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ON THE ABSOLUTE NÖRLUND SUMMABILITY OF THE FACTORED FOURIER SERIES

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1. Let $\{s_n\}$ denote the n-th partial sum of a given infinite series $\sum a_n$. Let $\{p_n\}$ be a sequence of constants, real or complex and let

$$P_n = p_0 + p_1 + \cdots\cdots\cdots\cdots + p_n,\ P_{-1} = p_{-1} = 0.$$

The sequence $\{t_n\}$, given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \frac{1}{P_n}\sum_{k=0}^{n} P_k a_{n-k} \tag{1.1}$$

defines the Nörlund means of the sequence $\{s_n\}$ generated by the sequence $\{p_n\}$.

Then, the series $\sum a_n$ is said to be summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, the series

$$\sum_n |t_n - t_{n-1}| \tag{1.2}$$

is convergent.

When the special cases in which $p_n = \dfrac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$, $\alpha > 0$, and $p_n = \dfrac{1}{n+1}$, summability $|N, p_n|$ are the same as the summability $|C, \alpha|$ and the absolute harmonic summability, respectively.

2. Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. We assume, without any loss of generality, that the constant term in the Fourier series of $f(t)$ is zero and that the Fourier series of $f(t)$ is given by

$$\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{2.1}$$

We write

$$\varphi_x(t) = \varphi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\},$$

$$\lambda(n) = \lambda_n \qquad \text{and} \qquad \Delta\lambda_n = \lambda_n - \lambda_{n+1}.$$

Dealing with the $|N,p_n|$ summability of Fourier series, T. Singh [6] proved the following theorem.

THEOREM A. *If* $\varphi(t)$ *is a function of bounded variation in* $(0, \pi)$ *then the series*

$$\sum \frac{(n+1)p_n}{P_n} A_n(t)$$

is summable $|N, p_n|$, *at* $t=x$, *where* $\{p_n\}$ *is a non-negative nonincreasing sequence such that* $\{(n+1)p_n/P_n\}$ *is of bounded variation and the sequence* $\{\Delta p_n\}$ *is non-increasing.*

In this paper, we prove the following theorem.

THEOREM. *Let* $\{p_n\}$ *and* $\{\Delta p_n\}$ *are both non-negative and non-increasing sequences. Let* $\lambda(t)$, $t>0$, *be a positive, non-decreasing function satisfying the condition* $\{\lambda_n/P_n\}$ *is non-increasing**⁾.

If the conditions

$$\sum_{n=k}^{\infty} \frac{\lambda_n p_n}{P_n^{\,2}} = O\left(\frac{\lambda_k}{P_k}\right) \tag{2.3}$$

and

$$\int_0^{\pi} \lambda\left(\frac{\kappa}{t}\right) |d\varphi(t)| < \infty \tag{2.4}$$

for some constant $\kappa>0$ *hold, then the series*

$$\sum_{n=0}^{\infty} \frac{(n+1)p_n}{P_n} \lambda_n A_{n+1}(t)$$

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*⁾ We may replace the condition "$\{\lambda_n/P_n\}$ is non-decreasing" by the conditions "$\lambda(2n) = O(\lambda(n))$ and $\lambda_n = o(P_n)$, as $n\to\infty$". The proof runs almost similar.

is summable $|N, p_n|$ *at* $t=x$.

If $\lambda(t)$ is a constant function the condition (2.3) is satisfied automatically, because

$$\sum_{n=k}^{\infty}\frac{p_n}{P_n^2}\leqq\sum_{n=k}^{\infty}\frac{P_n-P_{n-1}}{P_nP_{n-1}}=\sum_{n=k}^{\infty}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)=O\left(\frac{1}{P_k}\right).$$

Therefore, our theorem includes Theorem A. Applying this theorem, we can deduce several known and unknown theorems about Fourier series.

3. **Proof of Theorem.** We need some lemmas for the proof of our theorem.

LEMMA 1 [3]. *If* $\{p_n\}$ *is non-negative, non-increasing, then, for* $0\leqq a\leqq b\leqq\infty$, $0\leqq t\leqq\pi$ *and any* n, *we have*

$$\left|\sum_{k=a}^{b}p_k e^{i(n-k)t}\right|\leqq P_\tau,$$

where $\tau=[1/t]$ *and* $P_n=p_0+p_1+\cdots\cdots+p_n$.

LEMMA 2 [6]. *If* $\{p_n\}$ *and* $\{\Delta p_n\}$ *are both non-negative and non-increasing then the sequence* $\{(p_k-p_n)/(n-k)\}$ *is also non-increasing for* $k<n$.

LEMMA 3 [6]. *If* $\{p_n\}$ *is non-negative and non-increasing, then* $\{(P_n-P_k)/(n-k)\}$ *is a non-increasing sequence for* $k<n$.

LEMMA 4 [6]. *Under the same assumptions as those of Lemma* 1 *the sequeuce* $\{(p_k-p_n)/P_{n-k}\}$ *is non-increasing.*

PROOF OF THEOREM. Using (1.1) we have

where $$t_n=\frac{1}{P_n}\sum_{k=0}^{n}P_k v_{n-k}\lambda_{n-k}A_{n+1-k}(t),$$

$$v_n=\frac{(n+1)p_n}{P_n}\quad\text{and}\quad A_n(x)=\frac{2}{\pi}\int_0^{\pi}\varphi(t)\cos ntdt.$$

Therefore,

$$t_n - t_{n-1} = \sum_{k=0}^{n-1} \left(\frac{P_k}{P_n} - \frac{P_{k-1}}{P_{n-1}} \right) v_{n-k} \lambda_{n-k} A_{n+1-k}(t)$$

$$= \frac{2}{\pi} \int_0^\pi \varphi(t) \left\{ \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - P_k p_n) v_{n-k} \lambda_{n-k} \cos(n+1-k)t \right\} dt$$

$$= \frac{2}{\pi} \int_0^\pi d\varphi(t) \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - P_k p_n) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k}.$$

Thus, by (1.2), to prove our theorem, it is enough to show that

$$\sum_n |t_n - t_{n-1}|$$

$$\leqq \frac{2}{\pi} \int_0^\pi \left| d\varphi(t) \right| \left| \sum_{n=1}^\infty \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - P_k p_n) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k} \right| = O(1).$$

Considering the condition (2.4), it suffices for our purpose to prove that

$$\text{(3.1)} \qquad \sum = \sum_{n=1}^\infty \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - P_k p_n) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k} \right|$$

$$= O\left(\lambda \left(\frac{\kappa}{t} \right) \right), \quad \text{uniformly for } 0 < t < \pi.$$

Let us write $\tau = [\kappa/2t]$ and $m = [n/2]$, where $[x]$ denote the integral part of x.

Now, we observe that

$$\text{(3.2)} \qquad \sum \leqq \sum_{n=1}^{2\tau} \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - P_k p_n) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k} \right|$$

$$+ \sum_{n=2\tau+1}^\infty \frac{1}{P_n P_{n-1}} \left| \sum^{m} (P_n p_k - P_k p_n) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k} \right|$$

$$+ \sum_{n=2\tau+1}^\infty \frac{1}{P_n P_{n-1}} \left| \sum_{k=m+1}^{n-1} P_n (p_k - p_n) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k} \right|$$

$$+ \sum_{n=2\tau+1}^\infty \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=m+1}^{n-1} (P_n - P_k) v_{n-k} \lambda_{n-k} \frac{\sin(n+1-k)t}{n+1-k} \right|$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4, \quad \text{say}.$$

Since $P_n = p_0 + p_1 + \cdots\cdots\cdots\cdots + p_n > (n+1)p_n$, $|\sin(n-k)t| \leqq (n-k)t$ and λ_n is non-decreasing, we get

$$\sum\nolimits_1 \leqq \sum_{n=1}^{2\tau} \frac{P_n \lambda_n}{P_n P_{n-1}} \sum_{k=0}^{n-1} \frac{p_k(n+1-k)t}{n+1-k} = At \sum_{n=1}^{2\tau} \frac{\lambda_n P_{n-1}}{P_{n-1}} \tag{3.3}$$

$$= O\left(\lambda\left(\frac{\kappa}{t}\right)\right),$$

where A denote an absolute constant. For the inside summation of $\sum_2$, by Abel's transformation, we get

$$I = \sum_{k=0}^{m} (P_n p_k - P_k p_n) \frac{p_{n-k}}{P_{n-k}} \lambda_{n-k} \sin(n+1-k)t$$

$$= \sum_{k=0}^{m} \left(P_n - \frac{P_k p_n}{p_k}\right) \frac{p_{n-k}}{P_{n-k}} \lambda_{n-k} p_k \sin(n+1-k)t$$

$$= \sum_{k=0}^{m-1} \Delta\left[\left(P_n - \frac{P_k p_n}{p_k}\right) \frac{\lambda_{n-k} p_{n-k}}{P_{n-k}}\right] \sum_{\nu=0}^{k} p_\nu \sin(n+1-\nu)t$$

$$+ \left(P_n - \frac{P_m p_n}{p_m}\right) \frac{\lambda_{n-m}}{P_{n-m}} p_{n-m} \sum_{\nu=0}^{m} p_\nu \sin(n-\nu+1)t$$

$$= I_1 + I_2, \quad \text{say.}$$

By virtue of Lemma 1 and the hypotheses of our theorem, we have

$$\sum_{n=2\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} |I_2| \leqq P_\tau \sum_{n=2\tau+1}^{\infty} \frac{\lambda_{n-m} p_{n-m}}{P_n P_{n-m}} \tag{3.4}$$

$$= A P_\tau \sum_{n=\tau}^{\infty} \frac{\lambda_n p_n}{P_n^2} = O\left(\lambda\left(\frac{\kappa}{t}\right)\right).$$

Since

$$\Delta\left[\left(P_n - \frac{P_k p_n}{p_k}\right) \frac{\lambda_{n-k}}{P_{n-k}} p_{n-k}\right]$$

$$= \frac{\lambda_{n-k-1} p_{n-k-1}}{P_{n-k-1}} \Delta\left(P_n - \frac{P_k p_n}{p_k}\right) + \left(P_n - \frac{P_k p_n}{p_k}\right) p_{n-k} \Delta\left(\frac{\lambda_{n-k}}{P_{n-k}}\right)$$

$$+\left(P_n-\frac{P_k p_n}{p_k}\right)\frac{\lambda_{n-k-1}}{P_{n-k-1}}\Delta p_{n-k},$$

we obtain

$$\sum_{n=2\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}|I_1|=AP_\tau\Bigg[\sum_{n=2\tau+1}^{\infty}\frac{1}{P_nP_{n-1}}\Bigg\{p_n\sum_{k=0}^{m-1}\left|\Delta\left(\frac{P_k}{p_k}\right)\right|\frac{\lambda_{n-k-1}p_{n-k-1}}{P_{n-k-1}}$$

$$+\sum_{k=0}^{m-1}\left(P_n-\frac{P_kp_n}{p_k}\right)p_{n-k}\left|\Delta\left(\frac{\lambda_{n-k}}{P_{n-k}}\right)\right|+\sum_{k=0}^{m-1}\left(P_n-\frac{P_kp_n}{p_k}\right)\frac{\lambda_{n-k-1}}{P_{n-k-1}}|\Delta p_{n-k}|\Bigg\}\Bigg]$$

$$=\sum\nolimits_{2,1}+\sum\nolimits_{2,2}+\sum\nolimits_{2,3},\quad\text{say.}\tag{3.5}$$

First we consider $\sum_{2,1}$.

$$\sum\nolimits_{2,1}=AP_\tau\sum_{n=2\tau}^{\infty}\frac{p_n}{P_n^{\ 2}}\sum_{k=0}^{m-1}\frac{\lambda_{n-k-1}p_{n-k-1}}{P_{n-k-1}}\left(\frac{P_{k+1}}{p_{k+1}}-\frac{P_k}{p_k}\right)\tag{3.6}$$

$$=AP_\tau\sum_{n=2\tau}^{\infty}\frac{p_n}{P_n^{\ 2}}\frac{\lambda_np_{n-m}}{P_{n-m}}\sum_{k=0}^{m-1}\left(\frac{P_{k+1}}{p_{k+1}}-\frac{P_k}{p_k}\right)$$

$$=AP_\tau\sum_{n=\tau}^{\infty}\frac{p_n\lambda_n}{P_n^{\ 2}}=O\left(\lambda\left(\frac{\kappa}{t}\right)\right),$$

because $\{\lambda_n/P_n\}$ is non-increasing and $\{P_n/p_n\}$ is non-decreasing.

Obviously,

$$\sum\nolimits_{2,2}=AP_\tau\sum_{n=2\tau+1}^{\infty}\frac{P_np_{n-m-1}}{P_nP_{n-1}}\sum_{k=0}^{m-1}\left(\frac{\lambda_{n-k-1}}{P_{n-k-1}}-\frac{\lambda_{n-k}}{P_{n-k}}\right)\tag{3.7}$$

$$=AP_\tau\sum_{n=\tau}^{\infty}\frac{p_n\lambda_n}{P_n^{\ 2}}=O\left(\lambda\left(\frac{\kappa}{t}\right)\right).$$

It is easy to see that

$$\sum\nolimits_{2,3}=AP_\tau\sum_{n=2\tau+1}^{\infty}\frac{\lambda_{n-m}}{P_{n-1}P_{n-m}}\sum_{k=0}^{m-1}(p_{n-k-1}-p_{n-k})\tag{3.8}$$

$$=AP_\tau\sum_{n=\tau}^{\infty}\frac{\lambda_np_n}{P_n^{\ 2}}=O\left(\lambda\left(\frac{\kappa}{t}\right)\right).$$

Observing that

$$\sum_2 = \sum_{n=2\tau+1}^{\infty} \frac{|I|}{P_n P_{n-1}} \leqq \sum_{n=2\tau+1}^{\infty} \frac{1}{P_n P_{n-1}} (|I_1| + |I_2|),$$

we have, by (3. 4), (3. 5), (3. 6), (3. 7) and (3. 8),

$$\sum_2 = O\left(\lambda\left(\frac{\kappa}{t}\right)\right). \tag{3.9}$$

We now treat $\sum_3$. Since

$$\sum_3 \leqq \sum_{n=2\tau+1}^{\infty} \frac{1}{P_{n-1}} (|J| + |K|) = \sum_{3,1} + \sum_{3,2}, \quad \text{say},$$

where

$$J = \sum_{k=m+1}^{n-\tau} (p_k - p_n) \frac{\lambda_{n-k} p_{n-k}}{P_{n-k}} \sin(n+1-k)t$$

and

$$K = \sum_{k=n-\tau+1}^{n-} (p_k - p_n) \frac{\lambda_{n-k} p_{n-k}}{P_{n-k}} \sin(n+1-k)t,$$

it is enough to estimate $\sum_{3,1}$ and $\sum_{3,2}$ respectively.

Using Abel's transformation we have

$$\begin{aligned} |J| &\leqq \left| \sum_{k=m+1}^{n-\tau-1} \Delta\left[(p_k - p_n) \frac{\lambda_{n-k}}{P_{n-k}} p_{n-k}\right] \sum_{\nu=0}^{k} \sin(n+1-\nu)t \right| \\ &\quad + \left| (p_{n-\tau} - p_n) \frac{\lambda_\tau p_\tau}{P_\tau} \sum_{\nu=0}^{n-\tau} \sin(n+1-\nu)t \right| + \left| (p_{m+1} - p_n) \frac{\lambda_{n-m-1} p_{n-m-1}}{P_{n-m-1}} \sum_{k=0}^{m} \sin(n-k)t \right| \\ &= \frac{A}{t} \sum_{k=m+1}^{n-\tau-1} \left[p_{n-k} \Delta \left\{ \frac{(p_k - p_n)\lambda_{n-k}}{P_{n-k}} \right\} + (p_{k+1} - p_n) \frac{\lambda_{n-k-1}}{P_{n-k-1}} (p_{n-k-1} - p_{n-k}) \right] \\ &\quad + \frac{A p_\tau \lambda_\tau}{t P_\tau} (p_{n-\tau} - p_n) + \frac{A p_m p_{n-m-1} \lambda_{n-m-1}}{t P_{n-m-1}}, \end{aligned}$$

by Lemma 4 and the hypotheses of the theorem.

Applying this to $\sum_{3,1}$, we obtain

$$\sum_{3,1} = \frac{Ap_\tau}{t}\sum_{n=\tau}^{\infty}\frac{p_n\lambda_n}{P_n^2} + \frac{A\tau p_\tau\lambda_\tau}{tP_\tau^2}\sum_{n=2\tau}^{\infty}(p_{n-\tau}-p_{n-\tau+1}) + \frac{A}{t}\sum_{n=\tau}^{\infty}\frac{p_n^2\lambda_n}{P_n^2}$$

$$(3.10) \qquad = A\frac{\tau p_\tau\lambda_\tau}{P_\tau} + \frac{A\tau^2 p_\tau^2\lambda_\tau}{P_\tau^2} + A\tau p_\tau\sum_{n=\tau}^{\infty}\frac{p_n\lambda_n}{P_n^2}$$

$$= O\left(\frac{\tau p_\tau\lambda_\tau}{P_\tau}\right) = O\left(\lambda\left(\frac{\kappa}{t}\right)\right),$$

because

$$p_{n-\tau} - p_n = \sum_{k=1}^{\tau+1}(p_{n-k}-p_{n-k+1}) \leqq \tau(p_{n-\tau}-p_{n-\tau+1}).$$

Next,

$$\sum_{3,2} \leqq \sum_{n=2\tau+1}^{\infty}\frac{1}{P_{n-1}}\sum_{k=n-\tau+1}^{n-1} v_{n-k}\frac{p_k-p_n}{n+1-k}\lambda_{n-k}|\sin(n-k+1)t|$$

$$\leqq A\sum_{n=2\tau+1}^{\infty}\frac{1}{P_{n-1}}\sum_{k=n-\tau}^{n-1}\frac{(p_k-p_n)\lambda_{n-k}}{n+1-k} = A\lambda_\tau\sum_{n=2\tau+1}^{\infty}\frac{p_{n-\tau+1}-p_n}{\tau P_{n-1}}\sum_{k=n-\tau}^{n-1}1$$

$$(3.11) \qquad = A\tau\lambda_\tau\sum_{n=2\tau}^{\infty}\frac{p_{n-\tau}-p_{n-\tau+1}}{P_n} = A\frac{\tau\lambda_\tau}{P_\tau}\sum_{n=\tau}^{\infty}(p_n-p_{n+1}) = O\left(\lambda\left(\frac{\kappa}{t}\right)\right),$$

by Lemma 2 and $v_n = O(1)$.

We devide $\sum_4$ into two parts.

$$(3.12) \qquad \sum_4 \leqq \sum_{n=2\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}(|L|+|M|) = \sum_{4,1} + \sum_{4,2},$$

where

$$L = \sum_{k=m+1}^{n-\tau}(P_n-P_k)v_{n-k}\lambda_{n-k}\frac{\sin(n+1-k)t}{n+1-k}$$

and

$$M = \sum_{k=n-\tau+1}^{n-1}(P_n-P_k)v_{n-k}\lambda_{n-k}\frac{\sin(n+1-k)t}{n+1-k}.$$

By the reason that $\{\lambda_n\}$ is non-decreasing and $\{(P_n-P_k)/(n-k)\}$ is non-increasing,

$$\sum\nolimits_{4,2}=A\sum_{n=2\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{k=n-\tau+1}^{n-1}\frac{P_n-P_k}{n-k}\lambda_{n-k}=A\sum_{n=2\tau+1}^{\infty}\frac{\lambda_\tau p_n(P_n-P_{n-\tau+1})}{\tau P_{n-1}^{\,2}}\sum_{k=n-\tau+1}^{n-1}1$$

$$(3.13)\qquad =A\lambda_\tau\tau\sum_{n=\tau}^{\infty}\frac{p_n^{\,2}}{P_n^{\,2}}=A\tau\lambda_\tau\sum_{n=\tau}^{\infty}\frac{v_n^{\,2}}{n^2}=O\left(\lambda\left(\frac{\kappa}{t}\right)\right).$$

Before the estimation of $\sum_{4,1}$, we must calculate L.

By Abel's lemma we have

$$L=\sum_{k=m+1}^{n-\tau}(P_n-P_k)\frac{\lambda_{n-k}}{P_{n-k}}p_{n-k}\sin(n+1-k)t$$

$$=\sum_{k=m+1}^{n-\tau-1}\Delta\left\{(P_n-P_k)\frac{\lambda_{n-k}}{P_{n-k}}p_{n-k}\right\}\sum_{\nu=0}^{k}\sin(n+1-\nu)t$$

$$+\left\{(P_n-P_{n-\tau})\frac{p_\tau\lambda_\tau}{P_\tau}\sum_{k=0}^{n-\tau}\sin(n+1-k)t-(P_n-P_m)\frac{p_{n-m}\lambda_{n-m}}{P_{n-m}}\sum_{\nu=0}^{m}\sin(n+1-\nu)t\right\}$$

$$=A\tau\sum_{k=m+1}^{n-\tau-1}\frac{p_{n-k-1}\lambda_{n-k-1}}{P_{n-k-1}}(P_{k+1}-P_k)+A\tau\sum_{k=m+1}^{n-\tau-1}(P_n-P_k)p_{n-k}\left|\Delta\left(\frac{\lambda_{n-k}}{P_{n-k}}\right)\right|$$

$$+A\tau\sum_{k=m+1}^{n-\tau-1}(P_n-P_k)\frac{\lambda_{n-k-1}}{P_{n-k-1}}|\Delta p_{n-k}|+A\tau\left\{\frac{\tau p_\tau\lambda_\tau p_{n-\tau}}{P_\tau}+\frac{(n-m)p_m\lambda_{n-m}}{P_{n-m}}\right\}$$

$$=L_1+L_2+L_3+L_4,\text{ say.}$$

Therefore,

$$\sum_{n=2\tau+1}^{\infty}\frac{p_n|L_1|}{P_nP_{n-1}}=A\tau\sum_{n=2\tau+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{k=m+1}^{n-\tau-1}p_{k+1}\frac{p_{n-k-1}\lambda_{n-k-1}}{P_{n-k-1}}$$

$$(3.14)\qquad =A\tau\sum_{n=2\tau+1}^{\infty}\frac{p_np_{m+2}\lambda_{n-m-2}}{P_nP_{n-1}}\sum_{k=\tau}^{n-m}\frac{p_k}{P_k}=A\tau\sum_{k=\tau}\frac{p_k}{P_k}\sum_{n=2k}^{\infty}\frac{p_np_m\lambda_{n-m-1}}{P_nP_{n-1}}$$

$$=A\tau\sum_{k=\tau}^{\infty}\frac{\lambda_kp_k^{\,2}}{P_k^{\,2}}=O\left(\lambda\left(\frac{\kappa}{t}\right)\right).$$

And

$$\sum_{n=2\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} |L_2| = A\tau \sum_{n=2\tau+1}^{\infty} \frac{p_n}{P_{n-1}^2} \sum_{k=m+1}^{n-\tau-1} (n-k) p_k p_{n-k} \left| \Delta \left(\frac{\lambda_{n-k}}{P_{n-k}} \right) \right|$$

$$= A\tau \sum_{k=\tau}^{\infty} k p_k \Delta \left(\frac{\lambda_k}{P_k} \right) \sum_{n=k}^{\infty} \frac{p_n^2}{P_n^2} = A\tau \sum_{k=\tau}^{\infty} p_k \Delta \left(\frac{\lambda_k}{P_k} \right)$$

$$(3.15) \qquad = A\tau p_\tau \sum_{k=\tau}^{\infty} \left(\frac{\lambda_k}{P_k} - \frac{\lambda_{k+1}}{P_{k+1}} \right) = \frac{A\tau p_\tau \lambda_\tau}{P_\tau} = O\left(\lambda\left(\frac{\kappa}{t} \right) \right).$$

By the similar way, we have

$$\sum_{n=2\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} |L_3| = A\tau \sum_{n=2\tau}^{\infty} \frac{p_n p_m}{P_n P_{n-1}} \sum_{k=m+1}^{n-\tau-1} (n-k) \frac{\lambda_{n-k-1}}{P_{n-k-1}} |\Delta p_{n-k}|$$

$$= A\tau \sum_{k=\tau}^{\infty} \frac{\lambda_k}{P_k} \Delta p_k \sum_{n=k}^{\infty} \frac{p_n^2}{P_n^2} = A\tau \sum_{k=\tau}^{\infty} \frac{k\lambda_k}{P_k} (p_k - p_{k+1})$$

$$(3.16) \qquad = A \frac{\tau\lambda_\tau}{P_\tau} \sum_{k=\tau}^{\infty} (p_k - p_{k+1}) = \frac{A\tau p_\tau \lambda_\tau}{P_\tau} = O\left(\lambda\left(\frac{\kappa}{t} \right) \right).$$

At last,

$$\sum_{n=2\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} |L_4| = A\tau v_\tau \lambda_\tau \sum_{n=\tau}^{\infty} \frac{p_n^2}{P_n^2} + A\tau \sum_{n=2\tau+1}^{\infty} \frac{p_n^2 \lambda_n v_n}{P_n^2}$$

$$(3.17) \qquad = O(\lambda_\tau) + O\left(\frac{\tau p_\tau \lambda_\tau}{P_\tau} \right) = O\left(\lambda\left(\frac{\kappa}{t} \right) \right).$$

From the results of (3.14), (3.15), (3.16) and (3.17), we get

$$(3.18) \qquad \sum\nolimits_{4,1} = \sum_{n=2\tau+1}^{\infty} \frac{p_n}{P_n P_{n-1}} |L| = O\left(\lambda\left(\frac{\kappa}{t} \right) \right).$$

Summing up (3.2), (3.3), (3.9), (3.10), (3.11), (3.13) and (3.18), we obtain

$$\sum = O\left(\lambda\left(\frac{\kappa}{t} \right) \right).$$

This terminates the proof of our theorem.

4. Corollaries. Very recently, G. Dass and V. P. Srivastava [2] proved the next theorem.

THEOREM B. *Let* $\{\mu_n\}$ *be a positive non-decreasing sequence and let*

$$\{p_n\} \in \mathcal{M} : \frac{p_{n+1}}{p_n} \leqq \frac{p_{n+2}}{p_{n+1}} \leqq 1 \quad (n=0,1,2,\cdots\cdot).$$

If

$$\sum_{n=1}^{m} |t_n - t_{n-1}| = O(\mu_n), \tag{4.1}$$

then $\sum_{n=1}^{\infty} \varepsilon_n a_n$ *is summable* $|N, p_n|$, *where*

$$\varepsilon_n \mu_n = O(1) \tag{4.2}$$

$$\sum_{n=1}^{\infty} (n+1)\mu_n |\Delta^2 \varepsilon_n| < \infty. \tag{4.3}$$

Applying this theorem and our main theorem, we are able to obtain several known and unknown results.

We observe that $\{p_n\} \in \mathcal{M}$, then $\{\Delta p_n\}$ is non-decreasing because

$$\Delta p_n - \Delta p_{n+1} = p_n - 2p_{n+1} + p_{n+2} \geqq p_n + p_{n+2} - 2\sqrt{p_n p_{n+2}}$$
$$= (\sqrt{p_n} - \sqrt{p_{n+2}})^2 \geqq 0.$$

And, if $\mu_n = \text{constant}$ for $n=1,2,\cdots$, then the condition (4.1) is reduced to $\sum_{n=1}^{\infty} |t_n - t_{n-1}| = O(1)$, that is to say, the series $\sum_{n=1}^{\infty} a_n$ is summable $|N, p_n|$.

Considering the above mentions, we get the following corollaries.

COROLLARY 1 [4]. *If*

$$\int_0^{\pi} t^{-\alpha} |d\varphi(t)| < \infty,$$

then the series $\sum_{n=1}^{\infty} n^{\alpha} A_n(t)$ *is summable* $|C, \beta|$ *at* $t = x$, *where* $0 \leqq \alpha < \beta < 1$.

PROOF. In our theorem, we put

$$p_n = \frac{\Gamma(n+\beta)}{\Gamma(\beta)\Gamma(n+1)}, \quad \lambda_n = n^{\alpha},$$

then $\{p_n\} \in \mathcal{M}$ and $\sum \frac{(n+1)n^\alpha}{(n+\beta)} A_n(t)$ is summable $|C,\beta|$. Put, $\frac{(n+1)n^\alpha}{n+\beta} A_n(t) = a_n$ and $\varepsilon_n = \frac{n+\beta}{n+1}$ then it is easy to see that $\sum_{n=1}^{\infty} \Delta^2 \varepsilon_n = O(1)$ and $\sum_{n=1}^{\infty} \varepsilon_n a_n = \sum_{n=1}^{\infty} n^\alpha A_n(t)$ is summable $|C,\beta|$, by Theorem B. By the similar way, we get

COROLLARY 2. *If* $0<\alpha<1$, $\beta \geqq 0$ *and*

$$\int_0^\pi \left(\log \frac{\kappa}{t}\right)^\beta |d\varphi(t)| < \infty, \text{ where } \kappa > \pi,$$

then the series $\sum_{n=2}^{\infty} (\log n)^\beta A_n(t)$ *is summable* $|C,\alpha|$ *at* $t=x$.

This corollary coincides to L. S. Bosanquet [1] for $\beta = 0$ and R. Mohanty [5] for $\beta = 1$, respectively.

COROLLARY 3. *If,* $1>\alpha \geqq 0$, $\beta \geqq 0$, $\alpha+\beta<1$ *and*

$$\int_0^\pi \left(\log \frac{\kappa}{t}\right)^\beta |d\varphi(t)| < \infty,$$

then the series

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{\{\log(n+2)\}^{1-\beta}} \text{ is summable } \left|N, \frac{1}{(n+2)\{\log(n+2)\}^\alpha}\right|.$$

For $\alpha = \beta = 0$ this corollary is proved by O. P. Vershney [7].

PROOF. Putting

$$p_n = \frac{1}{(n+2)\{\log(n+2)\}^\alpha}, \quad \lambda_n = \{\log(n+2)\}^\beta,$$

we have

$$P_n = \frac{1}{2(\log 2)^\alpha} + \cdots\cdots + \frac{1}{(n+2)\{\log(n+2)\}^\alpha} \sim \frac{\{\log(n+2)\}^{1-\alpha}}{1-\alpha}$$

and λ_n/P_n is non-increasing.

Moreover, it is easy to see that $\{p_n\} \in \mathcal{M}$ and

$$\sum_{n=k}^{\infty}\frac{p_n\lambda_n}{P_n^2} = O(1)\sum_{n=k}^{\infty}\frac{1}{(n+2)\{\log(n+2)\}^{2-\alpha-\beta}} = O\left(\frac{1}{(\log(k+2))^{1-\alpha-\beta}}\right)$$

$$= O\left(\frac{\lambda_k}{P_k}\right), \text{ for } 1-\alpha-\beta>0.$$

Thus all assumptions of our theorem hold. Hence we have $\sum_{n=0}^{\infty}\frac{(n+2)p_n\lambda_n}{P_n}A_n(t)$ is summable $|N, p_n|$.

By some calculation, we see that

$$\gamma_n = P_n - \frac{(\log(n+2))^{1-\alpha}}{1-\alpha} + \frac{(\log 2)^{1-\alpha}}{1-\alpha}$$

is positive bounded and decreasing sequence such that

$$\Delta\gamma_n = O\left(\frac{1}{(n+2)^2\log(n+1)}\right) \text{and } \Delta^2\gamma_n = O\left(\frac{1}{(n+2)^2\log(n+2)}\right).$$

Setting

$$a_n = \frac{(n+2)p_n\lambda_n}{P_n}A_n(t), \qquad \varepsilon_n = \frac{P_n}{(n+2)p_n\log(n+2)},$$

we get $\varepsilon_n = O(1)$ and $\Delta^2\varepsilon_n = O\left(\frac{1}{(n+2)^2(\log(n+2))^{2-\alpha}}\right)$. Thus (4. 2) and (4. 3) hold. Therefore, by Theorem B, the proof is finished.

Following theorem holds, analogously.

COROLLARY 4. *If*

$$\int_0^\pi \left(\log\log\frac{\kappa}{t}\right)^\beta |d\varphi(t)| < \infty \text{ for } 0 \leqq \beta < 1$$

then the series $\sum_{n=0}^{\infty}\frac{A_n(t)}{\log(n+2)\{\log\log(n+2)\}^{1-\beta}}$ *is summable* $\left|N, \frac{1}{(n+2)\log(n+2)}\right|$ *at* $t=x$.

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