Tôhoku Math. Journ.
23(1971), 59-65.

SOME RESULTS FOR AN ORIENTABLE 5-DIMENSIONAL SUBMANIFOLD OF R^7

KATSUEI KENMOTSU

(Received Sept 18, 1970)

1. Statement of results. Let R^7 be a Euclidean space of dimension 7. M.Kobayashi [3] has shown that the properties of the vector cross product on R^7 induce an almost contact structure on an orientable 5-dimensional submanifold of R^7, and he proved that (1) if the submanifold is totally geodesic in R^7 (for the induced metric), then the almost contact structure is normal and as a partial converse (2) if the structure is normal and the immersion is totally umbilical, then the submanifold is totally geodesic. In the present paper we show that

THEOREM 1. *Let M be an orientable submanifold of codimension 2 in R^7. If the almost contact structure is normal, M is a minimal submanifold of R^7.*

THEOREM 2. *Let M be an orientable submanifold of codimension 2 in R^7. M is quasi-Sasakian and have the trivial normal connection if and only if M is totally geodesic.*

The new device to prove the above mentioned theorems is that we can take locally suitable normal vector fields relative to the almost contact structure on M. By virtue of Theorem 1, we see that no closed submanifold can satisfy the normality condition. Furthermore as the second application of the Theorem 1, we see that the 5-dimensional sphere have at least two different almost contact metric structures for the same induced metric, since it is well known that the sphere has a Sasakian structure (i.e., normal contact metric structure). For the later use, we list up the properties of the vector cross product of R^7 [2]:

$$A \times B = -B \times A, \tag{1.1}$$

$$\langle A \times B, C\rangle = \langle A, B \times C\rangle, \tag{1.2}$$

$$(A \times B) \times C + A \times (B \times C) = 2\langle A, C\rangle B - \langle B, C\rangle A - \langle A, B\rangle C, \tag{1.3}$$

$$\bar{\nabla}_A(B\times C)=\bar{\nabla}_A B\times C+B\times\bar{\nabla}_A C, \tag{1.4}$$

for any vector fields A, B and C on R^7, where $<,>$ and $\bar{\nabla}$ are the canonical Riemannian metric of R^7 and the Riemannian connection for $<,>$, respectively.

2. Types of almost contact Riemannian manifolds. Let $M=(M,g)$ be a Riemannian manifold and $V(M)$ the module of C^∞-vector fields on M. An almost contact Riemannian manifold M is a Riemannian manifold equipped with a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η which satisfy $\phi^2=-1+\xi\otimes\eta$, $\eta(\xi)=1$, $\phi(\xi)=0$, $\eta(X)=g(X,\xi)$ and $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\cdot\eta(Y)$. Such a manifold is orientable and odd dimensional. To describe the geometry of an almost contact Riemannian manifold M, we consider two special tensors. The first is a 2-form, w, and it is defined for $A,B\in V(M)$ by

$$w(A,B)=g(A,\phi B). \tag{2.1}$$

The second, called the torsion tensor of M, is a $(1,2)$ tensor field S^1 defined by

$$\begin{aligned} S^1(A,B)&=[A,B]+\phi[\phi A,B]+\phi[A,\phi B]-[\phi A,\phi B]\\ &\quad+\{B\eta(A)-A\eta(B)\}\xi. \end{aligned} \tag{2.2}$$

The following Proposition is used to prove the Proposition 5.1.

PROPOSITION 2.1. *Let $A,B,C\in V(M)$. Then*

$$\begin{aligned} &dw(A,B,C)-dw(A,\phi B,\phi C)+g(A,S^1(B,\phi C))\\ &=2(\nabla_A w)(B,C)-\eta(B)\{g(A,\nabla_{\phi C}\xi)-g(\phi C,\nabla_A\xi)\}\\ &\quad+\eta(C)g(A,S^2(B))-\eta(C)\{g(A,\nabla_{\phi B}\xi)+g(\phi B,\nabla_A\xi)\}, \end{aligned} \tag{2.3}$$

where $S^2(B)$ is, by definition,

$$S^2(B)=\nabla_\xi\phi\cdot B-\nabla_B\phi\cdot\xi+\nabla_{\phi B}\xi. \tag{2.4}$$

