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AUTOMORPHISMS OF L^*-ALGEBRAS*

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In this paper we are concerned with some properties of (algebraic)*-automorphisms and *-isomorphisms of semi-simple L^*-algebras. As a consequence of the inner product uniqueness theorem for L^*-algebras established earlier ([4], see Theorem 1 below), it follows that every *-isomorphism φ of a semi-simple L^*-algebra L is necessarily topological and moreover φ is a semi-L^*-isomorphism if L is simple (Corollary to Theorem 1). From these results we deduce that a *-isomorphism of a semi-simple L^*-algebra can be expressed in terms of partial semi-L^*-isomorphisms (Theorem 2).

We give some conditions under which a *-automorphism is automatically unitary. While a *-automorphism of any finite-dimensional simple L^*-algebra is unitary (Corollary to Proposition 2), this result holds for an infinite-dimensional simple L^*-algebra provided it is of classical type (Theorem 3). Under additional conditions on the automorphism, the same result holds also for the general simple L^*-algebra (see §2). Actually, it is our conjecture that the result is valid even without the additional conditions.

We introduce a notion of regularity for automorphisms of semi-simple L^*-algebras and show by means of a category argument that such automorphisms exist whenever the L^*-algebras are separable (Theorem 4). For automorphisms which are inner, a criterion for regularity is obtained (Proposition 7) which coincides with the one given by Gantmacher for the regularity of automorphisms of semisimple Lie algebras.

1. Preliminaries and structure of *-isomorphisms. Let L be a real or complex Lie algebra of arbitrary dimension. L is called an L^*-algebra if (i) L is equipped with an inner product relative to which it is a Hilbert space; (ii) L is closed for a *-operation $x \to x^*$ which satisfies the connecting relation

$$<[x,y],z> = <y,[x^*,z]> ,$$

where $[\cdot]$ as usual stands for the Lie bracket.

If the centre of L (as a Lie algebra) is zero, L is called semi-simple. L is called simple if it is of dimension greater than one and contains no closed ideals other than $\{0\}$ and L.

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If L is a real semi-simple L^*-algebra, its complexification $\tilde{L}=L+\sqrt{-1}\ L$ can be made into a complex (semi-simple) L^*-algebra by extending to $\tilde{L}$ the operations of L in the following way. If $z_i=x_i+\sqrt{-1}\ y_i,(i=1,2), z=x+\sqrt{-1}\ y$ belong to $\tilde{L}$, we set

(a) $[z_1, z_2] = [x_1, x_2] - [y_1, y_2] + \sqrt{-1}\ \{[x_1, y_2] + [y_1, x_2]\}$

(b) $<z_1, z_2] = <x_1, x_2> + <y_1, y_2> + \sqrt{-1}\ \{<y_1, x_2> - <x_1, y_2>\}$

(c) $z^*=x^*-\sqrt{-1}\ y^*$.

L is called a real form of the complex algebra $\tilde{L}$.

Every semi-simple L^*-algebra L has an orthogonal decomposition $L=\Sigma\oplus L_i$, with L_i simple. (This has been established for complex L by Schue [8]. Though his proof (involving theoıy of complex Banach algebras) cannot apparently be adapted for real L, a proof for this case is easily obtained by using the simple decomposition of the complexification $\tilde{L}$.)

Let L, L' be two semi-simple L^*-algebras (both real or both complex). A Lie algebra isomorphism φ of L onto L' is called a *-isomorphism if φ is a *-map, i. e., $(\varphi x)^* = \varphi x^*$ for all x in L. An isomorphism φ is called a *semi-L^*-isomorphism* if there exists a (positive) constant k such that

$$<\varphi x, \varphi y> = k<x, y> \text{ for all } x,y \text{ in } L;$$

$g=k$ is called the *gauge* of φ.

