AUTOMORPHISMS OF $L^*$-ALGEBRAS*

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(Received Dec. 18, 1968)

In this paper we are concerned with some properties of (algebraic)*-automorphisms and *-isomorphisms of semi-simple $L^*$-algebras. As a consequence of the inner product uniqueness theorem for $L^*$-algebras established earlier ([4], see Theorem 1 below), it follows that every *-isomorphism $\varphi$ of a semi-simple $L^*$-algebra $L$ is necessarily topological and moreover $\varphi$ is a semi-$L^*$-isomorphism if $L$ is simple (Corollary to Theorem 1). From these results we deduce that a *-isomorphism of a semi-simple $L^*$-algebra can be expressed in terms of partial semi-$L^*$-isomorphisms (Theorem 2).

We give some conditions under which a *-automorphism is automatically unitary. While a *-automorphism of any finite-dimensional simple $L^*$-algebra is unitary (Corollary to Proposition 2), this result holds for an infinite-dimensional simple $L^*$-algebra provided it is of classical type (Theorem 3). Under additional conditions on the automorphism, the same result holds also for the general simple $L^*$-algebra (see §2). Actually, it is our conjecture that the result is valid even without the additional conditions.

We introduce a notion of regularity for automorphisms of semi-simple $L^*$-algebras and show by means of a category argument that such automorphisms exist whenever the $L^*$-algebras are separable (Theorem 4). For automorphisms which are inner, a criterion for regularity is obtained (Proposition 7) which coincides with the one given by Gantmacher for the regularity of automorphisms of semisimple Lie algebras.

1. Preliminaries and structure of *-isomorphisms. Let $L$ be a real or complex Lie algebra of arbitrary dimension. $L$ is called an $L^*$-algebra if (i) $L$ is equipped with an inner product relative to which it is a Hilbert space; (ii) $L$ is closed for a *-operation $x \rightarrow x^*$ which satisfies the connecting relation

$$<[x,y],z> = <y,[x^*,z]>,\]$$

where $[\cdot,\cdot]$ as usual stands for the Lie bracket.

If the centre of $L$ (as a Lie algebra) is zero, $L$ is called semi-simple. $L$ is called simple if it is of dimension greater than one and contains no closed ideals other than $\{0\}$ and $L$.\footnote{The author is currently at the Institute for Advanced Study.}
If $L$ is a real semi-simple $L^*$-algebra, its complexification $\overline{L} = L + \sqrt{-1} L$ can be made into a complex (semi-simple) $L^*$-algebra by extending to $\overline{L}$ the operations of $L$ in the following way. If $z_i = x_i + \sqrt{-1} y_i, (i=1,2), z = x + \sqrt{-1} y$ belong to $\overline{L}$, we set

(a) $[z_1, z_2] = [x_1, x_2] - [y_1, y_2] + \sqrt{-1} \{[x_1, y_2] + [y_1, x_2]\}$

(b) $<z_1, z_2> = <x_1, x_2> + <y_1, y_2> + \sqrt{-1} \{<x_1, y_2> - <x_2, y_1>\}$

(c) $z^* = x^* - \sqrt{-1} y^*$.

$L$ is called a real form of the complex algebra $\overline{L}$.

Every semi-simple $L^*$-algebra $L$ has an orthogonal decomposition $L = \sum \Phi L_i$, with $L_i$ simple. (This has been established for complex $L$ by Schue [8]. Though his proof (involving theory of complex Banach algebras) cannot apparently be adapted for real $L$, a proof for this case is easily obtained by using the simple decomposition of the complexification $\overline{L}$.)

Let $L, L'$ be two semi-simple $L^*$-algebras (both real or both complex). A Lie algebra isomorphism $\phi$ of $L$ onto $L'$ is called a *-isomorphism if $\phi$ is a *-map, i.e., $(\phi x)^* = \phi x^*$ for all $x$ in $L$. An isomorphism $\phi$ is called a semi-$L^*$-isomorphism if there exists a (positive) constant $k$ such that

$<\phi x, \phi y> = k<x, y>$ for all $x, y$ in $L$;

$g=k$ is called the gauge of $\phi$.