PROOF. The proof of (2.3) follows from the facts that

$$dw(A,B,C)=\mathfrak{S}(\nabla_A w)(B,C), \tag{2.5}$$

$$\begin{aligned} S^1(B,C)&=\nabla_{\phi C}\phi\cdot B-\nabla_B\phi\cdot\phi C-\nabla_{\phi B}\phi\cdot C+\nabla_C\phi\cdot\phi B\\ &\quad+\eta(C)\nabla_B\xi-\eta(B)\nabla_C\xi, \end{aligned} \tag{2.6}$$

where $\mathfrak{S}$ is a cyclic sum for A, B, C and ∇ is a Riemannian connection for g. Q. E. D.

(ϕ, ξ, η, g)-structure is called normal if $S^1 = 0$. $S^1 = 0$ implies $S^2 = 0$ [4]. It is known [6] that $S^1 = 0$ if and only if

$$\phi\nabla_B\phi\cdot C - \nabla_{\phi B}\phi\cdot C - (\nabla_B\eta)(C)\cdot\xi = 0 . \tag{2.7}$$

(ϕ, ξ, η, g)-structure is called a quasi-Sasakian structure if it is normal and w is closed. In a quasi-Sasakian manifold ξ is a Killing vector field [1]. (S.Tanno has pointed out [5] that there are some gaps in the paper [1], but the above statement is true.)

3. Five dimensional submanifold of R^7. Let N_i $(i = 1, 2)$ be mutually orthogonal unit normal vector fields on a neighborhood of $x \in M$. An almost contact metric structure on M is defined by [3];

$$\xi = N_1 \times N_2 , \tag{3.1}$$

$$\phi A = A \times \xi , \tag{3.2}$$

$$\eta(X) = g(X, \xi) , \tag{3.3}$$

where g is an induced metric. The second fundamental forms h_i and the third fundamental form s is defined as follows :

$$\bar{\nabla}_A N_1 = -h_1 A + s(A) N_2 , \tag{3.4}$$

$$\bar{\nabla}_A N_2 = -h_2 A - s(A) N_1 . \tag{3.5}$$

Then we have

$$\bar{\nabla}_A B = \nabla_A B + g(h_1 A, B) N_1 + g(h_2 A, B) N_2 , \tag{3.6}$$

where we define a symmetric tensor $H(A, B)$ by

$$H(A, B) = g(h_1 A, B) N_1 + g(h_2 A, B) N_2 . \tag{3.7}$$

Let $\bar{R}$ be a curvature tensor of R^7. Calculating the normal part of $\bar{R}(A, B)C$, $A, B, C \in V(M)$, we see that the Codazzi-Mainardi's equation is

$$\nabla_A h_1\cdot B - \nabla_B h_1\cdot A - s(A) h_2 B + s(B) h_2 A = 0 , \tag{3.8}$$

$$\nabla_A h_2 \cdot B - \nabla_B h_2 \cdot A + s(A) h_1 B - s(B) h_1 A = 0 . \tag{3.9}$$

Let $\hat{R}$ be the curvature tensor of the normal connection $\hat{\nabla}$, that is, $\hat{\nabla}_A V = (\overline{\nabla}_A V)^N$ (= the normal component of $\overline{\nabla}_A V$) for a vector field V normal to M:

$$\hat{R}(A, B)V = \hat{\nabla}_A \hat{\nabla}_B V - \hat{\nabla}_B \hat{\nabla}_A V - \hat{\nabla}_{[A,B]} V . \tag{3.10}$$

It is easily verified that for unit vector fields N_i normal to M,

$$\hat{R}(A, B) N_1 = 2ds(A, B) N_2 , \tag{3.11}$$

$$\hat{R}(A, B) N_2 = -2ds(A, B) N_1 . \tag{3.12}$$

Let $(E_i, \phi E_i, \xi)$ $(i=1, 2)$ be an adapted frame on a neighborhood of $x \in M$. On account of (1. 1)$\sim$(1. 4), we see that $E_1 \times E_2$ and $E_1 \times \phi E_2$ are mutually orthogonal unit (local) vector fields normal to M. Throughout this paper we assume

$$N_1 = E_1 \times E_2 \text{ and } N_2 = E_1 \times \phi E_2 . \tag{3.13}$$

4. **Proof of Theorem 1**. As the preparation we give a necessary and sufficient condition for the normality of (ϕ, ξ, η, g)-structure on M.

