Tôhoku Math. Journ.
22(1970), 597-603.

A CONSTRUCTIVE DEFINITION OF AN INTEGRAL

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(Received January 19, 1970)

1. Introduction. In [6] and [7] a constructive definition is given for the Denjoy-Perron integral. In [7] the author remarks, "If, however, we wished to include also discontinuous integrals we should have to modify some details of the definition". Although many discontinuous integrals have been defined this approach does not seem to have been used. In this note it is shown that simple modifications of the definition in [6, 7] give a construction that includes many known integrals as special cases.

2. A General Constructive Definition. Unless otherwise stated functions in this paper will either be finite real-valued point functions with domain the fixed bounded non-empty closed interval, I_0, of the real line, or finite real-valued interval functions with domain all closed sub-intervals of some sub-interval of I_0. The symbols $I, J, I', J', \cdots$ etc. will denote closed sub-intervals; small letters $f, g \cdots$ will usually denote point functions, capital letters $F, G \cdots$ interval functions; script capitals $\mathcal{J}, \mathcal{L}, \mathcal{D} \cdots$ will denote various integrals. In particular $\mathcal{L}, \mathcal{D}, \mathcal{D}^*$ will denote the Lebesgue, Denjoy-Khintchine and Denjoy-Perron integrals respectively, [6, 7]; further $\mathcal{L}(f, I)$ will denote value of the $\mathcal{L}$-integral of f on the interval I with a similar notation for other integrals.

We will distinguish a class $\mathfrak{C}$ of interval functions, with domain all closed subintervals of I_0, and a non-negative function (not necessarily finite) V defined on couples (F, I), where I is in the domain of F. The following assumptions are made about the class $\mathfrak{C}$ and the function V.

(I) If $F \in \mathfrak{C}$, and $G \in \mathfrak{C}$ then $F+G \in \mathfrak{C}$.

(II) If $F \in \mathfrak{C}$ then $F(I)$ is completely determined by the values of $F(J)$, for $J \subset I$.

(III) If $F \in \mathfrak{C}$, $J \subset I$ then $0 \leqq V(F, J) \leqq V(F, I) < \infty$.

(IV) If $\sum_{n \in N} V(F, I_n) < \infty$ then $\sum_{n \in N} |F(I_n)| < \infty$.

(V) If $\sum_{n \in N} V(F, I_n) < \infty$ and $\sum_{n \in N} V(G, I_n) < \infty$ then $\sum_{n \in N} V(F+G, I_n) < \infty$.

(VI) Let Q be a closed set with end points those of I and contiguous intervals in I, $\{I_n\}_{n \in N}$; further suppose that for each n, I_n is in the domain of

F_n, and $F_n \in \mathfrak{C}$. If then $\sum_{n \in N} V(F_n, I_n) < \infty$ and if G is the interval function defined by $G(J) = \sum_{I_n \subset J} F_n(I_n)$ then $G \in \mathfrak{C}$.

$\mathfrak{C}$ can be considered as a class of continuous interval functions, V a form of oscillation Now we define a general integral following the classical lines in [6, 7], but using the notation in [9].

Let $\mathcal{J}$ be a real-valued function with domain, dom $\mathcal{J}$, a set of ordered pairs $\{(f, I)\}$, f being a real-valued point function with domain an interval I; we will put $\text{dom}_I \mathcal{J} = \{f;\ (f, I) \in \text{dom}\, \mathcal{J}\}$.

$\mathcal{J}$ is called an *integral,* or more precisely a $\mathfrak{C}$-*integral,* iff we have the following.

(A) If $f \in \text{dom}_I \mathcal{J}$ then $f \in \text{dom}_J \mathcal{J}$ for all $J \subset I$.

(B) If $f \in \text{dom}_I \mathcal{J}$ then $\mathcal{J}(f, J)$, $J \subset I$ is finitely additive and in $\mathfrak{C}$. (This implies that $\mathcal{J}(f, J) = 0$ if either $J = \emptyset$ or J is a singleton.)

(C) If $f \in \text{dom}_I \mathcal{J}$ and $f \in \text{dom}_J \mathcal{J}$, I and J abutting, then $f \in \text{dom}_K \mathcal{J}$, where $K = I \cup J$.

(D) If $f = 0$ on I then $f \in \text{dom}_I \mathcal{J}$ and $\mathcal{J}(f, I) = 0$.

When $f \in \text{dom}_I \mathcal{J}$, $\mathcal{J}$ an integral, then f is said to be $\mathcal{J}$-*integrable on* I. If A is any subset of I we will say that f is $\mathcal{J}$-*integrable on* A iff $f\, 1_A \in \text{dom}_I \mathcal{J}$, $(1_A(x) = 1$ if $x \in A$, $= 0$ if $x \notin A)$; the value of the $\mathcal{J}$-integral of f on A, $\mathcal{J}(f, A)$, is just $\mathcal{J}(f 1_A, I)$, a value independent of I, see [7].

Two such integrals $\mathcal{J}_1$ and $\mathcal{J}_2$ will be termed *compatible* iff $\mathcal{J}_1(f, A) = \mathcal{J}_2(f, A)$ whenever both sides exist. If $\mathcal{J}_1, \mathcal{J}_2$ are compatible and $\text{dom}\, \mathcal{J}_1 \subset \text{dom}\, \mathcal{J}_2$ we will write $\mathcal{J}_1 \subset \mathcal{J}_2$ and call $\mathcal{J}_2$ an *extension* of $\mathcal{J}_1$.

If $\mathfrak{C}$ is the class of continuous interval functions, which certainly satisfies (I) and (II) above, then $\mathcal{L}, \mathcal{D}, \mathcal{D}^*$ are all $\mathfrak{C}$-integrals and $\mathcal{L} \subset \mathcal{D}^* \subset \mathcal{D}$, [6, 7].

A point x is called a $\mathcal{J}$-*singular point of* f iff there is a sequence $\{I_n\}_{n \in N}$ such that $\lim_{n \to \infty} |I_n| = 0$, and for all $n \in N$, $x \in I_n$ and $(f, I_n) \notin \text{dom}\, \mathcal{J}$. Let us write S, or more precisely $S(f)$, or even $S(\mathcal{J}, f)$, for the set of $\mathcal{J}$-singular points of f. Clearly S is closed and if $I \cap S = \emptyset$ then $(f, I) \in \text{dom}\, \mathcal{J}$.

Two extensions of $\mathcal{J}$ are now defined, $\mathcal{J}^c$ and $\mathcal{J}^h$. We will say $(f, I) \in \text{dom}\, \mathcal{J}^c$ iff the following hold.

(E) $I \cap S(\mathcal{J}, f)$ is finite or empty.

(F) There is a finitely additive interval function F such that (i) $F \in \mathfrak{C}$, (ii) if $J \cap S(\mathcal{J}, f) = \emptyset$, $J \subset I$, then $\mathcal{J}(f, J) = F(J)$.

If $(f, I) \in \text{dom}\, \mathcal{J}^c$ then by (II) F is unique and if we define $\mathcal{J}^c(f, I) = F(I)$, $\mathcal{J}^c$ is clearly a $\mathfrak{C}$-integral and an extension of $\mathcal{J}$.

We will say $(f, I) \in \text{dom}\, \mathcal{J}^h$ iff we have the following.

(G) f is $\mathcal{J}$-integrable on $I \cap S(\mathcal{J}, f)$ and on each of the intervals $\{I_n\}$ contiguous in I to $I \cap S$.

(H) If $\{I_n\}$ is as in (G) then $\sum_n V(\mathcal{J}(f, \cdot), I_n) < \infty$

If $(f, I) \in \operatorname{dom} \mathcal{J}^h$ we define $\mathcal{J}^h(f, I) = \mathcal{J}(f, I \cap S) + \sum_n \mathcal{J}(f, I_n)$, then from (III), (IV), (V) and (VI) we see that $\mathcal{J}^h$ is a $\mathfrak{C}$-integral and clearly an extension of $\mathcal{J}$. We will write $\mathcal{J}^{ch}$ for $(\mathcal{J}^c)^h$.

If Ω is the first uncountable ordinal, if for each $\alpha < \Omega$, $\mathcal{J}_\alpha$ is an integral and if $\alpha < \beta$ implies $\mathcal{J}_\alpha \subset \mathcal{J}_\beta$ then for $\alpha < \beta \leqq \Omega$ we define $\sum_{\alpha<\beta} \mathcal{J}_\alpha$ to be the operation $\mathcal{J}$ with $\operatorname{dom} \mathcal{J} = \bigcup_{\alpha<\beta} \operatorname{dom} \mathcal{J}_\alpha$ and if $(f, I) \in \operatorname{dom} \mathcal{J}$ then $\mathcal{J}(f, I) = \mathcal{J}_\alpha(f, I)$ where α is the least ordinal $\gamma, \gamma < \beta$, such that $(f, I) \in \operatorname{dom} \mathcal{J}_\gamma$. Clearly $\mathcal{J}$ is an extension of $\mathcal{J}_\alpha$, for all $\alpha < \beta$ and $\mathcal{J} \subset \mathcal{J}_\beta$.

Such a transfinite sequence of integrals can be defined inductively as follows. Let $\mathcal{J}_0 = \mathcal{J}$, some given integral and suppose $\mathcal{J}_\alpha$ has been defined for all $\alpha < \beta < \Omega$; then put

$$\mathcal{J}_\beta = \left(\sum_{\alpha<\beta} \mathcal{J}_\alpha \right)^{ch},$$

$$\mathcal{J}_\Omega = \sum_{\beta<\Omega} \mathcal{J}_\beta .$$

Let us call such a sequence of integrals an *s-sequence*; it is the integral $\mathcal{J}_\Omega$ obtained from such an s-sequence that we wish to consider.

3. The Main Properties of an s-sequence. We assume in this section that we are given an s-sequence $\{\mathcal{J}_\alpha\}_{\alpha<\Omega}$. Our applications will all have $\mathcal{J}_0 = \mathcal{L}$ but it is convenient not to assume this directly; so we introduce the following concept that plays the role of the absolute continuity of the $\mathcal{L}$-integrals.

If Q is a closed set then F is said to be a *$\mathcal{J}$-integral on Q* iff there is some $(f, I) \in \operatorname{dom} \mathcal{J}$ with $Q \subset I$ and $F(J) = \mathcal{J}(f, J)$ whenever the end points of J lie in Q. We then assume

(VII) Let $Q, I, \{I_n\}_{n \in N}$ be as in (VI) and $(f, I) \in \operatorname{dom} \mathcal{J}_0$. Define G by $G(J) = \sum_{I_n \subset J} \mathcal{J}_0(f, I_n)$ then if $\sum_{n \in N} \mathcal{J}_0(f, I_n) < \infty$, G is a $\mathcal{J}_0$-integral on Q.

We now introduce sub-classes of $\mathfrak{C}$ that play the role of absolutely continuous interval functions.

DEFINITION 1. If $Q, I, \{I_n\}$ are as in (VI) then F is GAC on Q iff (a) $F \in \mathfrak{C}$, (b) F is a $\mathcal{J}_0$-integral on Q, (c) $\sum_n V(F, I_n) < \infty$.

In particular we have that if $F \in \mathfrak{C}$ then F is GAC on any $\{x\}$, with x in the domain of F.

DEFINITION 2. F is said to be σ-GAC on I iff (a) $F \in \mathfrak{C}$, (b) $I = \bigcup_{n \in N} Q_n$, $\{Q_n\}_{n \in N}$ a sequence of closed sets on which F is GAC.

We will say an integral $\mathcal{I}$ is (σ-)GAC when for all $(f, I) \in \operatorname{dom} \mathcal{I}$, $\mathcal{I}(f, \cdot)$ is (σ-)GAC on I.

THEOREM 3. *If $\mathcal{I}_0$ is GAC then $\mathcal{I}_\Omega$ is σ-GAC.*

PROOF. It suffices to check that $\mathcal{I}_0^c$ and $\mathcal{I}_0^h$ are σ-GAC.

(a) By (E) and (F), if $\mathcal{I}_0$ is GAC then $\mathcal{I}_0^c$ is σ-GAC.

(b) To prove that $\mathcal{I}_0^h$ is σ-GAC it suffices to check that G, defined by $G(J) = \sum_{I_n \subset J} \mathcal{I}_0(f, I_n)$, is σ-GAC. By (VI) and (H), $G \in \mathfrak{C}$ and by hypothesis G is GAC on each I_n and so it suffices to check that G is GAC on S, but this is immediate from (IV), (VII), and (H).

THEOREM 4. *If $\tilde{\mathcal{I}}$ is a $\mathfrak{C}$-integral that is σ-GAC and if $\mathcal{I}_\Omega \subset \tilde{\mathcal{I}}$ then $\tilde{\mathcal{I}} = \mathcal{I}_\Omega$.*

PROOF. Suppose f is $\mathcal{I}$-integrable, $\alpha < \Omega$; let $S_\alpha = S(\mathcal{I}_\Omega, f)$; then if $\alpha < \beta < \Omega$, $S_\alpha \supset S_\beta$, and further for some $\alpha < \Omega$, $S_\alpha = S_{\alpha+1}$, [7, p. 258]. Suppose $S_\alpha \neq \emptyset$.

Then f is $\mathcal{I}_\alpha$-integrable in every interval I with $I \cap S_\alpha = \emptyset$. Hence, by hypothesis f is $\mathcal{I}_{\alpha+1}$-integrable on every interval contig-ous to S_α; in particular S_α is perfect.

Since $\tilde{\mathcal{I}}$ is σ-GAC it follows that S_α contains a closed portion Q such that $\tilde{\mathcal{I}}$ is a $\mathcal{I}_0$-integral on Q and $\sum_{n \in N} V(\tilde{\mathcal{I}}(f, \cdot), I_n) < \infty$, where $\{I_n\}$ are the contiguous intervals on Q and so $\mathcal{I}_{\alpha+1}$-integrable on some interval containing Q. But this contradiction implies $S_\alpha = \emptyset$; that is f is $\mathcal{I}_\alpha$-integrable which completes the proof.

4. Some Examples. In this section we consider some examples of s-sequences $\{\mathcal{I}_\alpha\}_{\alpha < \Omega}$ with $\mathcal{I}_0 = \mathcal{L}$, in particular therefore we always have (VII) holding.

(1) Let $\mathfrak{C}$ be the class of continuous interval functions, V the usual oscillation (i.e. $V(F, I) = O(F, I)$). Then (I)-(VI) are easily seen to be satisfied, (for (VI) see for instance [6, p. 172]).

The classes GAC and σ-GAC are then just the classes of continuous AC^*, ACG^* functions, [7], and $\mathcal{I}_\Omega$ is just $\mathcal{D}^*$, [6, 7].

(2) Let $\mathfrak{C}$ be the class of continuous interval functions F with the property that if $\lim_{n \to \infty} |I_n| = 0$ then $\lim_{n \to \infty} O(F, I_n) = 0$ and let $V(F, I) = |F(I)|$. Then again (I)-(VI) are satisfied, [6, 7], and the classes GAC and σ-GAC are just the classes AC and

ACG respectively, [7]. The integral $\mathscr{I}_\Omega$ is just $\mathscr{D}$, [6, 7].

(3) Let $\mathfrak{C}$ be the class of continuous interval functions F with the property that if $\lim_{n\to\infty}|I_n|=0$ then $\lim_{n\to\infty}\dfrac{O(F, I_n)}{\rho(x, I_n)}=0$ for all x in the complement of $\bigcup_{n\in N} I_n$; let $V(F, I)=|F(I)|$. Then the integral $\mathscr{I}_\Omega$ is one discussed by Burkill, and lies between $\mathscr{D}$ and $\mathscr{D}^*$ [1, 7]. (Here of course $\rho(x, J)$ is the distance from x to J.)

(4) Let $\mathfrak{C}$ be the class of continuous interval functions F and $V(F, I)=\{O(F,I)\}^{1/p}$, $1\leqq p<\infty$. Then if we write $\mathscr{D}_p$ for the integral $\mathscr{I}_\Omega$ in this case we have $\mathscr{L}\subset\mathscr{D}_p\subset\mathscr{D}^*$. Let us write AC_p and ACG_p for the classes GAC and σ-GAC arising this case. We will show that $\mathscr{D}_p$ coincides with the integral introduced by Burkill and Gehring, [3], which we will denote by (p).

DEFINITION 5. f is $\mathscr{D}(p)$ integrable if f is $\mathscr{D}^*$-integrable and its indefinite integral F is in W_p (where by F in W_p we mean that $\sup \sum_{k=1}^{m}|F(x_k)-F(x_{k-1})|^{1/p}$ is finite, the sup being taken over all finite subdivisions).

By Theorem 4 it suffices to prove the following lemmas.

LEMMA 6. *If f is $\mathscr{D}(p)$ integrable and F is its indefinite integral then F is ACG_p.*

PROOF. Since f is $\mathscr{D}^*$-integrable, F is continuous, and ACG^* and in W_p and hence immediately in ACG_p.

LEMMA 7. *Let Q, I and $\{I_n\}$ be as in* (VI). *If f is $\mathscr{D}(p)$ integrable over Q as well as over each I_n and if $\sum_n\{O(F_k, I_n)\}^{1/p}<\infty$, F_k being the indefinite integral of f over I_k, then f is $\mathscr{D}(p)$ integrable over I.*

PROOF. Since the conditions imply f is $\mathscr{D}^*$-integrable, [7, p. 257] it suffices to prove that the indefinite $\mathscr{D}^*$-integral of f over I is in W_p. But this is just (2. 6. 2) of [3].

COROLLARY 8, $\mathscr{D}_p\subset\mathscr{D}(p)$.

PROOF. Immediate as in [7].

(5) We now show that the C_rP-integrals can be obtained this way, [2, 8]. The definition of the C_rP-integral is obtained by induction on r; the $r=0$ case is

just the $\mathcal{D}^*$-integral and let us assume C_kP-integrals have been defined $1 \leqq k \leqq r-1$, and proceed to define the C_rP-integral. In this connection we need the following taken from [2, 8] expressed in the notation of interval functions.

(i) If $f(x) = F([a, x])$ is C_{r-} P-integrable on $[a, b] = I$ write

$$C_r\text{-}F(I) = \frac{r}{|I|^r} C_{r-1}P\text{-}\int_a^b (b-t)^{r-1} F([a, t])dt\,.$$

(ii) If C_r-F is continuous we say F is C_r-continuous.

(iii) If we use the notation of (i) put

$$O_r(F, I) = \sup\{\overline{\text{bound}}_{a<x<b}\, C_r\text{-}F([a, x]), \overline{\text{bound}}_{a<x<b}\, C_r\text{-}F([x, b])\}\,.$$

Now suppose we take for $\mathfrak{C}$ the class of C_r-continuous interval functions and for V the above defined O_r. The properties (I)-(III), and (V) are obvious; (IV), (VI) are proved in [8]; see in particular Lemma III and Property B of [8].

The concepts of GAC and σ-GAC are just the C_r-continuous functions that are AC^*(C_r-sense) and ACG^* (C_r-sense) respectively of [8]; for this see Theorem II of [8].

The integral $\mathcal{I}_\Omega$ is just the C_rP-integral and to see this after Theorem 4 it suffices to remark that the C_rP-integral is a σ-GAC $\mathfrak{C}$-integral in the present sense and to prove that $\mathcal{I}_\Omega \subset C_rP$. For this it suffices as in [7] to prove the following generalization of Property B of [8].

LEMMA 9. *Let I, $\{I_n\}$ be as in* (VI). *If f is C_rP-integrable over Q as well as over each I_n and if $\sum_n O_r(F_n, I_n) < \infty$, F_n being the C_rP-integral of f over I_n, then f is C_rP-integrable on I and*

$$C_rP-\int_I f = C_rP\text{-}\int_Q f + \sum_n C_rP\text{-}\int_{I_n} f\,.$$

PROOF. Put $I = [a, b]$, $I(x) = [a, x]$ and define

$$F(x) = \sum_n C_rP\text{-}\int_{I_n \cap I(x)} f\,.$$

Then it is sufficient to show F is ACG^*(C_r-sense) on I; see [7, 8].

Clearly F is C_r-continuous on I and ACG^*(C_r-sense) on each I_n; it remains to show F is AC^*(C_r-sense) on Q. Let

$$g(x) = f(x),\ x \in Q$$

$$= \frac{F_k(b_k)}{|I_k|}, x \in I_k = [a_k b_k];$$

then g is $\mathcal{L}$-integrable on I and if $G(x) = \int_a^x g$, G coincides with F on Q, [8]. Hence F is AC^* (C_r-sense) on Q, by Theorem II, [8].

It follows then, as in [8], that $f\ l_{I\sim Q}$ has F as its C_rP-integral; but $f\ l_Q$ is C_rP-integrable, by hypothesis, and so we get the lemma.

(6) In a similar way we can obtain the M_r-integral as a special case of our general construction. The M_r-integral is also obtained by induction on r, with $r=0$ being the $\mathcal{D}$-integral, [4]. Let us assume the M_k-integrals have been defined, $0 \leqq k \leqq r-1$ and proceed to define the M_r-integral.

M_r-continuity is defined in the same way as C_r-continuity but with M_{r-1}-integral replacing the $C_{r-1}P$-integral.

We now take for $\mathfrak{C}$ the class of M_r-continuous interval functions F with the property that if $\lim_{n\to\infty}|I_n| = 0$ then $\lim_{n\to\infty} O_r(F, I_n) = 0$; and we define $V(F, I)$ to be $|F(I)|$ as in example (2).

The fact that then $\mathcal{I}_\Omega$ is just the M_r-integral follows from results in [4,5].

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