A semi-$L^*$-isomorphism is automatically a *-isomorphism (cf. [3, Lemma 4]). If $k$ (or $g$) = 1, the semi-$L^*$-isomorphism is called an $L^*$-isomorphism. Note that the $L^*$-isomorphisms of a semi-simple $L^*$ algebra $L$ are just its unitary Lie isomorphisms.

**THEOREM 1.** Let $L$ be a real or complex centre-free Lie algebra closed for a *-operation. Let $<\cdot, \cdot>_1, <\cdot, \cdot>_2$ be two inner products on $L$ such that relative to $<\cdot, \cdot>_1$, $<\cdot, \cdot>_2$ $L$ is a (semi-simple) $L^*$-algebra $L_1$, $L_2$). Then $<\cdot, \cdot>_1$ and $<\cdot, \cdot>_2$ are topologically equivalent. Further, if $L_1$ is simple so is $L_2$ and the two inner products are multiples of each other.

The proof of these assertions when $L$ is complex will be found in [4]. The extension of the first of these assertions for the case where $L$ is real is easily obtained by passing to the complexification $\overline{L}$, while that for the second, though not deducible\(^{(1)}\) from that for $\overline{L}$, can be proved ab initio in exactly the same way as for the complex case.

\(^{(1)}\) because, when $L$ is simple, $L$ need not always be simple.
COROLLARY. Let $L$, $L'$ be two semi-simple $L^*$-algebras. Every $^*$-isomorphism $\varphi$ of $L$ onto $L'$ is topological. Also, if $L$ is simple $\varphi$ is a semi-$L^*$-isomorphism.

PROOF. Introduce in $L$ a second inner product $\langle \cdot, \cdot \rangle_1$ by setting $\langle x, y \rangle_1 = \langle \varphi x, \varphi y \rangle$. Then, by Theorem 1, $\langle \cdot, \cdot \rangle_1$ is equivalent to the original inner product of $L$, which means $\varphi$ is topological. The second assertion of the corollary obviously follows from the corresponding assertion of Theorem 1.

DEFINITION 1. Let $L$, $L'$ be two semi-simple $L^*$-algebras (both real or both complex). A map $\varphi$ of $L$ into $L'$ is called a partial semi-$L^*$-isomorphism if there exists a closed ideal $I$ of $L$ such that the restriction $\varphi_I$ (of $\varphi$ to $I$) is a semi-$L^*$-isomorphism and if further $\varphi$ maps the orthogonal complement $I^\perp$ to $\{0\}$.

PROPOSITION 1. A partial semi-$L^*$-isomorphism $\varphi$ of $L$ is a $^*$-homomorphism of $L$ which is bounded; $\|\varphi\| = g$, $g$ being the gauge of $\varphi_I$.

PROOF. Since $I$, as a closed ideal of $L$, is a semi-simple $L^*$-subalgebra, it follows that $\varphi_I$, and hence $\varphi$, is a $^*$-map. Further, since $[I, I^\perp] = \{0\}$, if 

$$x_i = x_i + y_i (x_i \in I, \ y_i \in I^\perp), \ i = 1, 2$$

then

$$\varphi[x_i + y_i, x_i + y_i] = \varphi[x_i, x_i] + [y_i, y_i].$$

The homomorphism property of $\varphi$ now readily follows from this relation. Finally,

$$\|\varphi z\|^2 = \|\varphi x\|^2 + \|\varphi y\|^2 \leq \|\varphi_I x\|^2 + \|\varphi_I y\|^2 \leq g^2,$$

and consequently $\|\varphi\| = g$.

THEOREM 2. A $^*$-isomorphism $\varphi$ of a semi-simple $L^*$-algebra $L$ has the form

$$\varphi = \Sigma \varphi_i,$$

where $\varphi_i$ are partial semi-$L^*$-isomorphisms.
PROOF. Let $L = \Sigma \oplus L_i$ be the orthogonal decomposition of $L$ with $L_i$ simple. By Theorem 1, $\varphi$ is topological and its restriction to $L_i$ is a semi-$L^*$-isomorphism. Now define a linear mapping by setting

$$\varphi; x = \varphi(x) \quad \text{if} \quad x \in L_i, \quad \varphi; x = 0 \quad \text{if} \quad x \perp L_i.$$ 

Then it is clear that $\varphi_i$ is a partial semi-$L^*$-isomorphism of $L$ and $\varphi = \Sigma \varphi_i$.

2. Unitariness conditions for $*$-automorphisms. We begin with

PROPOSITION 2. If an automorphism $\varphi$ of a finite-dimensional simple $L^*$-algebra $L$ leaves the class of Cartan subalgebras (in the $L^*$-sense\(^{(1)}\)) invariant, then $\varphi$ is $*$-preserving and unitary.

PROOF. First of all, since $L$ is simple, by a result due to Schue [8, 2.5], the inner product $< \cdot , \cdot >$ of $L$ and the Cartan scalar product $\cdot \cdot$ are connected by the relation

$$(1) \quad <x, y^* > = \varepsilon(x, y), \quad (x, y \in L)$$

where $\varepsilon$ is some positive number independent of $x, y$. (Though this result has been established by Schue only when $L$ is complex, his proof applies equally to the real case.) We next observe that if $L$ is real then

$$(1') \quad <z, w^* > = \varepsilon(z, w), \quad (z, w \in \bar{L})$$

even though $\bar{L}$ may fail to be simple. This observation follows from $(1)$ and the first part of Lemma 6.1 [7, p. 154].

We now make the following notational convention. $\bar{L}$ will denote the complexification of $L$ if $L$ is real and $L$ itself if $L$ is complex. $\bar{\varphi}$ will denote accordingly the extension of $\varphi$ to $\bar{L}$ ($\bar{\varphi}(x + \sqrt{-1} y) = \varphi x + \sqrt{-1} \varphi y$) or $\varphi$ itself. It is now clearly sufficient to prove the assertions of Proposition 2 for $\bar{\varphi}$.

To prove $\bar{\varphi}$ is $*$-preserving it is enough, in view of linearity of $\bar{\varphi}$, to show that $\bar{\varphi}$ maps self-adjoint elements into self-adjoint elements. Let $z$ be a self-adjoint element of $\bar{L}$. Then there exists a Cartan subalgebra $\bar{H}$ containing $z$. Let $\Delta = \{\alpha\}$ be the root system relative to $\bar{H}$. Then there are elements $h_\alpha, \hat{h}_\alpha \in H$ with

$$(2) \quad \alpha(h) = <h, h_\alpha> = (h, \hat{h}_\alpha),$$

\(^{(1)}\) these are also Cartan subalgebras in the Lie algebra sense (see [8, p. 71]).
where \( h_\alpha \) is known to be self-adjoint \([8, \text{p. 72}]\). Since (1'), (2') imply \( \hat{h}_\alpha = \epsilon h_\alpha \), it follows that \( \hat{h}_\alpha \) is also self-adjoint. Now \( \varphi \) being an automorphism, \( \varphi \hat{h}_\alpha = \hat{h}_\alpha \), where \( \alpha' \) is a root relative to \( \varphi \hat{H} \). Thus \( \varphi \hat{h}_\alpha \) is self-adjoint for each \( \hat{h}_\alpha \), and since the \( \hat{h}_\alpha \) span \( \hat{H} \), it is clear that \( \varphi \alpha \) is self-adjoint, as we wished to show.

It remains to prove that \( \varphi \) is unitary. But this now readily follows from (1') since \( \varphi \) is \( \ast \)-preserving.

**COROLLARY.** Every \( \ast \)-automorphism of \( L \) is unitary.

The rest of the present section is concerned with some generalisations of the above corollary to infinite-dimensional simple \( L^\ast \)-algebras.

**THEOREM 3.** Let \( L \) be either a complex simple \( L^\ast \)-algebra of classical type or a real form of such an algebra. Then a \( \ast \)-automorphism \( \varphi \) of \( L \) is unitary.

**PROOF.** We adopt the notational convention introduced in Proposition 2. \( \overline{L} \) is therefore a complex simple \( L^\ast \)-algebra of classical type and so, by definition, is semi-\( L^\ast \)-isomorphic (say under a map \( \psi \)) to one of the standard algebras \( L_A, L_B, L_C \). (For the definitions of the standard algebras see \([5]\), or \([8, \text{Theorem 3}] \) (separable case).) By Theorem 1, \( \overline{\varphi} \) is a semi-\( L^\ast \)-automorphism of \( \overline{L} \).

Let \( \Delta = \{ \alpha \} \) be the root system of \( \overline{L} \) relative to a Cartan subalgebra \( \overline{H} \) of \( \overline{L} \), and \( g_\alpha \) the gauge of \( \psi \). Denote by \( \rho(\overline{H}) \) the range of values of \( ||\alpha|| (= ||h_\alpha||) \) as \( \alpha \) varies in \( \Delta \). Then, using explicitly the root systems for \( L_A, L_B, L_C \) determined in \([5]\), we obtain

\[
\rho(\overline{H}) = \begin{cases} 
\left( \frac{\sqrt{2}}{g_\alpha} \right) & \text{if } \overline{L} \text{ is of type } A, \\
\left( \frac{1}{g_\alpha}, \frac{1}{g_\alpha \sqrt{2}} \right) & \text{if } \overline{L} \text{ is of type } B \text{ and } \overline{H} \text{ of type 1}, \\
\left( \frac{1}{g_\alpha} \right) & \text{if } \overline{L} \text{ is of type } B \text{ and } \overline{H} \text{ of type 2}, \\
\left( \frac{1}{g_\alpha}, \frac{\sqrt{2}}{g_\alpha} \right) & \text{if } \overline{L} \text{ is of type } C. 
\end{cases}
\]

Since with \( \overline{H} \), \( \varphi \overline{H} \) is also a Cartan subalgebra (of the same type too), it follows that

\[
\rho(\varphi \overline{H}) = \rho(\overline{H}) .
\]
On the other hand, if $\phi h = h$, then

\[ \|\alpha\| = g\|\alpha'\|, \]

where $g$ is the gauge of $\phi$. The relations (1), (2) can clearly subsist only if $g = 1$. This means $\phi$, and hence $\phi$, is unitary.

**COROLLARY.** Every *-automorphism of a separable simple $L^*$-algebra is unitary.

This follows from Theorem 3 and Schue’s result that every separable (infinite-dimensional) simple $L^*$-algebra is of classical type [8, Theorem 3].

**PROPOSITION 3.** Let $\phi$ be a *-automorphism of a complex simple $L^*$-algebra $L$ such that $\phi$ leaves some Cartan subalgebra $H$ of $L$ set-wise invariant, $H = \phi H$. Then $\phi$ is unitary.

**PROOF.** As in Theorem 3, we obtain the relation

\[ \|\alpha\| = g\|\alpha'\| \]

where $\Delta = \{\alpha\}$ is the root system relative to $H = H$, $g$ the gauge of $\phi = \phi$ and $\alpha \rightarrow \alpha'$ is now a bijective mapping of $\Delta$ onto itself. By Corollary 1 to Proposition 2 of [1], we have for any two roots $\alpha, \beta \in \Delta(\|\alpha\| \geq \|\beta\|)$

\[ (2') \|\alpha\| = \|\beta\| \text{ or } \sqrt{2} \|\beta\| \]

(assuming here, as we may, that $L$ is infinite-dimensional). The relations (2), (2') clearly imply that $g = 1$, i.e., that $\phi$ is unitary.

**PROPOSITION 4.** Let $L$ be a complex simple $L^*$-algebra. A *-automorphism $\phi$ (of $L$) whose spectrum contains a number $\lambda_0$ of unit modulus is unitary. In particular, any *-automorphism $\phi$ admitting a non-zero fixed point is unitary.

**PROOF.** By the Corollary to Theorem 1, $\phi$ is a semi-$L^*$-automorphism:

\[ \langle \phi x, \phi y \rangle = g^2 \langle x, y \rangle, \ (x, y \in L). \]

It follows that $\phi$ is unitary, so that $\phi \phi^* = g^2 I = \phi^* \phi$, where $I$ is the identity operator. The last equations imply that $\phi$ is normal. Since $\lambda_0 \in \sigma(\phi)$, the spectrum
of \( \varphi \), it results from the spectral mapping theorem that

\[ |\lambda_0|^t \in \sigma(\varphi \varphi^*) = \{ g^t \}. \]

Therefore \( g = |\lambda_0| = 1 \), whence \( \varphi \) is unitary.

### 3. Semi-regular and regular automorphisms.

**Definition 2.** Let \( L \) be a semi-simple \( L^* \)-algebra. Let \( D \) be a bounded derivation (in the Lie algebra sense) of \( L \). We set

\[ e^D = I + D + \frac{D^2}{2!} + \cdots, \quad (I = \text{identity}) \]

Then \( e^D \) is a bounded operator which is moreover, by a standard reasoning, an automorphism of \( L \). In particular, for \( D = \text{ad} \alpha (\alpha \in L) \) we write \( \varphi_\alpha \) for \( e^{\text{ad} \alpha} \).

If \( \alpha \) is a normal element (i.e., \([\alpha, \alpha^*] = 0\)), we call \( \varphi_\alpha \) an inner automorphism.

**Definition 3.** An automorphism \( \varphi \) of \( L \) is called semi-regular if 1 is an eigenvalue of \( \varphi \) and further the 1-eigensubspace \( L_1 \) contains a maximal abelian subalgebra of \( L \). (Observe that the 1-eigensubspace of an automorphism is always a subalgebra.)

Let \( \varphi_\alpha \) be an inner automorphism. Since \( \alpha \) is normal, it is contained in a Cartan subalgebra \( H \). Since \( H \) is abelian it is clear that \( L_1 \supseteq H \), and a Cartan subalgebra being maximal abelian \([8, \text{p. 70}]\), \( \varphi_\alpha \) is semi-regular. More generally, if \( D \) is a bounded derivation annihilating some Cartan subalgebra, then \( e^D \) is semi-regular.

**Proposition 5.** Every semi-regular \( * \)-automorphism \( \varphi \) of a semi-simple \( L^* \)-algebra \( L \) is unitary.

**Proof.** The hypothesis on \( \varphi \) clearly implies that the 1-eigensubspace \( L_1 \) of \( L \) contains a Cartan subalgebra \( H \). Then, with the notational convention in Theorem 2, \( \tilde{H} \) is a Cartan subalgebra of \( \tilde{L} \). Let

\[ \tilde{L} = \tilde{H} \oplus \mathbb{R} \oplus \tilde{V}_s \quad (\oplus \text{ denoting orthogonal sum}) \]

be the root space (or Cartan) decomposition of \( \tilde{L} \) relative to \( \tilde{H} \) (see [9]). Since \( \tilde{\varphi} \) leaves \( \tilde{H} \) element-wise invariant, it follows that

\[ \tilde{\varphi} h_s = h_s, \quad \tilde{\varphi} \tilde{V}_s = \tilde{V}_s, \]
where \( h_a \) is the vector of \( \widetilde{H} \) such that \( \chi(h) = \langle h, h_a \rangle \) for all \( h \) in \( \widetilde{H} \). Now choose for each positive root \( \alpha \) a vector \( v_\alpha \in \widetilde{V}_\alpha \) with \( \|v_\alpha\| = 1 \), then \( v_\alpha^* \in \overline{\widetilde{V}}_{-\alpha} \) ([8, p. 73]). Let

\[
\varphi v_\alpha = \lambda_\alpha v_\alpha, \quad \varphi v_\alpha^* = \lambda_{-\alpha} v_\alpha^*.
\]

Then

\[
\lambda_\alpha \lambda_{-\alpha} [v_\alpha, v_\alpha^*] = \varphi [v_\alpha, v_\alpha^*] = \varphi h_\alpha = h_\alpha = [v_\alpha, v_\alpha^*].
\]

Therefore \( \lambda_\alpha \lambda_{-\alpha} = 1 \). Again \( \overline{\varphi} v_\alpha^* = (\overline{\varphi} v_\alpha)^* = \overline{\lambda_\alpha} v_\alpha^* \), whence \( \lambda_{-\alpha} = \overline{\lambda_\alpha} \). Hence \( |\lambda_\alpha| = 1 \), which means \( \|\overline{\varphi} v_\alpha\| = 1 \). On the other hand, since \( \overline{\varphi} h = h \), we have trivially \( \|\overline{\varphi} h\| = \|h\| \) for all \( h \in \widetilde{H} \). These conclusions plus the mutual orthogonality of the \( \widetilde{V}_\alpha \) and \( \widetilde{H} \) imply that \( \overline{\varphi} \) (and so \( \varphi \)) is unitary.

**PROPOSITION 6.** For an inner automorphism \( \varphi_{h_\alpha} \) of a semi-simple \( L^* \)-algebra \( L \), the following assertions are equivalent:

(i) \( \varphi_{h_\alpha} \) is a \(*\)-map;

(ii) \( \varphi_{h_\alpha} \) is unitary;

(iii) \( h_\alpha \) is skew-adjoint.

**PROOF.** That (i) \( \Rightarrow \) (ii) follows from the previous proposition, while (ii) \( \Rightarrow \) (i) is just a particular case of the general fact that an \( L^* \)-isomorphism (or even a semi-\( L^* \)-isomorphism) is automatically a \(*\)-map.

We shall now prove that (i) \( \Rightarrow \) (iii). With the previous notational convention, if (i) holds then \( \varphi_{h_\alpha} \) is a \(*\)-map of \( L \). Further, it is clear that if \( v_\alpha \in \widetilde{V}_\alpha \) then

\[
\varphi_{h_\alpha}(v_\alpha) = e^{\alpha(h_\alpha)} v_\alpha, \quad \varphi_{h_\alpha}(v_\alpha^*) = e^{-\bar{\alpha}(h_\alpha)} v_\alpha^*.
\]

But \( \varphi_{h_\alpha}(v_\alpha^*) = (\varphi_{h_\alpha}(v_\alpha))^* \), so that \( e^{\alpha(h_\alpha)} = e^{-\bar{\alpha}(h_\alpha)} \). Therefore

\[
\alpha(h_\alpha) + \overline{\alpha(h_\alpha)} = 0, \quad \text{or } \alpha(h + h_\alpha^*) = 0.
\]

The arbitrariness of the root \( \alpha \) and the ‘total’ property of the set \( \Delta = [\alpha] \) of roots [1, Lemma 6] now imply \( h_\alpha^* = -h_\alpha \).

To complete the proof of the theorem we have only to show that (iii) \( \Rightarrow \) (ii). But this readily follows since (assuming (iii))

\[
\varphi_{h_\alpha}^{-1} = \varphi_{-h_\alpha} = \varphi_{h_\alpha^*} = (\varphi_{h_\alpha})^*.
\]

q. e. d.

In view of the above proposition we call an inner automorphism \( \varphi_{h_\alpha} \) with
DEFINITION 4. A semi-regular automorphism \( \varphi \) of \( L \) is called regular if the 1-eigensubspace \( L_1 \) is a maximal abelian subalgebra.

PROPOSITION 7. An inner automorphism \( \varphi_h \) is regular if and only if the 1-eigensubspace \( L_1 \) of \( L \) is abelian (cf. [6, Theorems 5, 8]).

PROOF. Suffices to prove that if \( L_1 \) is abelian then \( \varphi_h \) is regular. Since \( h_0 \) is a normal element there exists a Cartan subalgebra \( H \) of \( L \) containing \( h_0 \). Since \( H \) is abelian, \( \varphi_h \) leaves \( H \) pointwise invariant, and therefore \( H = L_1 \). But \( H \) as a Cartan subalgebra is maximal abelian. Consequently \( H = L_1 \) and \( \varphi_h \) is regular.

COROLLARY. For a regular inner automorphism \( \varphi_h \) the 1-eigensubspace \( L_1 \) is a Cartan subalgebra.

PROPOSITION 8. An inner automorphism \( \varphi_h \) of a semi-simple \( L \) is regular if and only if for some Cartan subalgebra \( H \) (of \( L \)) containing \( h_0 \) we have

\[
\frac{\alpha(h_0)}{2\pi\sqrt{-1}} \equiv 0 \pmod{1}, \text{ for all } \alpha \in \Delta,
\]

where \( \Delta \) is the root system of \( \bar{L} \) relative to \( \bar{H} \). In particular, if \( \varphi_h \) is regular then \( h_0 \) is a regular element in the sense of [2] (i.e., the null space \( N_0 \) of \( \text{ad} \ h_0 \) in \( \bar{L} \) is a Cartan subalgebra).

PROOF. Suppose first that \( \varphi_h \) is regular. Then by Corollary to Proposition 7, \( \bar{L}_1 = \bar{H} \) is a Cartan subalgebra of \( \bar{L} \). If the condition in Proposition 8 is not satisfied for \( H \), we must have

\[
e^{\alpha(h_0)} = 1 \text{ for some } \alpha \in \Delta.
\]

It follows that if \( v_\alpha \in \bar{V}_\alpha \), then

\[
e^{\alpha(h_0)}v_\alpha = e^{\alpha(h_0)}v_\alpha = v_\alpha.
\]

This means \( \bar{V}_\alpha \subseteq \bar{L}_1 = \bar{H} \), which is absurd. Hence the condition must hold.

Next, suppose the condition holds relative to some Cartan subalgebra \( \bar{H} \). Then in particular, \( \alpha(h_0) \neq 0 \) for all \( \alpha \in \Delta \) so that by Theorem 1 of [2], \( h_0 \) is
a regular element. Further $\tilde{N}_0 \supset \tilde{H}$, whence by maximality property of Cartan subalgebras $\tilde{N}_0 = \tilde{H}$. This clearly implies that $\bar{L}_1 \supset \bar{H}$. Let now

$$\bar{L} = \bar{H} \oplus \Sigma \oplus \bar{V}.$$ 

be the root space decomposition of $\bar{L}$. For $x \in \bar{L}_1$, we can write

$$x = h + \Sigma \nu_\alpha (\nu_\alpha \in \bar{V}).$$

Then

$$h + \Sigma \nu_\alpha = x = e^{\text{ad } h} x = h + \Sigma e^{\alpha(h)} \nu_\alpha.$$ 

It follows that $e^{\alpha(h)} \nu_\alpha = \nu_\alpha$ for all $\alpha$. But by our supposition $e^{\alpha(h)} \neq 1$. Hence $\nu_\alpha = 0$, $x = h \in \bar{H}$, and $\bar{L}_1 = \bar{H}$. Thus $\varphi_h$ is regular as we wished to show.

**Theorem 4.** Let $L$ be a separable semi-simple $L^*$-algebra. Then there exist regular inner automorphisms of $L$. If $L$ is compact$^{(3)}$ or complex, then there exist even regular inner $L^*$-automorphisms.

**Proof.** First let $L$ be real and $\bar{L}$ be its complexification. Let $\bar{H}$ be a Cartan subalgebra which is the complexification of a Cartan subalgebra $H$ of $L$, and $\Delta$ the root system of $\bar{L}$ relative to $\bar{H}$. Since $L$ is separable, so is $\bar{L}$, and consequently $\Delta = \{\alpha_i\}$ is countable. Define

$$P_{n,i} = \{ h \in H : \alpha_i(h) = 2n\pi\sqrt{-1} \},$$

where $n$ runs through all integers. Each $P_{n,i}$ is either empty or a hyperplane of $H$. In any case $P_{n,i}$ is non-dense (see footnote in [2, p. 162]). It follows by Baire’s category theorem that we can choose an $h_i \in H$ with $\alpha_i(h_i) \neq 2n\pi\sqrt{-1}$ for any $i$ or $n$. Then by Proposition 8, $\varphi_{h_i}$ is a regular inner automorphism of $L$.

Next, let $L$ be compact. Then $h_i^* = -h_i$ and $\varphi_{h_i}$ is a regular inner $L^*$-automorphism (Proposition 6). Finally, if $L$ is complex we take its compact from $L_k$, i.e., the real $L^*$-algebra $L_k$ of all skew-adjoint elements of $L$ ([3, p. 523]). Choose a $h_i \in L_k$ as above. Then $\varphi_{h_i} = e^{\text{ad } h_i}$ taken over $L$, gives a regular inner $L^*$-automorphism of $L$.

**Remark.** It was shown in [2] that a non-separable type $A$ complex simple $L^*$-algebra $L_A$ contains no regular element. More generally, it can be shown,

$^{(3)}$ i.e., every element of $L$ is skew-adjoint.
using Bessel's inequality and the criterion for regular element [2, Theorem 1], that every complex semi-simple \(L^*-\)algebra admitting an uncountable subset of mutually orthogonal roots contains no regular element. The \(L^*-\)algebras \(L\) admitting such an orthogonal subset of roots include besides the non-separable simple algebras \(L_A, L_B, L_C\) also all semi-simple \(L\) with uncountably many simple components. In view of Proposition 8 none of these algebras — which are all, of course, non-separable — has a regular inner automorphism.

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