A semi-L^*-isomorphism is automatically a *-isomorphism (cf. [3, Lemma 4]). If k (or g) $=1$, the semi-L^*-isomorphism is called an *L^*-isomorphism*. Note that the L^*-isomorphisms of a semi-simple L^* algebra L are just its unitary Lie isomorphisms.

THEOREM 1. *Let L be a real or complex centre-free Lie algebra closed for a *-operation. Let* $<\cdot>_1$, $<\cdot>_2$ *be two inner products on L such that relative to* $<\cdot>_1$ $(<\cdot>_2)$ *L is a (semi-simple) L^*-algebra* L_1 (L_2). *Then* $<\cdot>_1$ *and* $<\cdot>_2$ *are topologically equivalent. Further, if* L_1 *is simple so is* L_2 *and the two inner products are multiples of each other.*

The proof of these assertions when L is complex will be found in [4]. The extension of the first of these assertions for the case where L is real is easily obtained by passing to the complexification $\tilde{L}$, while that for the second, though not deducible[1] from that for $\tilde{L}$, can be proved *ab initio* in exactly the same way as for the complex case.

(1) because, when L is simple, L need not always be simple.

COROLLARY. *Let L, L' be two semi-simple L^*-algebras. Every *-isomorphism φ of L onto L' is topological. Also, if L is simple φ is a semi-L^*-isomorphism.*

PROOF. Introduce in L a second inner product $<\cdot>_1$ by setting $<x,y>_1 = <\varphi x, \varphi y>$. Then, by Theorem 1, $<\cdot>_1$ is equivalent to the original inner product of L, which means φ is topological. The second assertion of the corollary obviously follows from the corresponding assertion of Theorem 1.

DEFINITION 1. Let L, L' be two semi-simple L^*-algebras (both real or both complex). A map φ of L into L' is called a *partial semi-L^*-isomorphism* if there exists a closed ideal I of L such that the restriction φ_I (of φ to I) is a semi-L^*-isomorphism and if further φ maps the orthogonal complement $I^{\perp}$ to $\{0\}$.

PROPOSITION 1. *A partial semi-L^*-isomorphism φ of L is a *-homomorphism of L which is bounded*; $\|\varphi\| = g$, *g being the gauge of φ_I.*

PROOF. Since I, as a closed ideal of L, is a semi-simple L^*-subalgebra, it follows that φ_I, and hence φ, is a *-map. Further, since $[I, I^{\perp}] = \{0\}$, if

$$z_i = x_i + y_i (x_i \in I,\ \ y_i \in I^{\perp}), i = 1, 2$$

then

$$\varphi[x_1 + y_1, x_2 + y_2] = \varphi([x_1, x_2] + [y_1, y_2]) .$$

The homomorphism property of φ now readily follows from this relation.

Finally,

$$\frac{\|\varphi z\|^2}{\|z\|^2} = \frac{\|\varphi_I x\|^2}{\|x\|^2 + \|y\|^2} \leqq \frac{\|\varphi_I x\|^2}{\|x\|^2} \leqq g^2 ,$$

and consequently $\|\varphi\| = g$.

THEOREM 2. *A *-isomorphism φ of a semi-simple L^*-algebra L has the form*

$$\varphi = \Sigma \varphi_i$$

where φ_i are partial semi-L^*-isomorphisms.

PROOF. Let $L=\Sigma\oplus L_i$ be the orthogonal decomposition of L with L_i simple. By Theorem 1, φ is topological and its restriction to L_i is a semi-L^*-isomorphism. Now define a linear mapping by setting

$$\varphi_i x = \varphi x \quad \text{if} \quad x \in L_i, \quad \varphi_i x = 0 \quad \text{if} \quad x \perp L_i.$$

Then it is clear that φ_i is a partial semi-L^*-isomorphism of L and $\varphi = \Sigma\varphi_i$.

2. **Unitariness conditions for *-automorphisms.** We begin with

PROPOSITION 2. *If an automorphism φ of a finite-dimensional simple L^*-algebra L leaves the class of Cartan subalgebras (in the L^*-sense)*[2] *invariant, then φ is *-preserving and unitary.*

PROOF. First of all, since L is simple, by a result due to Schue [8, 2.5], the inner product $<\cdot>$ of L and the Cartan scalar product $(\cdot)$ are connected by the relation

$$(1) \qquad <x, y^*> = \varepsilon(x, y), \quad (x, y \in L)$$

where ε is some positive number independent of x, y. (Though this result has been established by Schue only when L is complex, his proof applies equally to the real case.) We next observe that if L is real then

$$(1') \qquad <z, w^*> = \varepsilon(z, w), \quad (z, w \in \widetilde{L})$$

even though $\widetilde{L}$ may fail to be simple. This observation follows from (1) and the first part of Lemma 6.1 [7, p. 154].

We now make the following notational convention. $\widetilde{L}$ will denote the complexification of L if L is real and L itself if L is complex. $\widetilde{\varphi}$ will denote accordingly the extension of φ to $\widetilde{L}$ ($\widetilde{\varphi}(x+\sqrt{-1}\ y)=\varphi x+\sqrt{-1}\ \varphi y$) or φ itself. It is now clearly sufficient to prove the assertions of Proposition 2 for $\widetilde{\varphi}$.

To prove $\widetilde{\varphi}$ is *-preserving it is enough, in view of linearity of $\widetilde{\varphi}$, to show that $\widetilde{\varphi}$ maps self-adjoint elements into self-adjoint elements. Let z be a self-adjoint element of $\widetilde{L}$. Then there exists a Cartan subalgebra $\widetilde{H}$ containing z. Let $\Delta=\{\alpha\}$ be the root system relative to $\widetilde{H}$. Then there are elements h_α, $\hat{h}_\alpha \in H$ with

$$(2) \qquad \alpha(h) = <h, h_\alpha> = (h, \hat{h}_\alpha),$$

[2] these are also Cartan subalgebras in the Lie algebra sense (see [8, p. 71]).

where h_α is known to be self-adjoint [8, p. 72]. Since (1′), (2) imply $\widehat{h}_\alpha = \varepsilon h_\alpha$, it follows that $\widehat{h}_\alpha$ is also self-adjoint. Now $\widetilde{\varphi}$ being an automorphism, $\widetilde{\varphi}\widehat{h}_\alpha = \widehat{h}_{\alpha'}$ where α' is a root relative to $\widetilde{\varphi}\widetilde{H}$. Thus $\widetilde{\varphi}\widehat{h}_\alpha$ is selfadjoint for each $\widehat{h}_\alpha$, and since the $\widehat{h}_\alpha$ span $\widetilde{H}$, it is clear that $\widetilde{\varphi}z$ is self-adjoint, as we wished to show.

It remains to prove that $\widetilde{\varphi}$ is unitary. But this now readily follows from (1′) since $\widetilde{\varphi}$ is *-preserving.

COROLLARY. *Every *-automorphism of L is unitary.*

The rest of the present section is concerned with some generalisations of the above corollary to infinite-dimensional simple L^*-algebras.

THEOREM 3. *Let L be either a complex simple L^*-algebra of classical type or a real form of such an algebra. Then a *-automorphism φ of L is unitary.*

PROOF. We adopt the notational convention introduced in Proposition 2. $\widetilde{L}$ is therefore a complex simple L^*-algebra of classical type and so, by definition, is semi-L^*-isomorphic (say under a map ψ) to one of the standard algebras L_A, L_B, L_C. (For the definitions of the standard algebras see [5], or [8, Theorem 3] (separable case).) By Theorem 1, $\widetilde{\varphi}$ is a semi-L^*-automorphism of $\widetilde{L}$.

Let $\Delta = \{\alpha\}$ be the root system of $\widetilde{L}$ relative to a Cartan subalgebra $\widetilde{H}$ of $\widetilde{L}$, and g_0 the gauge of ψ. Denote by $\rho(\widetilde{H})$ the range of values of $\|\alpha\|$ $(=\|h_\alpha\|)$ as α varies in Δ. Then, using explicitly the root systems for L_A, L_B, L_C determined in [5], we obtain

$$\rho(\widetilde{H}) = \begin{cases} \left(\dfrac{\sqrt{2}}{g_0}\right) & \text{if } \widetilde{L} \text{ is of type } A, \\[2ex] \left(\dfrac{1}{g_0},\ \dfrac{1}{g_0\sqrt{2}}\right) & \text{if } \widetilde{L} \text{ is of type } B \text{ and } \widetilde{H} \text{ of type 1}, \\[2ex] \left(\dfrac{1}{g_0}\right) & \text{if } \widetilde{L} \text{ is of type } B \text{ and } \widetilde{H} \text{ of type 2}, \\[2ex] \left(\dfrac{1}{g_0},\ \dfrac{\sqrt{2}}{g_0}\right) & \text{if } \widetilde{L} \text{ is of type } C. \end{cases}$$

Since with $\widetilde{H}$, $\widetilde{\varphi}\widetilde{H}$ is also a Cartan subalgebra (of the same type too), it follows that

$$(1) \qquad \rho(\widetilde{\varphi}\widetilde{H}) = \rho(\widetilde{H}).$$

On the other hand, if $\tilde{\varphi} h_\alpha = h_{\alpha'}$, then

$$(2) \qquad \|\alpha\| = g\|\alpha'\|,$$

where g is the gauge of φ. The relations (1), (2) can clearly subsist only if $g=1$. This means $\tilde{\varphi}$, and hence φ, is unitary.

COROLLARY. *Every *-automorphism of a separable simple L^*-algebra is unitary.*

This follows from Theorem 3 and Schue's result that every separable (infinite-dimensional) simple L^*-algebra is of classical type [8, Theorem 3].

PROPOSITION 3. *Let φ be a *-automorphism of a complex simple L^*-algebra L such that φ leaves some Cartan subalgebra H of L set-wise invariant, $H=\varphi H$. Then φ is unitary.*

PROOF. As in Theorem 3, we obtain the relation

$$(2) \qquad \|\alpha\| = g\|\alpha'\|$$

where $\Delta = \{\alpha\}$ is the root system relative to $\tilde{H}=H, g$ the gauge of $\tilde{\varphi}=\varphi$ and $\alpha \to \alpha'$ is now a bijective mapping of Δ onto itself. By Corollary 1 to Proposition 2 of [1], we have for any two roots $\alpha, \beta \in \Delta(\|\alpha\| \geqq \|\beta\|)$

$$(2') \qquad \|\alpha\| = \|\beta\| \text{ or } \sqrt{2}\ \|\beta\|$$

(assuming here, as we may, that L is infinite-dimensional). The relations (2), (2′) clearly imply that $g=1$, i.e., that φ is unitary.

PROPOSITION 4. *Let L be a complex simple L^*-algebra. A *-automorphism φ (of L) whose spectrum contains a number λ_0 of unit modulus is unitary. In particular, any *-automorphism φ admitting a non-zero fixed point is unitary.*

PROOF. By the Corollary to Theorem 1, φ is a semi-L^*-automorphism:

$$<\varphi x, \varphi y> = g^2 <x, y>, \quad (x, y \in L).$$

It follows that $\frac{\varphi}{g}$ is unitary, so that $\varphi\varphi^* = g^2 I = \varphi^*\varphi$, where I is the identity operator. The last equations imply that φ is normal. Since $\lambda_0 \in \sigma(\varphi)$, the spectrum of φ, it results from the spectral mapping theorem that

$$|\lambda_0|^2 \in \sigma(\varphi\varphi^*) = \{g^2\} .$$

Therefore $g = |\lambda_0| = 1$, whence φ is unitary.

3. Semi-regular and regular automorphisms.

DEFINITION 2. Let L be a semi-simple L^*-algebra. Let D be a bounded derivation (in the Lie algebra sense) of L. We set

$$e^D = I + D + \frac{D^2}{2!} + \cdots, \quad (I = \text{identity})$$

Then e^D is a bounded operator which is moreover, by a standard reasoning, an automorphism of L. In particular, for $D = \text{ad}\ a$ $(a \in L)$ we write φ_a for $e^{\text{ad}\ a}$. If a is a normal element (i. e., $[a, a^*] = 0$), we call φ_a an *inner* automorphism.

DEFINITION 3. An automorphism φ of L is called *semi-regular* if 1 is an eigenvalue of φ and further the 1-eigensubspace L_1 contains a maximal abelian subalgebra of L. (Observe that the 1-eigensubspace of an automorphism is always a subalgebra.)

Let φ_a be an inner automorphism. Since a is normal, it is contained in a Cartan subalgebra H. Since H is abelian it is clear that $L_1 \supset H$, and a Cartan subalgebra being maximal abelian [8, p. 70], φ_a is semi-regular. More generally, if D is a bounded derivation annihilating some Cartan subalgebra, then e^D is semi-regular.

PROPOSITION 5. *Every semi-regular *-automorphism φ of a semi-simple L^*-algebra L is unitary.*

PROOF. The hypothesis on φ clearly implies that the 1-eigensubspace L_1 of L contains a Cartan subalgebra H. Then, with the notational convention in Theorem 2, $\tilde{H}$ is a Cartan subalgebra of $\tilde{L}$. Let

$$\tilde{L} = \tilde{H} \oplus \Sigma \oplus \tilde{V}_\alpha \quad (\oplus \text{ denoting orthogonal sum})$$

be the root space (or Cartan) decomposition of $\tilde{L}$ relative to $\tilde{H}$ (see [9]). Since $\tilde{\varphi}$ leaves $\tilde{H}$ element-wise invariant, it follows that

$$\tilde{\varphi} h_\alpha = h_\alpha, \ \tilde{\varphi} \tilde{V}_\alpha = \tilde{V}_\alpha ,$$

where h_α is the vector of $\widetilde{H}$ such that $\alpha(h) = <h, h_\alpha>$ for all h in $\widetilde{H}$. Now choose for each positive root α a vector $v_\alpha \in \widetilde{V}_\alpha$ with $\|v_\alpha\| = 1$, then $v_\alpha^* \in \widetilde{V}_{-\alpha}$ ([8, p. 73]). Let

$$\widetilde{\varphi} v_\alpha = \lambda_\alpha v_\alpha , \quad \widetilde{\varphi} v_\alpha^* = \lambda_{-\alpha} v_\alpha^* .$$

Then

$$\lambda_\alpha \lambda_{-\alpha} [v_\alpha, v_\alpha^*] = \widetilde{\varphi} [v_\alpha, v_\alpha^*]$$
$$= \widetilde{\varphi} h_\alpha = h_\alpha = [v_\alpha, v_\alpha^*] .$$

Therefore $\lambda_\alpha \lambda_{-\alpha} = 1$. Again $\widetilde{\varphi} v_\alpha^* = (\widetilde{\varphi} v_\alpha)^* = \overline{\lambda}_\alpha v_\alpha^*$, whence $\lambda_{-\alpha} = \overline{\lambda}_\alpha$. Hence $|\lambda_\alpha| = 1$, which means $\|\widetilde{\varphi} v_\alpha\| = 1$. On the other hand, since $\widetilde{\varphi} h = h$, we have trivially $\|\widetilde{\varphi} h\| = \|h\|$ for all $h \in \widetilde{H}$. These conclusions plus the mutual orthogonality of the $\widetilde{V}_\alpha$ and $\widetilde{H}$ imply that $\widetilde{\varphi}$ (and so φ) is unitary.

PROPOSITION 6. *For an inner automorphism φ_{h_0} of a semi-simple L^*-algebra L, the following assertions are equivalent*:

(i) *φ_{h_0} is a $*$-map*;
(ii) *φ_{h_0} is unitary*;
(iii) *h_0 is skew-adjoint.*

PROOF. That (i) $\Rightarrow$ (*ii*) follows from the previous proposition, while (ii) $\Rightarrow$ (i) is just a particular case of the general fact that an L^*-isomorphism (or even a semi-L^*-isomorphism) is automatically a $*$-map.

We shall now prove that (i) $\Rightarrow$ (iii). With the previous notational convention, if (i) holds than φ_{h_0} is a $*$-map of $\widetilde{L}$. Further, it is clear that if $v_\alpha \in \widetilde{V}_\alpha$ then

$$\varphi_{h_0}(v_\alpha) = e^{\alpha(h_0)} v_\alpha, \quad \varphi_{h_0}(v_\alpha^*) = e^{-\alpha(h_0)} v_\alpha^* .$$

But $\varphi_{h_0}(v_\alpha^*) = (\varphi_{h_0}(v_\alpha))^*$, so that $e^{\overline{\alpha(h_0)}} = e^{-\alpha(h_0)}$. Therefore

$$\alpha(h_0) + \overline{\alpha(h_0)} = 0 , \quad \text{or} \quad \alpha(h + h_0^*) = 0 .$$

The arbitrariness of the root α and the 'total' property of the set $\Delta = \{\alpha\}$ of roots [1, Lemma 6] now imply $h_0^* = -h_0$.

To complete the proof of the theorem we have only to show that (iii) $\Rightarrow$ (ii). But this readily follows since (assuming (iii))

$$\varphi_{h_0}^{-1} = \varphi_{-h_0} = \varphi_{h_0^*} = (\varphi_{h_0})^*. \qquad \text{q. e. d.}$$

In view of the above proposition we call an inner automorphism φ_{h_0}, with

h_0 skew-adjoint, an *inner L^*-automorphism.*

DEFINITION 4. A semi-regular automorphism φ of L is called *regular* if the 1-eigensubspace L_1 is a maximal abelian subalgebra.

PROPOSITION 7. *An inner automorphism φ_{h_0} is regular if and only if the* 1-*eigensubspace L_1 of L is abelian* (cf. [6, Theorems 5, 8]).

PROOF. Suffices to prove that if L_1 is abelian then φ_{h_0} is regular. Since h_0 is a normal element there exists a Cartan subalgebra H of L containing h_0. Since H is abelian, φ_{h_0} leaves H pointwise invariant, and therefore $H \subset L_1$. But H as a Cartan subalgebra is maximal abelian. Consequently $H = L_1$ and φ_{h_0} is regular.

COROLLARY. *For a regular inner automorphism φ_{h_0}, the* 1-*eigensubspace L_1 is a Cartan subalgebra.*

PROPOSITION 8. *An inner automorphism φ_{h_0} of a semi-simple L is regular if and only if for some Cartan subalgebra $\widetilde{H}$ (of $\widetilde{L}$) containing h_0 we have*

$$\frac{\alpha(h_0)}{2\pi\sqrt{-1}} \not\equiv 0 \ (mod\ 1),\ for\ all\ \alpha \in \Delta,$$

where Δ is the root system of $\widetilde{L}$ relative to $\widetilde{H}$. In particular, if φ_{h_0} is regular then h_0 is a regular element in the sense of [2] (*i.e., the null space $\widetilde{N}_0$ of* ad h_0 *in $\widetilde{L}$ is a Cartan subalgebra*).

PROOF. Suppose first that φ_{h_0} is regular. Then by Corollary to Proposition 7, $\widetilde{L}_1 = \widetilde{H}$ is a Cartan subalgebra of $\widetilde{L}$. If the condition in Proposition 8 is not satisfied for H, we must have

$$e^{\alpha(h_0)} = 1 \text{ for some } \alpha \in \Delta .$$

It follows that if $v_\alpha \in \widetilde{V}_\alpha$, then

$$e^{\operatorname{ad} h_0} v_\alpha = e^{\alpha(h_0)} v_\alpha = v_\alpha .$$

This means $\widetilde{V}_\alpha \subset \widetilde{L}_1 = \widetilde{H}$, which is absurd. Hence the condition must hold.

Next, suppose the condition holds relative to some Cartan subalgebra $\widetilde{H}$. Then in particular, $\alpha(h_0) \neq 0$ for all $\alpha \in \Delta$ so that by Theorem 1 of [2], h_0 is a regular element. Further $\widetilde{N}_0 \supset \widetilde{H}$, whence by maximality property of Cartan subalgebras $\widetilde{N}_0 = \widetilde{H}$. This clearly implies that $\widetilde{L}_1 \supset \widetilde{H}$. Let now

$$\widetilde{L} = \widetilde{H} \oplus \Sigma \oplus \widetilde{V}_\alpha$$

be the root space decomposition of $\widetilde{L}$. For $x \in \widetilde{L}_1$, we can write

$$x = h + \Sigma v_\alpha (v_\alpha \in \widetilde{V}_\alpha) .$$

Then

$$h + \Sigma v_\alpha = x = e^{\mathrm{ad}\, h_0} x = h + \Sigma e^{\alpha(h_0)} v_\alpha .$$

It follows that $e^{\alpha(h_0)} v_\alpha = v_\alpha$ for all α. But by our supposition $e^{\alpha(h_0)} \neq 1$. Hence $v_\alpha = 0$, $x = h \in \widetilde{H}$, and $\widetilde{L}_1 = \widetilde{H}$. Thus φ_{h_0} is regular as we wished to show.

THEOREM 4. *Let L be a separable semi-simple L^*-algebra. Then there exist regular inner automorphisms of L. If L is compact*(3) *or complex, then there exist even regular inner L^*-automorphisms.*

PROOF. First let L be real and $\widetilde{L}$ be its complexification. Let $\widetilde{H}$ be a Cartan subalgebra which is the complexification of a Cartan subalgebra H of L, and Δ the root system of $\widetilde{L}$ relative to $\widetilde{H}$. Since L is separable, so is $\widetilde{L}$, and consequently $\Delta = \{\alpha_i\}$ is countable. Define

$$P_{n,i} = \{h \in H \colon \alpha_i(h) = 2n\pi\sqrt{-1}\} ,$$

where n runs through all integers. Each $P_{n,i}$ is either empty or a hyperplane of H. In any case $P_{n,i}$ is non-dense (see footnote in [2, p. 162]). It follows by Baire's category theorem that we can choose an $h_1 \in H$ with $\alpha_i(h_1) \neq 2n\pi\sqrt{-1}$ for any i or n. Then by Proposition 8, φ_{h_1} is a regular inner automorphism of L.

Next, let L be compact. Then $h_1^* = -h_1$ and φ_{h_1} is a regular inner l^*-automorphism (Proposition 6). Finally, if L is complex we take its compact from L_k, i. e., the real L^*-algebra L_k of all skew-adjoint elements of L ([3, p. 523]). Choose a $h_1 \in L_k$ as above. Then $\varphi_{h_1} = e^{\mathrm{ad}\, h_1}$ taken over L, gives a regular inner L^*-automorphism of L.

REMARK. It was shown in [2] that a non-separable type A complex simple L^*-algebra L_A contains no regular element. More generally, it can be shown,

(3) i. e., every element of L is skew-adjoint.

using Bessel's inequality and the criterion for regular element [2, Theorem 1], that every complex semi-simple L^*-algebra admitting an uncountable subset of mutually orthogonal roots contains no regular element. The L^*-algebras L admitting such an orthogonal subset of roots include besides the non-separable simple algebras L_A, L_B, L_C also all semi-simple L with uncountably many simple components. In view of Proposition 8 none of these algebras —— which are all, of course, non-separable —— has a regular inner automorphism.

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