PROPOSITION 4. 1. *The* (ϕ, ξ, η, g)*-structure on* M *is normal if and only if*

$$\phi \nabla_{\phi X} \xi + \nabla_X \xi = 0 , \tag{4.1}$$

$$H(\phi X, \xi) = \xi \times H(X, \xi) . \tag{4.2}$$

PROOF. (Necessity): By virtue of (1. 4), (3. 2) and (3.6), we obtain

$$g(A, \nabla_C \phi \cdot B) = g(A \times B, \overline{\nabla}_C \xi) . \tag{4.3}$$

From (2. 7) and (4. 3) we see that $S^1 = 0$ if and only if, for any $A, B, C \in V(M)$,

$$g(\phi A \times B, \overline{\nabla}_C \xi) + g(A \times B, \overline{\nabla}_{\phi C} \xi) + \eta(A) (\nabla_C \eta)(B) = 0 . \tag{4.4}$$

On account of $\phi A \times B = \xi \times (A \times B) - 2\eta(B) A + \eta(A) B + g(A, B)\xi$, (4.4) is rewritten as follows:

$$\begin{aligned} g(A \times B, \phi \nabla_C \xi + \nabla_{\phi C} \xi + H(\phi C, \xi) - \xi \times H(C, \xi)) & \\ - 2\eta(B) g(A, \nabla_C \xi) + 2\eta(A) g(B, \nabla_C \xi) = 0 . & \end{aligned} \tag{4.5}$$

Setting $B=\xi$, we obtain

$$\nabla_C\xi + \phi\nabla_{\phi C}\xi = 0 . \tag{4.6}$$

If A and B are orthogonal to ξ, by (3.13), $A\times B$ have the form

$$A \times B = a\xi + \sum_{i=1}^{2} b^i N_i , \tag{4.7}$$

where a and b^i are scalars.

From (4.5) and (4.7) we also have

$$H(\phi C, \xi) = \xi \times H(C, \xi) . \tag{4.8}$$

The sufficiency follows from (4.5), (4.6), (4.7) and (4.8) by a direct calculation.

Q. E. D.

By virtue of (1.4), (3.1), (3.4) and (3.5) we have (c.f. [3])

$$\begin{aligned} \nabla_A\xi + \phi\nabla_{\phi A}\xi = & -h_1 A \times N_2 + h_2 A \times N_1 + h_1\phi A \times N_1 \\ & + h_2\phi A \times N_2 - H(A, \xi) - \xi \times H(\phi A, \xi) . \end{aligned} \tag{4.9}$$

Then we obtain

PROPOSITION 4.2. *Let H be a mean curvature vector of M. Then*

$$\begin{aligned} g(H, N_1) = & \; g(\nabla_{E_1}\xi + \phi\nabla_{\phi E_1}\xi, \phi E_2) - g(\nabla_{E_2}\xi + \phi\nabla_{\phi E_2}\xi, \phi E_1) \\ & + g(H(\xi, \xi), N_1) , \end{aligned} \tag{4.10}$$

$$\begin{aligned} g(H, N_2) = & -g(\nabla_{E_1}\xi + \phi\nabla_{\phi E_1}\xi, E_2) + g(\nabla_{E_2}\xi + \phi\nabla_{\phi E_2}\xi, E_1) \\ & + g(H(\xi, \xi), N_2) . \end{aligned} \tag{4.11}$$

PROOF. From the properties of the vector cross-product on R^7, we have

$$\begin{cases} N_1 \times \phi E_2 = \phi E_1, \; N_2 \times \phi E_2 = -E_1 , \\ N_1 \times \phi E_1 = -\phi E_2, \; N_2 \times \phi E_1 = E_2 . \end{cases} \tag{4.12}$$

The mean curvature vector H is, by definition,

$$H=\sum_{i=1}^{2}\{H(E_i, E_i)+H(\phi E_i, \phi E_i)\}+H(\xi, \xi) \tag{4.13}$$

$$=\sum_{i,j=1}^{2}\{g(h_jE_i, E_i)+g(h_j\phi E_i, \phi E_i)\}N_j+\sum_{j=1}^{2}g(h_j\xi, \xi)N_j\,.$$

Since the $g(h_jA, B)$ is symmetric, Proposition 4.2 follows from (4.9), (4.12) and (4.13). Q. E. D.

