \Pi \) such that \( m_i = 1 \) and further, if \( \Delta \) is of type \( A_n, \alpha_i \) is not orthogonal to \( \alpha_0 \) and the latter case, we set \( \alpha_0 \) one of the roots \( \alpha_0 \in \Pi \) such that \( m_i = 2 \) and that \( \alpha_0 \) is not orthogonal to \( \alpha_i \) and orthogonal to all roots in \( \Pi \) different from \( \alpha_i \). (There exists exactly one root which has these properties.) Then, the diagram of \( \Pi - \{ \alpha_i \} \) is connected. Further, we have

**Lemma.** Let \( \Delta \) be of type \( E_6, F_4 \) or \( G_2 \) and \( \alpha = \sum_{i=1}^{l} m_i \alpha_i \) be a root. Then, \( m_i = 2 \) if and only if \( \alpha \) is the highest root.

**Proof.** If \( \alpha = \alpha_0 \), then \( m_i = 2 \). Conversely, if \( \alpha = \sum_{i=1}^{l} m_i \alpha_i \) is a root such
that \( m_1 = 2 \) and \( \alpha \neq \alpha_\circ \), then we have \( \alpha_\circ - \alpha_{i(1)} - \cdots - \alpha_{i(k)} = \beta \) for some \( \alpha_{i(\ell)} \)
where \( i(\ell) \neq 1 \) \((1 \leq j \leq k)\). This is a contradiction, for \( \alpha_\circ - \alpha_i \in \Delta \) for all
\( i > 1 \). q.e.d.

3.8. We define a subset \( \Delta_\circ \) of \( \Delta \) closed under addition of roots and an
irreducible subsystem \( \Delta_\circ \) of \( \Delta \) as follows

\[
\Delta_\circ = \left\{ \alpha \in \Delta \mid \alpha = \sum_{i=1}^{l} m_i \alpha_i, \quad m_1 > 0 \right\}.
\]

Let \( \Delta_\circ = \{ \beta_1, \beta_2, \cdots, \beta_m \} \) where \( \beta_i < \beta_{i+1} \) and \( \beta_m = \alpha_\circ \) by a regular order
of \( \Delta \). Then from (3.7), we have

**COROLLARY.** In a group \( G(R) \) whose root system is \( \Delta \), for any roots
\( \beta_i \) and \( \beta_j \) of \( \Delta_\circ \) and for any elements \( s \) and \( t \) of \( R \),

\[
(x_\beta(s), x_\beta(t)) = 1 \quad \text{or} \quad x_\alpha(u) \quad \text{for some} \quad u \in R.
\]

3.9. **LEMMA.** Let \( \gamma \) be a dominant root in \( \Delta_\circ \), then \( \gamma - \alpha_i \in \Delta \) and
\( \gamma + \alpha_i \in \Delta \).

**PROOF.** Since \( \gamma \) is positive and is not a dominant root in \( \Delta \), from (3.6),
\( \gamma + \alpha_i \) is a root for some \( \alpha_i \in \Pi \). On the other hand, \( \gamma + \alpha_i \) is not a root for
all \( \alpha_i \in \Pi \), \( i > 1 \), for \( \gamma \) is a dominant root in \( \Delta_\circ \). Thus \( \alpha + \alpha_1 \) is a root. It is
clear that \( \alpha - \alpha_1 \) is not a root. q.e.d.

3.10. Now, let \( N \) be a subgroup of \( G(R) \) and \( N' \) be its subset stated in
(2.16). Let \( x = x_{\gamma_1(s_1)}x_{\gamma_2(s_2)} \cdots x_{\gamma_n(s_n)} \) be an element of \( N \)
where \( \gamma_i \in \Delta(1 \leq i \leq n) \) and \( \{i(1), i(2), \cdots, i(k)\} \) be the set of all indices such that \( s_{i(\ell)} \notin \alpha_{i(\ell)\alpha_0}(1 \leq j \leq k), \)
\( 1 \leq i(1) < i(2) < \cdots < i(k) \leq n \). Then a simple calculation shows that \( x = x_{\gamma_1(s_1)}x_{\gamma_2(s_2)} \cdots x_{\gamma_n(s_n)} \)
is also an element of \( N' \). We call \( x \) the reduced form of \( x \).

For a subset \( \Delta \) of \( \Delta \), we denote by \( U(\Delta') \) the subgroup of \( U(R) \) generated by
\( x_\alpha(t) \) for all positive roots \( \alpha \) in \( \Delta \) and for all \( t \in R \). Then, we have

3.11 **LEMMA.** Let \( G \) be not of type \( G_2 \). If there exists an element
\( x \in N \cap U(\Delta_\circ) \), then starting from \( x \) by a finite process of taking a commutator
with an element of \( U(\Delta_\circ) \) (resp. \( U(\Delta) \)) and taking its reduced form, we
obtain an element of \( N' \) of the form \( x_{\beta}(t)x_{\beta'}(t') \) (resp. \( x_{\beta}(t)x_{\beta}(t') \)), where
\( \beta, \beta' \) are dominant roots of \( \Delta \), \( \beta' \) the highest root and \( \beta' \) is a positive root
such that \( \beta' + \alpha_i = \beta' \).
We may assume that $x$ is of the form $x_{\beta_1}(t_1)x_{\beta_2}(t_{i+1}) \cdots x_{\beta_m}(t_m)$ where $1 \leq i \leq m$ and $t_i \in a_{i}(a_0)$. We prove the lemma by induction on $i$. If $i = m$, then the assertion is trivial. Suppose $i < m$ and assume that for any element

(1) \[ x = x_{\beta}(t_k)x_{\beta_1}(t_{k+1}) \cdots x_{\beta_m}(t_m), \quad k > i, \quad t_k \in a_{i}(a_0), \]

of $N$ the lemma is true. If $\beta_i$ is not dominant, then, by (3.6, ii), there exists a root $\alpha_i \in \Pi$ such that $\alpha + \alpha_i \in \Delta$ and $\alpha - \alpha_i \notin \Delta$. Therefore, if $\alpha_i = \alpha_i$, then, by (3.8), $(x_{\alpha_i}(1), x) = x'$ can be reduced to an element of $N'$ of the form (1). If $\alpha_i = \alpha$, or $\beta_i$ is dominant, we may assume that $(x_{\beta_i}(t_i), x_i(1)) \in E(R, \alpha_i, a_0)$ for all $\alpha_i \in \Pi \cap \Delta_0$. For, if there exists a root $\alpha_j (j > 1)$ such that $x' = (x_{\beta_i}(t_i), x_{\alpha_j}(1)) \not\in E(R, \alpha_i, a_0)$, then $x'$ can be reduced to an element of $N$ of the form (1). Now, we set $x = x_{\beta_i}(t_i)x'$ where $x' = x_{\beta_i}(t_{i+1}) \cdots x_{\beta_m}(t_m)$. Then we may apply induction assumption to $x'$. Thus we obtain an element stated in the lemma. q.e.d.

**3.12. Corollary.** Let $G$ be not of type $G_2$. If there exists an element $x \in N \cap U(\Delta)$, then starting from $x$ by a finite process of taking a commutator with an element of $U(\Delta)$ and taking its reduced form, we obtain an element of $N'$ of the form $x_{\beta}(t)x_{\beta'}(t')$ where $\beta$, $\beta'$ are dominant roots of $\Delta$.

**Proof.** We prove by induction on the rank of $\Delta$. If $\Delta$ is of rank = 1, then this is trivial. Assume that the lemma holds for the groups of rank less than that of $\Delta$. We set $x = x_1x_0$ with $x_1 \in U(\Delta_i)$ and $x_0 \in U(\Delta_0)$ (cf. 2.5). If $x_0 \in N'$, then by induction assumption, we obtain an element $x' = x_1x_0(s)x_0(s')$ of $N'$ where $x_1 \in U(\Delta_i)$ and $\gamma$, $\gamma'$ are dominant roots of $\Delta_0$. For, the group $U(\Delta_i)$ is stable by taking a commutator with an element of $U(\Delta_0)$. Then, by (3.9), $(x_1, x_0(1)) = x''$ is an element of $U(\Delta_i) \cap N'$. Thus, we may apply (3.11) to $x'$. If $x_0 \notin N'$, then $x_1 \in U(\Delta_l) \cap N'$. We may also apply (3.11) to $x_1$. q.e.d.

**3.13. Proof of (2.19) for the group of not type $G_2$.** If $G$ is of type $A_1$, it is known by Klingenberg (cf. [5], 2.7). Therefore, we can assume that the rank of $G$ is $> 1$. Let $x = xhy \in B(R) \cap N$, where $x \in U(m)$, $h \in T(R)$ and $y \in V(R)$. If $x$ and $y$ are in $E(R, a_1, a_0)$, then $z = h(x) \in N'$. Therefore, there exists a root $\alpha$ such that $\chi(\alpha) \equiv 1$ (mod $a_{i(\alpha)}$). Then, $(x_h(1), h(\chi)) = x(x(\alpha)^{-1} - 1)$ is an element of $N'$. Thus, we may assume that $x \notin E(R, a_1, a_0)$ or $y \notin E(E, a_1, a_0)$. Note that, for an element $z = xhy \in N'$, if $x$ and $y$ are the reduced forms of $x$ and $y$, then $x' = x'hy'$ is also an element of $N'$ which we call the reduced form of $z$. For a subsystem $\Delta'$ of $\Delta$, denote by $G(\Delta')$ the subgroup of $G(R)$ generated by $x_\alpha(t)$ for all $\alpha \in \Delta'$ and all $t \in R$ and by $T(R)$. Now, we prove the
following \((P_n)\) \(n \geq 2\) by induction on \(n\).

\((P_n)\) Let \(G\) be not of type \(G_2\). Suppose there exists an element \(z = xh\gamma\) of \(N' \cap B(R)\) such that \(x \in U(\Delta') \cap U(m)\), \(h \in T(R)\) and \(\gamma \in V(\Delta')\) and that \(x \not\in E(R, a_1, a_2)\) or \(\gamma \not\in E(R, a_1, a_2)\), where \(\Delta'\) is a subsystem of \(\Delta\) of rank \(n\). Then, starting from \(z\), by a finite process of taking its reduced form, taking a conjugate in \(G(\Delta)\) or taking a commutator with an element of \(G(\Delta')\), we obtain an element of the form \(x_1(s)x_2(s')\) in \(N'\), where \(\gamma, \gamma'\) are dominant roots of \(\Delta'\).

### 3.14. Proof of \((P_2)\) for the group of type \(A_2\). Let \(\Delta^+\) be the roots \((4)\) and denote

\[ z = x_\alpha(s_1)x_\beta(s_2)x_{\alpha+\beta}(s_3)\chi(h)x_{-\alpha}(t_2)x_{-\alpha}(t_1). \]

By \((3.12)\), it suffices to show that we obtain an element of \(U(R) \cap N'\) or \(V(R) \cap N'\). If \(x \not\in E(R, a_1)\), the argument is clear. Suppose \(x \in E(R, a_1)\).

(i) If \(s_1 \not\in a_1\) and \(s_3 \not\in a_1\), we have

\[ x_{-\alpha}(1)^{-1}z_{-\alpha}(1) = x_{\alpha}(s_2 \pm s_3)x_{\alpha+\beta}(s_3)\chi(h)x_{-\alpha}(1-x(\alpha))x_{-\alpha}(t_2)x_{-\alpha}(t_1). \]

Therefore \((z', x_{-\alpha}(1))\) is conjugate to \(z'' = x_\alpha(s_2 \pm s_3)x_{\alpha+\beta}(u)x_{-\alpha}(v)\) for some \(u, v \in R\). Then \(z'' = \omega_\alpha \omega_\beta z' \omega_{\alpha-1} \omega_{\beta-1}\) is an element of \(U(R) \cap N'\).

(ii) If \(s_1 \in a_1\) and \(s_3 \not\in a_1\), then we have \((z', x_{-\alpha}(1))\) is conjugate to \(x_{-\alpha}(\pm s_2)x_{-\alpha}(v)\) for some \(v \in R\) which is an element of \(V(R) \cap N'\).

(iii) If \(s_1 \not\in a_1\), then we have

\[ x_{\alpha}(1)^{-1}z_{\beta}(1) = x_\alpha(s_1)x_\beta(s_2)x_{\alpha+\beta}(s_3)\chi(\beta) - 1)h(\chi)x_{-\alpha}(t_2)x_{-\alpha}(t_1). \]

Therefore, \((z, x_{\beta}(1))\) is conjugate to \(z'' = x_{\alpha+\beta}(\pm s_2)x_{\alpha+\beta}(\beta')\chi(\beta)\) for some \(\beta' \in V(R)\) and \(\beta' \in R\). Then \(z''\) is an element of \(N'\) and a similar calculation as one of the above cases applies to \(z''\).

q. e. d.

### 3.15. Proof of \((P_2)\) for the groups of type \(B_2\). Let \(\Delta^+\) be the roots \((6)\) and we denote

\[ z = x_\alpha(s_1)x_\beta(s_2)x_{\alpha+\beta}(s_3)x_{3\alpha+\beta}(s_4)\chi(h)x_{2\alpha}(t_2)x_{-\alpha}(t_2)x_{-\alpha}(t_1). \]

Suppose \(x \not\in E(R, a_1, a_2)\). (i) If \(s_1 \not\in a_1\) and \(s_4 \not\in a_2\), then a direct calculation shows that \((z', x_{-\alpha}(1))\) is conjugate to \(z'' = x_{\alpha+\beta}(\pm 2s_3 \pm s_4)x_{\alpha+\beta}(s_4)y'\) for some \(y' \in V(R)\) and \((z'', x_{-\alpha}(1))\) is conjugate to \(z''' = x_{-\alpha}(\pm s_3)x_{\beta}(\pm s_4)\). Then \(\omega_\alpha \omega_{\beta-1} \omega_{\beta-1} \in U(R) \cap N'\).

(ii) If \(s_1 \in a_1\), \(s_4 \not\in a_1\) and \(s_4 \in a_2\), then we have
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\[ x_{-2a-\beta}(1)^{-1} z' x_{-2a-\beta}(1) = x_\beta(s_1) x_\alpha(s_2) x_{-\alpha}(s_3) x_\beta(s_4) h(\chi) x_{-2a-\beta}(1 - \chi(2\alpha + \beta)) y, \]

and \((z', x_{-2a-\beta}(1))\) is conjugate to \(z'' = x_{-a}(\pm s_1) x_\beta(\pm s_2) x_{-a-\beta}(u)\) for some \(u \in R\). Then \(\omega_{s_5 + \beta} \omega_{s_5}^{-1} \in U(R) \cap N\).

Then \((i)\) If \(s_1 \in a_1, \ s_2 \in a_2, \ s_3 \in a_2, \text{ and } s_4 \in a_3, \) then \((z', x_{-a-\beta}(1))\) is conjugate to \(x_{-a+\beta}(t_1) x_{-2a-\beta}(u)\) for some \(u \in R\) which is an element of \(V(R) \cap N'\).

\((iv)\) If \(s_1 \not\in a_1, \) we have

\[ x_\beta(1)^{-1} z' x_\beta(1) = x_\alpha(s_1) x_\alpha + \beta(\pm s_2) x_\alpha \beta(\pm s_3) x_\beta(s_4) x_\alpha + \beta(\pm s_5) x_{-2a-\beta}(1) x_{-2a-\beta}(1) \]

\[ x_\beta(\chi(\beta) - 1) h(\chi) x_{-2a-\beta}(t_1) x_{-a-\beta}(t_2) x_{-2a-\beta}(2t_1) x_{-2a-\beta}(\pm 2t_3) x_\beta(v)^{-1} x_\beta(\tau) x_\beta(\omega) x_{-a-\beta}(t_1) \]

and \((z, x_\beta(1))\) is conjugate to \(z'' = x_{a+\beta}(\pm s_1) x_{2a+\beta}(\pm s_2) x_\beta(v') \) for some \(v' \in R, \ h \in T(R)\) and \(y' \in V(R)\). A similar calculation as one of the above cases applies to \(z\).

3.16. PROOF OF \((P_{n-1}) \implies (P_n)\) for \(n \geq 3\). No loss of generality, we may assume \(n = l\). Denote \(z = x_{a} y_1 h_1 \) where \(z_1 \in U(\Delta), \ h_1 \in U(\Delta_0), \ h = h(\chi) \in \mathbb{T}(R), \ y_0 \in V(\Delta_0) \) and \(y_1 \in V(\Delta_1)\) (cf. 2.5, 3.7 and 3.8). Suppose \(z_0 = x_{a} y_0 h_0 \in E_\Delta^+(R, a_1, a_2)\) and \(z_0 \in E(R, a_1, a_3) \) or \(y_0 \in E(R, a_1, a_3)\). Then, by \((P_{n-1})\), we obtain an element \(x_\beta(\chi) x_\beta(s) y_1\) of \(N'\) such that \(x_\beta(s) x_\beta(s') \in E(R, a_1, a_3)\) where \(\gamma, \gamma'\) are dominant roots in \(\Delta_0\). For \(U(\Delta_1)\) and \(V(\Delta_1)\) are stable by taking a conjugate by an element of \(E(\Delta_0)\) or a commutator with an element of \(E(\Delta_0)\). Therefore, we may assume that \(z = x_{a} y_1\) for \(z_1 \in U(\Delta)\) and \(y_1 \in V(\Delta_1)\). Then, by (3.11), we obtain an element \(z' = x_\beta(t_1) x_\beta(t') x_\beta(t') y_1\) where \(\beta, \beta'\) are dominant roots and \(\beta''\) is a positive root such that \(\alpha_1 + \beta'' = \beta\) is the highest root and where \(y_1 \in V(\Delta_1)\), for \(V(\Delta_1)\) is stable by taking a commutator with an element of \(U(\Delta_0)\) or taking a reduced form. Further, we may assume that \(x_\beta(t)\) is commutative modulo \(E(R, a_1, a_3)\) for all \(x_\beta(t), \ t > 1\) (cf.proof of 3.11) and that \(z'\) is a reduced form. Now, let \(\Delta'\) be the set of roots \(\gamma\) such that \(x_{-\gamma}(u)\) is a factor of \(y_1\) for \(u \in a_1(\gamma)\). If \(\Delta' = \emptyset\), then \(z' \in U(R) \cap N\). If \(\Delta' \neq \emptyset\), we may assume that there exists a root \(\gamma \neq \alpha_1\) of \(\Delta\). For, otherwise, \(\omega_{a_1} z \omega_{a_1}^\prime \in U(R) \cap N\). For a root \(\gamma' \in \Delta', \) if there exists \(\alpha_2 \in \Pi(i > 1)\) such that \(-\gamma + \alpha_2 \in \Delta\) and \(-\gamma - \alpha_2 \in \Delta\), then \((z', x_{a_2}(1)) \in V(R) \cap N'\). Otherwise, by (3.6. i), for any root \(\gamma \neq \alpha_1\) of \(\Delta', \) \(-\gamma + \alpha_2 \in \Delta\) and \(-\gamma - \alpha_2 \in \Delta\). Therefore, we may assume that \(x_\beta(t)\). For, if \(x_\beta(t)\) is a factor of \(x_{a_2}(t)\), \((z', x_{a_2}(1)) \in U(R) \cap N'\) and further if \(x_\beta(t')\) is a factor of \(x_{a_2}(1)\), \((z', x_{a_2}(1)) \in V(R) \cap N'\). Thus we have proved \((P_n)\).

This completes the proof of (2.19) for the groups of not type \(G_2\).

3.17. PROOF OF (2.19) FOR THE GROUP OF TYPE \(G_2\). Let \(z = x_{a} y_1 \in B(\Delta) \cap N'\). We may assume that \(x \in E(R, a_1, a_3)\) or \(y \in E(R, a_1, a_3)\). Further,
since \( k \neq F \), we may assume that \( u = 1 \). Let \( \Delta^+ \) be the roots (9) and denote

\[
x = x_a(s_1)x_b(s_2)x_{a+b}(s_3)x_{2a+b}(s_4)x_{3a+b}(s_5)x_{3a+2b}(s_6)
\]

\[
y = y_{-3a-3b}(t_8)x_{-3a-3b}(t_8)x_{-2a-3b}(t_5)x_{-2a-3b}(t_5)x_{-a-3b}(t_3)x_{-a}(t_3)
\]

Let \( u \) be a unit of \( R \) such that \( u-1 \) is also a unit, and \( \chi_{a,u}(\alpha) \) be an element of \( \text{Hom}(Z[T], R) \) such that \( \chi_{a,u}(\alpha) = u, \chi_{a,u}(\beta) = 1 \) (resp. \( \chi_{a,u}(\alpha) = 1, \chi_{a,u}(\beta) = u \)). We denote by \( z' \) the reduced form of \( z \).

(i) If \( s_1 \in a_1, s_2 \in a_3 \) and \( s_3 \in a_1 \), then \( h(\chi_{a,u}(\alpha), z) \) is conjugate to \( z = x_{3a+b}(u^{-1}-1)s_4)x_{3a+2b}(u^{-1}s_5)y' \), for some \( y' \in V(R) \). Therefore, if \( s_4 \notin a_4 \), then \( (x_{a+b}(1), z') \) is conjugate to \( z'' = x_a + (u-1)s_4)y' \) and \( (x_{-3a-2b}(1), z'') \) is conjugate to \( x_{-3a+b}(u^{-1}-1)s_4) \). If \( s_5 \notin a_3 \) and \( s_5 \notin a_5 \), \( (x_{-3a-3b}(1), z') \) is conjugate to \( x_{-3a-b}(u^{-1}-1)s_4) \). Finally, if \( s_4 \notin a_4 \) and \( s_5, s_6 \in a_3 \), then \( (x_{-3a-2b}(1), z') \) is conjugate to

\[
z'' = x_{-a-b}(\pm s_4)x_{a+\beta}(\pm s_5)x_{a+b}(\pm s_6)x_{-a}(\pm s_1)
\]

and we have

\[
z''' = \omega_a+b\omega_{a+\beta}^{-1}x_{a+\beta}(\pm s_5)x_{2a+\beta}(\pm s_2)x_{3a+b}(\pm s_1).
\]

Then, \( h(\chi_{a,u}(\alpha), z''') \) is conjugate to \( z^{(4)} = x_{a+\beta}(u-1)s_4)x_{a+\beta}(u-1)s_5)y' \) and \( (h(\chi_{a,u}(\alpha), z^{(4)}) \) is conjugate to \( z^{(5)} = x_{a+\beta}(\pm s_4)x_{a+\beta}(\pm s_5)y' \) where \( s_4 \in a_1, \) If \( v' \notin a_1, \) we have \( (x_{a+b}(1), z^{(5)}) = x_{3a+2b}(\pm v') \).

(ii) If \( s_1 \in a_1, s_2 \in a_3 \) and \( s_3 \notin a_1, \) we may assume that \( s_4 \in a_2, s_5 \in a_4 \) and \( s_6 \in a_3 \). For, if it does not hold, then \( h(\chi_{a,u}(\alpha), z') \) has the form of the case (i). Now let \( z'' = x_{a+\beta}(s_4)y', \) then \( (x_{-3a-3b}(1), z') \) is conjugate to

\[
z'' = x_{-3a-b}(\pm s_4)x_{-a}(\pm s_5)x_{a}(\pm s_6)x_{-3a-b}(\pm s_1)
\]

and we have

\[
z''' = \omega_{a+\beta}\omega_{a-b}^{-1}x_{a+b}(\pm s_5)x_{2a+b}(\pm s_2)x_{3a+2b}(\pm s_1).
\]

Then, \( h(\chi_{a,u}(\alpha), z''') \) is conjugate to \( z^{(4)} = x_{a+\beta}(u^{-1}-1)s_4)x_{a+\beta}(v)x_{a+\beta}(\pm v') \) and \( (h(\chi_{a,u}(\alpha), z^{(4)}) \) is conjugate to \( z^{(5)} = x_{a+\beta}(s_4)x_{a+\beta}(v) \), where \( s_4 \notin a_1, \) If \( v' \notin a_1, \) we have \( (x_{a+\beta}(1), z^{(5)}) = x_{3a+2b}(\pm v') \).

(iii) If \( s_1 \in a_1 \) or \( s_2 \in a_3 \), taking a conjugate of \( h(\chi_{a,u}(\alpha), z') \) or \( h(\chi_{a,u}(\alpha), z') \) if necessary, we may assume that either \( s_1 \notin a_1 \) and \( s_2 \in a_3 \) or \( s_1 \in a_1 \) and \( s_2 \notin a_3 \). Then a conjugate of \( (x_{a+b}(1), z') \) or \( (x_{a+b}(1), z') \) has the form of the case (ii).

\( q. e. d. \)

3.18. PROOF OF (2.20). If the roots of \( \Delta \) have all length 1, then it is
clear. Assume that $\Delta$ has two roots whose lengths are different. If $G$ is of rank $> 2$, then there exists a root $\gamma$ linearly independent to $\beta$, $\beta'$ such that (arranging $\beta$, $\beta'$ in a suitable order) $\beta+\gamma$ is a root and $\beta-\gamma$, $2\beta+\gamma$, $2\beta-\gamma$ and $\beta'+\gamma$ are not roots (cf. [1], Lemma 13, P. 60). Then, if $t \not\equiv a_{\beta}(\beta)$, we have \((x_{\beta}(1), x_{\beta}(t)x_{\beta}'(t)) = x_{\beta+\gamma}(\pm t) \in N'$. Now, let $G$ be of type $B_2$. Since $k \neq F_4$, there exists a unit $u$ of $R$ such that $u^{-1}$ is also a unit. Let $x = x_{a+b}(t)x_{a+b}(t')$. If $t \not\equiv a_{\beta}(\beta)$, we have $y = \omega_{a+b}(t)x_{a+b}(\pm t)$ and $(h(x, y)) = x_{a+b}(\pm (u-1)t) \in N'$. If $G$ is of type $G_2$, since $X = P$, we can prove easily. q. e. d.

\section*{3.19. Proof of (2. 21).} Since $G^*(R, m) \supset E^*(R, a_1, a_2)$, by (3.3), we have $B(R) \supset E^*(R, a_1, a_2)$. Now, assume that $E^*(R, a_1, a_2) \supset E^*(R, a_1, a_2)$. Then, (2.18), (2.19) and (2.20) apply to $N = E^*(R, a_1, a_2)$, we have an element $x_\alpha(t)$ of $E^*(R, a_1, a_2)$ not contained in $E(R, a_1, a_2)$. This is a contradiction. q. e. d.

\section*{3.20. Proof of (1. 9).} If $R$ is a field, then the theorem is a well known result of Chevalley (cf. [1], [10]). Further, if the rank of $G$ is $> 1$, the result has been given by Klingenberg (cf. [5]). If $N$ is a central subgroup of $G(R)$, the theorem is trivial, for $E^*(R, 0)$ contains the center of $G(R)$ and $E(R, 0) = 1$. Therefore, we may assume that the rank of $G$ is $> 1$, $R$ is not a field and $N$ is not central. Let $a_1$ and $a_2$ be the ideal of $R$ and the special submodule of $R$ associated with $(G, a_1)$ which are maximal satisfying $N \supset E(R, a_1, a_2)$. If $a_2 = R$, then by definition $a_1 = R$ and we have $E^*(R, a_1, a_2) \supset E(R, a_1, a_2)$. Therefore, we may assume $a_1$ is proper. Now, assume that $E^*(R, a_1, a_2) \supset N$. Then, by (2.17), there exists an element $x_\alpha(t) \in N$ which is not contained in $E(R, a_1, a_2)$. Further, if $G$ is of symplectic type and $\lambda(\alpha) = \lambda$, then $x_\alpha(rt) \in N$ for any $r \in R$ and otherwise, we have $x_\alpha(rt) \in N$ for any $r \in R$. Now, let $a_1'$ be the ideal of $R$ generated by $a_1$ and $t$, and $a_2'$ be the special submodule associated with $a_2'$ generated by $a_2$ and $t$. Then $N$ contains $E(R, a_1', a_2')$ (cf. 2. 4). This contradicts to the maximality of $a_1$ and $a_2$. Thus, we have $E^*(R, a_1, a_2) \supset N \supset E(R, a_1, a_2)$. Note that if $N \supset E(R, a_1, a_2)$ and $N \supset E(R, a_2)$, where $a_1, b_1$ are ideals of $R$ and $a_1, b_1$ are special submodules associated with $a_1, b_1$, respectively, then $N \supset E(R, a_1, a_2)$. Therefore, $a_1$ and $a_2$ are uniquely determined by $N$. Finally, the result shows that $N$ is a normal subgroup of $G(R)$. q. e. d.

\section*{3.21. Proof of (1.10) and (1.11).} From (1.9), we have $E^*(R, a) \supset G(R, a) \supset E(R, a)$. If $G$ is simply connected, $T(R, a) = T(R, a)$. Therefore, from (3.3), we have $G(R, a) = E(R, a)$. This shows (1.10). (1.11) follows from (1.10) and (2. 4). q.e.d.
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