ON THE STONE-WEIERSTRASS THEOREM OF $C^*$-ALGEBRAS

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1. Introduction. Let $A$ be the $C^*$-algebra of all complex valued continuous functions vanishing at infinity on a locally compact space. The Stone-Weierstrass theorem gives the conditions under which a $C^*$-subalgebra $B$ coincides with $A$. A plausible non-commutative extension of the Stone-Weierstrass theorem is

Conjecture. Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{B}$ be a $C^*$-subalgebra of $\mathcal{A}$. Let $P(\mathcal{A})$ be the set of all pure states of $\mathcal{A}$ and let $0$ be the identically zero function on $\mathcal{A}$. Suppose that $\mathcal{B}$ separates $P(\mathcal{A}) \cup \{0\}$, then $\mathcal{A} = \mathcal{B}$.


The purpose of this paper is to present another consideration to the conjecture. Unfortunately, we can not solve the problem completely; but the author feels that the results obtained here indicate strongly that the conjecture will be true for all separable $C^*$-algebras. Throughout the present paper, we shall deal with separable $C^*$-algebras only. The main tool to attack the problem is the reduction theory. As corollaries of our results, we shall show: (1) Let $\mathcal{A}$ be a separable $C^*$-algebra and let $\mathcal{B}$ be a uniformly hyperfinite $C^*$-subalgebra of $\mathcal{A}$. Suppose that $\mathcal{B}$ separates $P(\mathcal{A}) \cup \{0\}$, then $\mathcal{A} = \mathcal{B}$; (2) A new proof of Kaplansky’s theorem in the separable case; (3) Let $\mathcal{A}$ be a separable $C^*$-algebra and let $\mathcal{B}$ be a $C^*$-subalgebra of $\mathcal{A}$. Suppose that there exists a $*$-representation $\{\pi, \mathcal{B}\}$ of $\mathcal{A}$ such that $\pi(\mathcal{B}) = \pi(\mathcal{A})$ and the commutant of $\pi(\mathcal{B})$ is hyperfinite, where $\pi(\cdot)$ is the weak closure of $\pi(\cdot)$. Then, $\mathcal{B}$ can not separate $P(\mathcal{A}) \cup \{0\}$; (4) Let $\mathcal{A}$ be a separable $C^*$-algebra and let $\mathcal{B}$ be a $C^*$-subalgebra of $\mathcal{A}$. Suppose that there exists a $*$-representation $\{\pi, \mathcal{B}\}$ of $\mathcal{A}$ such that $\pi(\mathcal{A})$ is a finite $W^*$-agebra and $\pi(\mathcal{B}) = \pi(\mathcal{A})$, where $\pi(\cdot)$ is the weak closure of $\pi(\cdot)$. Then, $\mathcal{B}$ can not separate $P(\mathcal{A}) \cup \{0\}$.

2. Theorems. Let $\mathcal{A}$ be a $C^*$-algebra and let $\mathcal{B}$ be a $C^*$-subalgebra of $\mathcal{A}$. Let $P(\mathcal{A})$ be the set of all pure states of $\mathcal{A}$, and let $0$ be the identically zero function on $\mathcal{A}$. Throughout this section, we shall assume that $\mathcal{B}$ separates $P(\mathcal{A}) \cup \{0\}$—namely, for any two different $\varphi_1, \varphi_2 \in P(\mathcal{A}) \cup \{0\}$, there exists an element $b$ such that $\varphi_1(b) \neq \varphi_2(b)$.
If \( \mathfrak{A} \) has not the unit, we shall consider the \( C^* \)-algebra \( \mathfrak{A}_1 = \mathfrak{A} + \lambda \mathfrak{I} \) and the subalgebra \( \mathfrak{B}_1 = \mathfrak{B} + \lambda \mathfrak{I} \) obtained by adjoining the unit 1, where \( \lambda \) are complex numbers. Any pure state \( \varphi \) on \( \mathfrak{A} \) can be uniquely extended to a pure state \( \tilde{\varphi} \) on \( \mathfrak{A}_1 \); therefore \( P(\mathfrak{A} + \lambda \mathfrak{I}) = P(\mathfrak{A}) + \lambda \rho_\varphi \), where \( \rho_\varphi \) is the pure state of \( \mathfrak{A}_1 \) such that \( \varphi(\mathfrak{I}) = 0 \). Then, clearly \( \mathfrak{B}_1 \) separates \( P(\mathfrak{A}_1) \cup \{0\} \); therefore it is enough to assume that \( \mathfrak{A} \) has the unit 1.

**Lemma 1.** \( \mathfrak{B} \) contains the unit 1.

**Proof.** Suppose that \( 1 \in \mathfrak{B} \). Then \( \|b + 1\| \geq 1 \) for \( b \in \mathfrak{B} \)—in fact, if \( \|b + 1\| < 1 \), \( -b \) is invertible and \( (-b)^{-1} \) is in \( \mathfrak{B} \); hence \( 1 \in \mathfrak{B} \). Therefore, there exists a bounded linear functional \( f \) on \( \mathfrak{A} \) such that \( f(\mathfrak{I}) = 0 \) and \( \|f\| = f(1) = 1 \); hence \( f \) is a state (cf. [4], [11]). Let \( \mathfrak{I} = \{ x \mid f(x^*x) = 0, x \in \mathfrak{A} \} \), then \( \mathfrak{I} \) is a closed left ideal of \( \mathfrak{A} \) and \( \mathfrak{B} \subseteq \mathfrak{I} \). Let \( \mathfrak{S} \) be a maximal left ideal of \( \mathfrak{A} \) such that \( \mathfrak{A} \subseteq \mathfrak{S} \); then there exists a pure state \( \varphi \) on \( \mathfrak{A} \) such that \( \mathfrak{S} = \{ x \mid \varphi(x^*x) = 0, x \in \mathfrak{A} \} \) (cf. [4], [8]); this implies that \( \mathfrak{S} \) can not separate \( \varphi \) and 0. This is a contradiction and completes the proof.

Henceforward, we shall assume that \( \mathfrak{A} \) has the unit and so \( \mathfrak{B} \) contains the unit. In this case, the separation of \( P(\mathfrak{A}) \cup \{0\} \) by \( \mathfrak{B} \) is equivalent to the separation of \( P(\mathfrak{A}) \) by \( \mathfrak{B} \).

**Definition 1.** A \( W^* \)-algebra \( M \) is said to be atomic, if it is a direct sum of type I-factors.

**Definition 2.** Let \( A \) be a \( C^* \)-algebra and let \( \{ \pi, \mathfrak{H} \} \) be a \( * \)-representation of \( A \) on a Hilbert space \( \mathfrak{H} \). By \( \pi(A) \), we shall denote the weak closure of \( \pi(A) \) on \( \mathfrak{H} \). The representation \( \{ \pi, \mathfrak{H} \} \) is called to be atomic, if the \( W^* \)-algebra \( \pi(A) \) is atomic.

**Definition 3.** Let \( \varphi \) be a state on a \( C^* \)-algebra \( A \), \( \{ \pi_\varphi, \mathfrak{H}_\varphi \} \) the \( * \)-representation of \( A \) on a Hilbert space \( \mathfrak{H}_\varphi \), constructed via \( \varphi \). \( \varphi \) is called to be atomic, if the representation \( \{ \pi_\varphi, \mathfrak{H}_\varphi \} \) is atomic.

**Lemma 2.** Let \( \varphi_1, \varphi_2 \) be two states on \( \mathfrak{A} \) such that the restriction \( \varphi_1|\mathfrak{B}, \varphi_2|\mathfrak{B} \) on \( \mathfrak{B} \) are atomic. Suppose that \( \varphi_1 = \varphi_2 \) on \( \mathfrak{B} \), then \( \varphi_1 = \varphi_2 \) on \( \mathfrak{A} \).

**Proof.** Put \( \varphi = \varphi_1 + \varphi_2 \) and consider the \( * \)-representation \( \{ \pi_{\varphi}, \mathfrak{H}_{\varphi} \} \) of \( \mathfrak{A} \). Let \( \varphi(x) = \langle \pi_{\varphi}(x)\xi, \xi \rangle \) for \( x \in \mathfrak{A} \), where \( \langle \cdot, \cdot \rangle \) is the inner product of \( \mathfrak{H}_\varphi \) and \( \xi \) is a vector in \( \mathfrak{H}_\varphi \), and let \( e \) be the projection of \( \mathfrak{H}_\varphi \) onto the closed subspace \( \pi_{\varphi}(\mathfrak{B})\xi \) generated by \( \pi_{\varphi}(\mathfrak{B})\xi \); then the representation \( b \mapsto \pi_{\varphi}(b)e(b \in \mathfrak{B}) \) is
ON THE STONE-WEIERSTRASS THEOREM

atomic. Let $z$ be the central envelope of $e'$ in the commutant $\pi_\phi(\mathfrak{B})'$ of $\pi_\phi(\mathfrak{B})$, then the mapping $yz \mapsto ye'$ of $\pi_\phi(\mathfrak{B})$ onto $\pi_\phi(\mathfrak{B})e'$ is a *-isomorphism; hence $\pi_\phi(\mathfrak{B})$ contains a direct summand of an atomic $W^*$-algebra. Let $p'$ be a minimal projection in $\pi_\phi(\mathfrak{B})'$, then $b \mapsto \pi_\phi(b)p'(b \in \mathfrak{B})$ is irreducible. Take $\eta (\|\eta\|=1) \in p'\hat{\mathcal{D}}$, and consider a state $\psi_\phi(x) = \langle \pi_\phi(x)\eta, \eta \rangle$ for $x \in \mathfrak{A}$. Then, $\psi_\phi|\mathfrak{B}$ is pure; we shall show that $\psi_\phi$ is pure on $\mathfrak{A}$. Let $\Gamma = \{ \psi | \psi = \psi_\phi \text{ on } \mathfrak{B}, \psi \text{ states on } \mathfrak{A} \}$, then $\Gamma$ is a $\sigma(\mathfrak{A}^*, \mathfrak{A})$-compact convex set in $\mathfrak{A}^*$, where $\mathfrak{A}^*$ is the dual Banach space of $\mathfrak{A}$. Arbitrary extreme point in $\Gamma$ is also extreme in the state space of $\mathfrak{A}$; hence it is pure. If $\Gamma$ contains two points, there are two different pure states $\psi_1, \psi_2$ on $\mathfrak{A}$ such that $\psi_1 = \psi_2$ on $\mathfrak{B}$; hence $\Gamma$ consists of only one point and it is pure.

Now suppose that $p'\hat{\mathcal{D}} \cong [\pi_\phi(\mathfrak{A})\eta]$, and let $V$ be the orthocomplement of $p'\hat{\mathcal{D}}$ in $[\pi_\phi(\mathfrak{B})\eta]$. Let $x_1(\neq 0) \in p'\hat{\mathcal{D}}$, $x_1(\neq 0) \in V$ and $\|x_1 + x_2\|=1$. Then, $g_1(x) = \langle \pi_\phi(x)(x_1 + x_2), (x_1 + x_2) \rangle$ and $g_2(x) = \langle \pi_\phi(x)(x_1 - x_2), (x_1 - x_2) \rangle$ for $x \in \mathfrak{A}$ are pure states of $\mathfrak{A}$ and $g_1 = g_2$ on $\mathfrak{B}$. Hence $g_1 = g_2$ on $\mathfrak{A}$. Since the restriction of $\pi_\phi(\mathfrak{B})$ on $[\pi_\phi(\mathfrak{A})\eta]$ is irreducible, $x_1 + x_2 = \lambda x_1 - x_2$ for some complex number $\lambda (|\lambda| =1)$. This is a contradiction; hence $[\pi_\phi(\mathfrak{B})\eta] = [\pi_\phi(\mathfrak{B})\eta]$ and so $p' \in \pi_\phi(\mathfrak{B})'$. Let $c$ be the greatest central projection of $\pi_\phi(\mathfrak{B})'$ such that $\pi_\phi(\mathfrak{B})'c$ is atomic; then any non-zero projection of $\pi_\phi(\mathfrak{B})'$ is a sum of mutually orthogonal minimal projections; hence $c \in \pi_\phi(\mathfrak{B})'$. Since $\xi \in c\hat{\mathcal{D}}$, $[\pi_\phi(\mathfrak{B})\xi] \subset c\hat{\mathcal{D}}$; hence $c\hat{\mathcal{D}} = \hat{\mathcal{D}}$ and so $c = 1_{\hat{\mathcal{D}}}$, where $1_{\hat{\mathcal{D}}}$ is the identity operator on $\hat{\mathcal{D}}$; therefore $\pi_\phi(\mathfrak{B})' \subset \pi_\phi(\mathfrak{B})'$ and so $\pi_\phi(\mathfrak{B}) = \pi_\phi(\mathfrak{B})$. Since $\varphi_1, \varphi_2 \leq 2\varphi$, there exists vectors $\eta_1, \eta_2$ such that $\varphi_1(x) = \langle \pi_\phi(x)\eta_1, \eta_1 \rangle$ and $\varphi_2(x) = \langle \pi_\phi(x)\eta_2, \eta_2 \rangle$ for $x \in \mathfrak{A}$. For $a \in \mathfrak{A}$, there exists a direct set $\{ \pi_\phi(b_a) \}$ ($b_a \in \mathfrak{B}$) such that $\pi_\phi(b_a) \rightarrow \varphi_1(a)$ (strongly); hence $\varphi_1(b_a) \rightarrow \varphi_2(a)$ and $\varphi_2(b_a) \rightarrow \varphi_2(a)$; $\varphi_1(b_a) = \varphi_2(b_a)$ implies $\varphi_1(a) = \varphi_2(a)$. This completes the proof.

**Lemma 3.** Let $\varphi_1, \varphi_2$ be two states on $\mathfrak{A}$ and suppose that one of them is atomic and $\varphi_1 = \varphi_2$ on $\mathfrak{B}$, then $\varphi_1 = \varphi_2$ on $\mathfrak{A}$.

**Proof.** Suppose that $\varphi_1$ is atomic. Consider the *-representation $\{ \pi_\phi, \hat{\mathcal{D}} \}$ of $\mathfrak{A}$, then $\pi_\phi(\mathfrak{B})$ is atomic; hence, there exists a family of mutually orthogonal minimal projections $(e_i | i = 1, 2, \ldots )$ in $\pi_\phi(\mathfrak{B})'$ such that $\sum e_i = 1_{\pi_\phi}$. Let $\varphi_1(x) = \langle \pi_\phi(x)e_i^\sharp, e_i^\sharp \rangle$, then $\varphi_1(x) = \sum_i \langle \pi_\phi(x)e_i^\sharp, e_i^\sharp \rangle = \sum_i \|e_i^\sharp\|^2 \langle \pi_\phi(x) e_i^\sharp, e_i^\sharp \rangle$, $\|e_i^\sharp\|^2$. Since $\langle \pi_\phi(x) e_i^\sharp, e_i^\sharp \rangle$ is pure, its restriction on $\mathfrak{B}$ is also pure (cf. the proof of Lemma 2); hence $\varphi_1|\mathfrak{B}$ is atomic and so by Lemma 2, $\varphi_1 = \varphi_2$ on $\mathfrak{A}$. This completes the proof.
Now we shall explain some results of the reduction theory (cf. [3], [11], [12]). Let $M$ be a type I $W^*$-algebra on a separable Hilbert space, $M_*$ the predual of $M$. Then, $M = \bigoplus_{i=1}^\infty M_i$, where $M_i$ is a homogenous type $I_i$ $W^*$-algebra ($n_i \leq \aleph_0$). Moreover, $M_i = B_i \otimes Z_i$, where $B_i$ is a type $I_i$ factor, and $Z_i$ is the center of $M_i$. Let $B_{i*}$ be the predual of $B_i$, then we can consider the weak *-topology $\sigma(B_i, B_{i*})$ on $B_i$.

Then, we have the realization $B_i \otimes Z_i = L^*(B_i, \Omega_i, \mu_i)$, where $(\Omega_i, \mu_i)$ is a measure space with a probability measure $\mu_i$ and $L^*(B_i, \Omega_i, \mu_i)$ is the $W^*$-algebra of all essentially bounded $B_i$-valued weakly*-measurable functions on $\Omega_i$. For $a \in B_i \otimes Z_i$, the corresponding element of $L^*(B_i, \Omega_i, \mu_i)$ is denoted by $\int a(t)\,d\mu_i(t)$. Therefore, we have the realization $M_{i*} = L^*(B_{i*}, \Omega_i, \mu_i)$.

For $g \in M_{i*}$, the corresponding element in $L^*(B_{i*}, \Omega_i, \mu_i)$ is denoted by $\int g(t)\,d\mu_i(t)$. Then we have: $\|g\| = \int \|g(t)\| \,d\mu_i(t)$, $g_1 + g_2 = \int (g_1(t) + g_2(t))\,d\mu_i(t)$, $\lambda g_1 = \int \lambda g_1(t)\,d\mu_i(t)$, and if $\varphi$ is a normal state on $M_i$, $\varphi(t)$ is a normal state on $B_i$ for almost all $t$; moreover let $D$ be a separable $C^*$-subalgebra of $M_i$, then we can choose a null set $Q_i$ such that $d \rightarrow d(t)$ ($d \in D$) is a *-homomorphism of $D$ into $B_i$ for all $t \in \Omega_i - Q_i$; moreover, if the $W^*$-subalgebra $(D, Z_i)$ of $M_i$ generated by $D$ and $Z_i$ coincides with $M_i$, the weak closure $\overline{D(t)} = B_i$ for all $t \in \Omega_i - Q_i$, where $D(t) = \{d(t) \mid d \in D\}$ and $\overline{D(t)}$ is the weak closure of $D(t)$.

Since $M = \bigoplus_{i=1}^\infty M_i$, by considering the direct sum $\bigoplus_{i=1}^\infty (\Omega_i, \mu_i)$ of the measure spaces $(\Omega_i, \mu_i)$, $M$ can be realized as the $W^*$-algebra of vector valued functions $\int x(t)\,d\mu_i(t)$ such that $x_i \in L^*(B_i, \Omega_i, \mu_i)$, $\|x\| = \sup_i \|x_i\|$, where $x_i$ is the restriction of $x$ on $\Omega_i$. This realization will be denoted by $M = \bigoplus_{i=1}^\infty L^*(B_i, \Omega_i, \mu_i)$.

Now let $E$ be a separable $C^*$-subalgebra of $M$ such that the $W^*$-subalgebra of $M$ generated by $E$ and $Z$ coincides with $M$, where $Z$ is the center of $M$. Then $E z_i$ and $Z_i$ generate $M_i$, where $z_i$ is the identity of $M_i$; hence there exists a null set $Q$ in $\Omega$ such that $a \rightarrow a(t)$ ($a \in E$) is a *-homomorphism and $E(t) = B_i$ for all $t \in \Omega_i - Q$ and all $i$.

Henceforward, the algebra $\mathcal{A}$ will be assumed to be separable. Let $\{\pi, \mathcal{B}\}$
be a *-representation of \( \mathfrak{A} \) on a separable Hilbert space \( \mathfrak{H} \). Put \( \mathfrak{A}_0 = \pi(\mathfrak{A}) \) and \( \mathfrak{B}_0 = \pi(\mathfrak{B}) \) and let \( \mathfrak{A}_0 \) (resp. \( \mathfrak{B}_0 \)) be the commutant of \( \mathfrak{A}_0 \) (resp. \( \mathfrak{B}_0 \)). Let \( C \) be a maximal abelian *-subalgebra of \( \mathfrak{A}_0 \), then the \( W^* \)-algebra \( (\mathfrak{A}_0, C) \) generated by \( \mathfrak{A}_0 \) and \( C \) is of type I and \( C \) is the center of \( (\mathfrak{A}_0, C) \), because \( (\mathfrak{A}_0, C) = \mathfrak{A}_0 \cap C = C \).

By putting \( (\mathfrak{A}_0, C) = M \), we can apply the reduction theory.

**Theorem 1.** Let \( T \) be a linear mapping of \( \mathfrak{A}_0 \) into \( (\mathfrak{A}_0, C) \) such that \( \|T(x)\| \leq \|x\| \) for \( x \in \mathfrak{A}_0 \); (\( \theta \) \( T(y) = y \) for \( y \in \mathfrak{B}_0 \). Then, \( T(x) = x \) for \( x \in \mathfrak{A}_0 \).

**Proof.** Suppose that \( T(x_0) \neq x_0 \) for some \( x_0 \in \mathfrak{A}_0 \). Then, there exists a normal state \( \psi \) of \( (\mathfrak{A}_0, C) \) such that \( \psi(T(x_0)) \neq \psi(x_0) \). \( (\mathfrak{A}_0, C) = \sum_{i=1}^n L^\infty(B_i, \Omega_i, \mu_i) \).

Now let \( D \) be the \( C^* \)-subalgebra of \( (\mathfrak{A}_0, C) \) generated by \( \mathfrak{A}_0 \) and \( T(x_0) \), then \( D \) is separable.

By the previous considerations, we can assume that \( x \rightarrow x(t) \) (\( t \in D \)) is a *-homomorphism of \( D \) into \( B_i \) and \( \mathfrak{A}_0(t) = B_i \) for all \( t \in \Omega_i \setminus \mathfrak{M} \) with \( \mu(\mathfrak{M}) = 0 \), where \( \mathfrak{M} = \{x(t) | x \in \mathfrak{A}_0 \} \).

Let \( \psi = \int \psi(t) \cdot d(t) \) and \( \psi(T(x_0)) = \int \psi(t)(T(t)) \cdot d(t) \). Since \( \psi(x_0) \neq \psi(T(x_0)) \), there exists a set \( \mathfrak{M} \) with \( \mu(\mathfrak{M}) > 0 \) such that \( \psi(t)(x_0(t)) \neq \psi(t)(T(x_0(t))) \) for all \( t \in \mathfrak{M} \). Therefore, there exists a \( t_0 \) such that \( \psi(t_0) \) is a positive linear functional on \( \mathfrak{A}_0(t_0) \) and \( \psi(t_0)(x_0(t_0)) \neq \psi(t_0)(T(x_0(t_0))) \), \( x \rightarrow x(t_0) \) (\( t \in D \)) is a *-homomorphism of \( D \) into \( B_i(t_0) \) and \( \mathfrak{A}_0(t_0) = B_i \). Now we shall define a linear functional \( \psi_1 \) on \( \mathfrak{A}_0 \) as follows: \( \psi_1(a) = \psi(t_0)(\pi(a)(t_0)) \) for \( a \in \mathfrak{A}_0 \). Then, \( \psi_1 \) is an atomic state on \( \mathfrak{A}_0 \). Let \( x_0 = \pi(a_0) \) for some \( a_0 \in \mathfrak{A}_0 \); we shall define a linear functional \( \psi_2 \) on \( \mathfrak{A}_0 + \lambda a_0 \) (\( \lambda \) complex numbers) as follows: \( \psi_2(b + \lambda a_0) = \psi(t_0)(\pi(b)(t_0) + \lambda T(x_0)(t_0)) \) for \( b \in \mathfrak{A}_0 \). Then,

\[
|\psi_2(b + \lambda a_0)| \leq \|\psi(t_0)\| \cdot \|\pi(b) + \lambda T(x_0)\| = \|\psi(t_0)\| \cdot \|T(\pi(b) + \lambda a_0)\| \leq \|\psi(t_0)\| \cdot \|\pi(b) + \lambda a_0\|.
\]

Therefore, \( \psi_2 \) is well-defined and bounded. Let \( \psi_3 \) be a linear functional on \( \mathfrak{A}_0 \) such that \( \|\psi_3\| = \|\psi_2\| \) and \( \psi_2 = \psi_3 \) on \( \mathfrak{A}_0 + \lambda a_0 \). Since \( \psi_3(1) = \psi_2(1) = \psi(t_0) \), \( \psi_3 \) is positive and clearly \( \psi_1 = \psi_3 \) on \( \mathfrak{A}_0 \); hence \( \psi_2(a_0) = \psi(t_0)(\pi(a_0)(t_0)) = \psi(t_0)(x_0(t_0)) = \psi(t_0)(T(x_0)(t_0)). \) This is a contradiction and completes the proof.

Let \( B(\mathfrak{H}) \) be the \( W^* \)-algebra of all bounded operators on \( \mathfrak{H} \). For any \( w \in B(\mathfrak{H}) \), let \( K(w) \) be the weakly closed convex subset of \( B(\mathfrak{H}) \) generated by \( \{u^*wu | u \in C_0\} \), where \( C_0 \) is the set of all unitary elements in \( C \). A family of
weakly continuous linear mappings \( \{ w \to u^*wu \mid u \in \mathcal{C} \} \) on \( B(\mathcal{S}) \) is commutative; hence by the theorem of Kakutani-Markoff (cf. [2]), \( K(w) \) contains at least one fixed point \( w_0 \)---namely, \( u^*w_0u = w_0 \) for all \( u \in \mathcal{C} \); hence \( w_0 \in C = (\mathcal{A}, C) \). Therefore, there exists a projection \( P \) with norm one of \( B(\mathcal{S}) \) onto \( (\mathcal{A}, C) \) (cf. [14]).

Now we shall show

**Theorem 2.** For \( x \in \mathcal{A} \), let \( \Gamma(x) \) be the weakly closed convex subset of \( B(\mathcal{S}) \) generated by \( \{ u^*xu' \mid u' \in \mathcal{B}_{u.u} \} \), where \( \mathcal{B}_{u,u} \) is the set of all unitary elements of the commutant \( \mathcal{B}_u \) of \( \mathcal{B}_u \). Then, \( P(r) = x \) for all \( r \in \Gamma(x) \).

**Proof.** Let \( L(B(\mathcal{S})) \) be the algebra of all bounded operators of \( B(\mathcal{S}) \) into \( B(\mathcal{S}) \). Then, \( L(B(\mathcal{S})) \) is the dual of \( B(\mathcal{S}) \otimes_B B(\mathcal{S}) \), where \( \gamma \) is the greatest cross norm and \( B(\mathcal{S})_\gamma \) is the predual of \( B(\mathcal{S}) \) (cf. [7]). We shall consider the weak \( \ast \)-topology on \( L(B(\mathcal{S})) \). Then, the unit sphere \( S \) of \( L(B(\mathcal{S})) \) is compact. The linear mapping \( V_w : u \to u^*wu \) (\( w \in B(\mathcal{S}) \)) belongs to \( S \); let \( S_0 \) be the weakly \( \ast \)-closed convex subset of \( S \) generated by \( \{ V_{u^*} \mid u^* \in \mathcal{B}_{u,u} \} \), then for arbitrary \( r \in \Gamma(x) \), there exists a \( V \in S_0 \) such that \( V(x) = r \).

Now, consider a linear mapping \( d \to P(V(d)) \) (\( d \in \mathcal{A} \)) of \( \mathcal{A} \) into \( (\mathcal{A}, C) \), then \( P(V(y)) = y \) for \( y \in \mathcal{B} \); hence by Theorem 1, \( P(V(x)) = P(r) = x \). This completes the proof.

**Corollary 1.** Let \( \mathcal{B}_0 \) be the weak closure of \( \mathcal{B}_0 \), then \( \| w - r \| = \| w - x \| \) for \( w \in \mathcal{B}_0 \) and \( r \in \Gamma(x) \), where \( x \in \mathcal{A} \).

**Proof.** For \( u' \in \mathcal{B}_{u,u} \), \( \| w - u'xu^* \| = \| u^*wu' - x \| = \| w - x \| \); therefore \( \| w - \sum_{i=1}^n \lambda_i u_i^*xu_i^* \| \leq \| w - x \| \), where \( \lambda_i \geq 0 \) and \( \sum_{i=1}^n \lambda_i = 1 \), \( u_i \in \mathcal{B}_{u,u} \); hence \( \| w - r \| \leq \| w - x \| \).

On the other hand, if \( \| w_0 - r_0 \| < \| w_0 - x \| \) for some \( w_0 \in \mathcal{B}_0 \) and \( r_0 \in \Gamma(x) \), then \( \| P(w_0 - r_0) \| = \| w_0 - P(r_0) \| \leq \| w_0 - r_0 \| \). But, \( w_0 - P(r_0) = w_0 - x \). This is a contradiction and completes the proof.

**Corollary 2.** \( \| v - r \| \geq \| v - x \| \) for \( v \in (\mathcal{A}, C) \) and \( r \in \Gamma(x) \), where \( x \in \mathcal{A} \).

The proof is quite similar with the second part of the proof of Corollary 1.

3. Applications. We shall show some applications of the results in the section 2.
**Definition 4.** Let $M$ be a $W^*$-algebra. $M$ is called to be hyperfinite, if there exists an increasing sequence of type $I_{n_t}$-factors $\{M_t\}$ $(n_t < +\infty)$ containing the unit of $M$ in $M$ such that $\bigcup_{t=1}^{\infty} M_t = M$, where $\bigcup$ is the weak closure of $\bigcup$.

**Proposition 1.** Let $\mathcal{A}$ be a separable $C^*$-algebra and $\mathcal{B}$ a $C^*$-subalgebra of $\mathcal{A}$. Suppose that there exists a $*$-representation $\{\pi, \mathcal{B}\}$ of $\mathcal{A}$ such that $\pi(\mathcal{B}) \subseteq \pi(\mathcal{A})$ and the commutant $\pi(\mathcal{B})'$ of $\pi(\mathcal{B})$ is hyperfinite. Then, $\mathcal{B}$ cannot separate $P(\mathcal{A}) \cup \{0\}$.

**Proof.** Suppose that $\mathcal{B}$ separates $P(\mathcal{A}) \cup \{0\}$. Put $\mathcal{A}_0 = \pi(\mathcal{A})$ and $\mathcal{B}_0 = \pi(\mathcal{B})$. By the result of Schwartz (cf. [14]), $\Gamma(x) \cap \mathcal{B}_0 \neq \{\phi\}$ for $x \in \mathcal{A}_0$; hence by Corollary 1, $\inf_{w \in \mathcal{B}_0^*} \|x - w\| = 0$ and so $x \in \mathcal{B}_0$. This is a contradiction and completes the proof.

**Definition 5.** Let $A$ be a $C^*$-algebra. $A$ is called to be uniformly hyperfinite, if there exists an increasing sequence of type $I_{n_t}$-factors $\{A_t\}$ $(n_t < +\infty)$ containing the unit of $A$ in $A$ such that the uniform closure of $\bigcup_{t=1}^{\infty} A_t = A$.

**Proposition 2.** Let $\mathcal{A}$ be a separable $C^*$-algebra and let $\mathcal{B}$ be a uniformly hyperfinite $C^*$-subalgebra of $\mathcal{A}$. Suppose that $\mathcal{B}$ separates $P(\mathcal{A}) \cup \{0\}$, then $\mathcal{A} = \mathcal{B}$.

**Proof.** Suppose that $\mathcal{B} \subseteq \mathcal{A}$ and let $f$ be a bounded selfadjoint linear functional on $\mathcal{A}$ such that $f(\mathcal{B}) = 0$ and $f \neq 0$. Let $f = f^* - f^-$ be the orthogonal decomposition such that $f^+, f^- \geq 0$, and $\|f^+\| + \|f^-\| = \|f\|$. Put $\varphi = f^* + f^-$ and take the $*$-representation $\{\pi, \mathcal{B}\}$ of $\mathcal{A}$ as the $\{\pi, \mathcal{B}\}$ in §2. Then, $\mathcal{B}_0 = \mathcal{A}_0$. Since $\mathcal{B}_0$ is uniformly hyperfinite, there exists an increasing sequence of type $I_{n_t}$-factors $\{B_t\}$ $(n_t < +\infty)$ in $\mathcal{B}_0$ such that the uniform closure of $\bigcup_{t=1}^{\infty} B_t = \mathcal{B}_0$. We can easily find a projection $Q$, with norm 1 of $B(\mathcal{B}_0)$ on $B_0$, because $B(\mathcal{B}_0) = B_0 \otimes B_0'$. Let $Q$ be an accumulate point of the set $\{Q_i| i = 1, 2, \cdots\}$ in $L(B(\mathcal{B}_0))$ with $Q(L(B(\mathcal{B}_0)), B(\mathcal{B}_0) \otimes B(\mathcal{B}_0)*_0)$, then clearly $Q(y) = y$ for $y \in \mathcal{B}_0$; moreover $Q(\mathcal{A}_0) \subseteq \bigcup_{t=1}^{\infty} B_t = \mathcal{B}_0 \subseteq (\mathcal{A}_0, C)$; hence by Theorem 1, $Q(x) = x$ for $x \in \mathcal{A}_0$ and so $\mathcal{A}_0 \subseteq \mathcal{B}_0$. This is a contradiction and completes the proof.
PROPOSITION 3. Let $\mathfrak{A}$ be a separable $C^*$-algebra and let $\mathfrak{B}$ be a $C^*$-subalgebra of $\mathfrak{A}$. Suppose that there exists a $*$-representation $\{\pi, \mathfrak{B}\}$ of $\mathfrak{A}$ such that $\pi(\mathfrak{A})$ is a finite $W^*$-algebra and $\pi(\mathfrak{B}) = \pi(\mathfrak{A})$. Then, $\mathfrak{B}$ can not separate $P(\mathfrak{A}) \cup (0)$.

PROOF. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A}) \cup (0)$. By the result of Umegaki (cf. [15]), there exists a projection $Q$ with norm 1 of $\pi(\mathfrak{A})$ onto $\pi(\mathfrak{B})$. On the other hand, by Theorem 1, $Q(\pi(a)) = \pi(a)$ for $a \in \mathfrak{A}$; hence $\pi(\mathfrak{A}) = \pi(\mathfrak{B})$. This is a contradiction and completes the proof.

PROPOSITION 4 (Kaplansky [9]). Let $\mathfrak{A}$ be a separable $C^*$-algebra and let $\mathfrak{B}$ be a type I $C^*$-subalgebra of $\mathfrak{A}$. Suppose that $\mathfrak{B}$ separates $P(\mathfrak{A}) \cup (0)$, then $\mathfrak{A} = \mathfrak{B}$.

PROOF. Suppose that $\mathfrak{B} \subsetneq \mathfrak{A}$. Take a $*$-representation $\{\pi, \mathfrak{B}\}$ of $\mathfrak{A}$ such that $\pi(\mathfrak{A}) \subsetneq \pi(\mathfrak{B})$. Since $\mathfrak{B}$ is a type I $C^*$-algebra, $\pi(\mathfrak{B})'$ is a type I $W^*$-algebra. By the theorem of Kakutani-Markoff, the structure theorem of type I $W^*$-algebras and the considerations of Schwartz (cf. [14]), we can easily see that $\Gamma(x) \cap \overline{\mathfrak{B}}_0 \neq (\phi)$ for $x \in \mathfrak{A}$; hence by Corollary 1, $x \notin \overline{\mathfrak{B}}_0$. This is a contradiction and completes the proof.

REFERENCES


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