The proof of Theorem 1 follows from the Proposition 4.1 and 4.2.

COROLLARY 4.3. *Let M be an orientable 5-dimensional submanifold of R^7. Then if the almost contact structure induced from the vector cross product is normal, M cannot be compact.*

PROOF. M must be a minimal submanifold, but it is well known that there are no compact minimal submanifold of R^7.

5. Proof of Theorem 2.

PROPOSITION 5.1. *Let M be an orientable 5-dimensional submanifold of R^7 with the almost contact structure (ϕ, ξ, η, g). The following conditions are equivalent:*

(1) (ϕ, ξ, η, g)*-structure is a quasi-Sasakian structure,*

(2) $$\overline{\nabla}_A\xi=0\,,$$

(3) $$h_1=\phi h_2\,.$$

PROOF. ((1)→(2)): From (2.3) and the last of §2, we have

$$(\nabla_A w)(B, C)-\eta(B)g(A, \nabla_{\phi C}\xi)=0\,. \tag{5.1}$$

Putting $C=\xi$ in this equation and using (4.3), we get $\overline{\nabla}_A\xi=0$.

(2)→(1): By (4.3) we have $\nabla_C w=0$ and so $dw=0$. From the Proposition 4.1 $S^1=0$ is clear.

(2)↔(3): On account of $\overline{\nabla}_A\xi=-h_1A\times N_2+h_2A\times N_1$ (c. f. [3]), $\overline{\nabla}_A\xi=0$ is equivalent to $h_1A\times N_2=h_2A\times N_1$. By $N_1=N_2\times\xi$, this equation is equivalent to

$h_1 = \phi h_2$. Q. E. D.

PROPOSITION 5.2. *Under the same assumption as Proposition* 5.1, *we have*

$$\hat{R}(A, B)N_1 = 2g(h_2A, \phi h_2B)N_2,$$
$$\hat{R}(A, B)N_2 = -2g(h_2A, \phi h_2B)N_1.$$

PROOF. Since the curvature tensor of R^7 is zero, we obtain

$$\begin{aligned} 0 &= \bar{\nabla}_A\bar{\nabla}_B N_1 - \bar{\nabla}_B\bar{\nabla}_A N_1 - \bar{\nabla}_{[A,B]} N_1 \\ &= -\nabla_A h_1\cdot B + \nabla_B h_1\cdot A - H(A, h_1B) + H(B, h_1A) \\ &\quad + \{A(s(B)) - B(s(A)) - s([A, B])\}N_2 + s(B)\bar{\nabla}_A N_2 \\ &\quad - s(A)\bar{\nabla}_B N_2 \qquad \text{(by (3.4))} \\ &= \hat{R}(A, B)N_1 - 2g(h_2A, \phi h_2B)N_2 \qquad \text{(by (3.8) and (3.11))}. \end{aligned}$$

The latter half of the Proposition 5.2 can be shown by a similar fashion. Q.E.D.

Since $g(h_1A, B)$ is symmetric, $h_1 = \phi h_2$ implies $h_i\phi = -\phi h_i$. Thus the proof of Theorem 2 follows directly from the Proposition 5.1 and 5.2.

REFERENCES

[1] D. E. BLAIR, The theory of quasi-Sasskian structure, J. Diff. Geom., 1(1967), 331–345.

[2] E. CALABI, Construction and properties of some 6-dimensional almost complex manifolds, Trans. Amer. Math. Soc., 87(1958), 407–438.

[3] M. KOBAYASHI, 5-dimensional orientable submanifold of R^7, I, II, Proc. Japan Acad., 45 (1969), 582–585, 670–674.

[4] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure, II, Tôhoku Math. J., 13(1961), 281–294.

[5] S. TANNO, Quasi-Sasakian structures of rank $2p+1$, preprint.

[6] S. TANNO, Almost complex structures in bundle spaces over almost contact manifolds, J. Math. Soc. Japan, 17(1965), 167–186.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN