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ON (f, g, u, v, λ)-STRUCTURES INDUCED ON A HYPERSURFACE OF AN ODD-DIMENSIONAL SPHERE

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An orientable differentiable submanifold M^{2n} of codimension 2 with globally defined normal vectors of an even-dimensional Euclidean space E^{2n+2} admits what we call an (f, g, u, v, λ)-structure, [1, 2, 3, 4, 8, 9]. In [7] the present author studied (f, g, u, v, λ)-structures induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1} and special metric f-structures with complemented frames which are closely related to these (f, g, u, v, λ)-structures. The main purpose of the present paper is to generalize some of results obtained in [7] and study further (f, g, u, v, λ)-structures with $\lambda=0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}.

In § 1, we review some of known results on (f, g, u, v, λ)-structures naturally induced on hypersurfaces M^{2n} of an odd-dimensional unit sphere S^{2n+1}. In § 2, we study (f, g, u, v, λ)-structures naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1} and obtain generalizations of some of results in [7]. In the last § 3, we study (f, g, u, v, λ)-structures with $\lambda = 0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}.

1. Preliminaries. Let M^{2n} be an orientable differentiable submanifold of codimension 2 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} and assume that there exist two globally defined mutually orthogonal unit normals C and D to M^{2n}. Then the natural Kählerian structure of E^{2n+2} induces a structure on M^{2n} defined by a tensor field of type (1, 1), a Riemannian metric g, two 1-forms u and v, and a function λ such that [1, 2, 8, 9]

$$(1.1)\qquad \begin{aligned} f^2X &= -X+u(X)U+v(X)V,\\ g(fX, fY) &= g(X, Y)-u(X)u(Y)-v(X)v(Y),\\ u(fX) &= \lambda v(X),\quad v(fX) = -\lambda u(X),\\ u(U) &= v(V) = 1-\lambda^2,\quad u(V) = v(U) = 0, \end{aligned}$$

for any vector fields X and Y, vector fields U and V being defined by

$$(1.2)\qquad u(X) = g(U, X),\qquad v(X) = g(V, X)$$

for any vector field X. Thus the third equations of (1. 1) can also be written as

$$fU = -\lambda V, \quad fV = \lambda U. \tag{1. 3}$$

We call an (f, g, u, v, λ)-structure the set of f, g, u, v and λ satisfying (1. 1). When the tensor field of type (1, 2) defined by

$$S(X, Y) = [f, f](X, Y) + du(X, Y)U + dv(X, Y)V \tag{1. 4}$$

vanishes, where $[f, f]$ is the Nijenhuis tensor formed with f, we say that the (f, g, u, v, λ)-structure is *normal*.

Now suppose that the M^{2n} is a hypersurface of an odd-dimensional sphere S^{2n+1} of radius 1 and choose the first normal C of M^{2n} as the opposite of the radius vector of S^{2n+1}. In this case, we say that the (f, g, u, v, λ)-structure is naturally induced on M^{2n}. Then, for the second fundamental tensor h and the Weingarten tensor H with respect to C and the third fundamental tensor l, we have

$$h(X, Y) = g(X, Y), \ HX = X, \ l(X) = 0, \tag{1. 5}$$

for any vector fields X and Y and consequently we obtain

$$\begin{aligned}(\nabla_X f)Y &= -g(X, Y)U + u(Y)X - k(X, Y)V + v(Y)KX,\\ (\nabla_X u)(Y) &= \omega(X, Y) - \lambda k(X, Y),\\ (\nabla_X v)(Y) &= -k(X, fY) + \lambda g(X, Y),\\ \nabla_X \lambda &= -v(X) + u(KX),\end{aligned} \tag{1. 6}$$

where k is the second fundamental tensor, K the Weingarten tensor with respect to D, and

$$\omega(X, Y) = g(fX, Y) \tag{1. 7}$$

a 2-form and consequently the tensor S defined by (1. 4) takes the form

$$S(X, Y) = v(X)(fK - Kf)Y - v(Y)(fK - Kf)X. \tag{1. 8}$$

In [7], we have proved

THEOREM A. *In order that the (f, g, u, v, λ)-structure naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1} be normal, it is necessary and sufficient that f and K commute, K being* the *Weingarten tensor.*

If $\lambda = 0$ on M^{2n}, then (1. 1) takes the form

$$
(1.9) \quad
\begin{aligned}
f^2X &= -X + u(X)U + v(X)V, \\
g(fX, fY) &= g(X, Y) - u(X)u(Y) - v(X)v(Y), \\
u(fX) &= 0, \qquad v(fX) = 0, \\
u(U) = v(V) &= 1, \qquad u(V) = v(U) = 0,
\end{aligned}
$$

and (1. 3) the form

$$
(1.10) \qquad fU = 0, \qquad fV = 0
$$

and consequently the set (f, g, u, v, λ) defines a metric f-structure with complemented frames [5, 6] and (1. 6) becomes

$$
(1.11) \quad
\begin{aligned}
(\nabla_X f)Y &= -g(X, Y)U + u(Y)X - k(X, Y)V + v(Y)KX, \\
(\nabla_X u)(Y) &= \omega(X, Y) \quad \text{or} \quad \nabla_X U = fX, \\
(\nabla_X v)(Y) &= -k(X, fY) \quad \text{or} \quad \nabla_X V = fKX, \\
u(KX) &= v(X) \quad \text{or} \quad KU = V.
\end{aligned}
$$

In [7], we proved

THEOREM B. *Consider the (f, g, u, v, λ)-structure with $\lambda = 0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}. In order that f and K commute, it is necessary and sufficient that V is a Killing vector field, or that the tensor field k satisfies*

$$
(1.12) \qquad k(X, Y) = k(fX, fY) + u(X)v(Y) + u(Y)v(X) + k(V, V)v(X)v(Y)
$$

for any vector fields X and Y.

THEOREM C. *Consider the (f, g, u, v, λ)-structure with $\lambda = 0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}. In order that f and K anticommute, it is necessary and sufficient that v is a harmonic 1-form, or that the tensor field k satisfies*

$$
(1.13) \qquad k(X, Y) = -k(fX, fY) + u(X)v(Y) + u(Y)v(X) + k(V, V)v(X)v(Y)
$$

for any vector fields X and Y. In this csae, M^{2n} is a minimal hypersurface of S^{2n+1} if and only if $k(V, V) = 0$.

2. (f, g, u, v, λ)-structure induced on a hypersurface of an odd-dimensional unit sphere. Assume that f and K commute : $fK-Kf=0$. This is equivalent to $k(X, fY)+k(Y, fX)=0$ for any vector fields X and Y, and consequently we have, from the third equation of (1. 6),

$$(\nabla_X v)(Y) + (\nabla_Y v)(X) = 2\lambda g(X, Y),$$

which shows that V is a conformal Killing vector field.

Conversely, suppose that V is a conformal Killing vector field, then we have

$$(\nabla_X v)(Y) + (\nabla_Y v)(X) = 2\rho g(X, Y),$$

ρ being a function. Thus from the third equation of (1. 6) and this equation, we find

$$-k(X, fY) - k(Y, fX) + 2\lambda g(X, Y) = 2\rho g(X, Y),$$

from which, by contraction $\lambda=\rho$, since k is symmetric and ω is skew-symmetric, and consequently $k(X, fY)+k(Y, fX)=0$, that is, $fK-Kf=0$.

Thus we have

THEOREM 2. 1. *For the (f, g, u, v, λ)-structure naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, in order that f and K commute, it is necessary and sufficient that V is a conformal Killing vector field.*

Assume next that f and K anticommute: $fK+Kf=0$. This is equivalent to $k(X, fY)-k(Y, fX)=0$, and consequently we have, from the third equation of (1. 6),

$$(\nabla_X v)(Y) - (\nabla_Y v)(X) = 0,$$

which shows that the 1-form v is closed. The converse being evident, we have

THEOREM 2. 2. *For the (f, g, u, v, λ)-structure naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, in order that f and K anticommute, it is necessary and sufficient that v is closed.*

Suppose that $\lambda=0$ on M^{2n}, then we have, from the third equation of (1. 6), $\delta v=0$, that is, v is coclosed. Conversely, if v is coclosed, we have, from the third equation of (1. 6), $\lambda=0$. Thus we have

THEOREM 2. 3. *For the (f, g, u, v, λ)-structure naturally induced on a* *hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, in order that $\lambda = 0$, it is necessary and sufficient that v is coclosed.*

From Theorems 2. 2 and 2. 3, we have

THEOREM 2. 4. *For the (f, g, u, v, λ)-structure naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, in order that f and K anticommute and $\lambda = 0$, it is necessary and sufficient that v is harmonic.*

We now assume that the hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1} is compact, and $\lambda = 0$. Then we have, from the fourth equation of (1. 6), $KU = V$.

Conversely, if this holds, then we have $\nabla_X \lambda = 0$, from which $\lambda =$ constant. But from the third equation of (1. 6), we have, by contraction, div $V = 2n\lambda$, since $k(X, Y)$ is symmetric and $\omega(X, Y)$ is skew-symmetric, from which

$$0 = \int_{M^{2n}} \mathrm{div} V dS = 2n\lambda \int_{M^{2n}} dS,$$

dS being the surface element of M^{2n}, and consequently $\lambda = 0$.

Thus we have

THEOREM 2. 5. *For the (f, g, u, v, λ)-structure naturally induced on a compact orientable hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, in order that $\lambda = 0$, it is necessary and sufficient that $KU = V$.*

3. (f, g, u, v, λ)-structure with $\lambda = 0$ induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}. We now assume that the (f, g, u, v, λ)-structure induced on a hypersurface M^{2n} of an odd-dimensional sphere S^{2n+1} satisfies $\lambda = 0$.

If we denote by F the natural Kähler structure of the ambient E^{2n+2}, then $FC \cdot D = \lambda$, the dot denoting the inner product in E^{2n+2} and consequently $\lambda = 0$ means that FC is orthogonal to C and D, and is tangent to M^{2n}. If we denote by (φ, ξ, η) the Sasakian structure induced on S^{2n+1}, then FC is the vector field ξ on S^{2n+1} and $\lambda = 0$ means that ξ is always tangent to M^{2n}.

From the second equation of (1. 11), we have, by taking account of $fU = 0$ and $fV = 0$,

$$\nabla_U U = 0, \quad \nabla_V U = 0. \tag{3.1}$$

From the third equation of (1. 11), we have, by taking account of $fV = 0$ and $KU = V$,

$$\nabla_U V = 0, \quad \nabla_V V = fKV. \tag{3.2}$$

Thus we have

THEOREM 3.1. *For the (f, g, u, v, λ)-structure with $\lambda=0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, we have*

$$[U, V] = \nabla_U V - \nabla_V U = 0. \tag{3.3}$$

We now assume that $\nabla_V V = 0$, then we have $fKV = 0$, from which, applying f, we find $-KV+u(KV)U+v(KV)V=0$. But, from $KU=V$, we have $u(KV)=k(U, V)=v(KU)=v(V)=1$, and consequently we have $KV=U+K(V,V)V$. Conversely, if KV has this form, then we have

$$\nabla_V V = fKV = f(U+K(V, V)V) = 0.$$

Thus we have

THEOREM 3.2. *For the (f, g, u, v, λ)-structure with $\lambda=0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, the vector field U is parallel in the direction of U and in the direction of V. The vector field V is parallel in the direction of U. In order that V is parallel in the direction of V, it is necessary and sufficient that*

$$KV = U + k(V, V)V. \tag{3.4}$$

Suppose that the vector field V is parallel in M^{2n}, then we have, from the third equation of (1.11),

$$fKX = 0 \tag{3.5}$$

for any vector field X, from which, applying f, we have

$$-KX + u(KX)U + v(KX)V = 0,$$

that is,

$$KX = v(X)U + k(X, V)V \tag{3.6}$$

or

$$k(X, Y) = v(X)u(Y) + k(X, V)v(Y), \tag{3.7}$$

from which, $k(X, Y)$ being symmetric,

$$v(X)u(Y) - v(Y)u(X) + k(X, V)v(Y) - k(Y, V)v(X) = 0.$$

Thus putting $Y=V$ in this equation, we find

$$k(X, V) = u(X) + k(V, V)v(X). \tag{3.8}$$

Thus substituting (3. 8) into (3. 7), we find

$$k(X, Y) = u(X)v(Y) + u(Y)v(X) + k(V, V)v(X)v(Y) \tag{3.9}$$

or

$$KX = v(X)U + u(X)V + k(V, V)v(X)V. \tag{3.10}$$

Conversely, if K has the form (3. 10), then (3. 5) is satisfied. Thus we have

THEOREM 3. 3. *For the (f, g, u, v, λ)-structure with $\lambda=0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, in order that the vector field V is parallel, it is necessary and sufficient that k has the form* (3. 9) *or K has the form* (3. 10).

Now, from the first equation of (1. 11), we have, by putting $X=U$,

$$(\nabla_U f)Y = -u(Y)U + u(Y)U - k(U, Y)V + v(Y)KU = 0$$

since $KU=V$ and, by putting $X=V$,

$$(\nabla_V f)Y = -v(Y)U + u(Y)V - k(V, Y)V + v(Y)KV. \tag{3.11}$$

Thus, in order that $\nabla_V f=0$, we should have

$$-v(Y)U + u(Y)V - k(V, Y)V + v(Y)KV = 0,$$

from which, putting $Y=V$, $KV=U+k(V, V)V$.

Conversely if this is satisfied, then we see, from (3. 11), that $\nabla_V f=0$. Thus we have

THEOREM 3. 4. *For the (f, g, u, v, λ)-structure with $\lambda=0$ naturally induced on a hypersurface M^{2n} of an odd-dimensional unit sphere S^{2n+1}, the tensor field f is parallel in the direction of U. In order that the tensor field f is parallel also in the direction of V, it is necessary and sufficient that K satisfies*

$$KV = U + k(V, V)V.$$

Now, using (1. 11), we have

$$\begin{aligned}(\mathcal{L}_U f)Y &= (\nabla_U f)Y + f\nabla_Y U - \nabla_{fY}U \\ &= -g(U, Y)U + u(Y)U - k(U, Y)V \\ &\quad + v(Y)KU + f^2Y - f(fY) = 0,\end{aligned} \tag{3.12}$$

since $KU=V$, $\mathcal{L}_U$ denoting the operator of Lie derivation with respect to U. On the other hand, we have

$$(\mathcal{L}_V f)Y = (\nabla_V f)Y + f\nabla_Y V - \nabla_{fY} V$$
$$= -g(V, Y)U + u(Y)V - k(V, Y)V + v(Y)KV + f^2KY - f(KfY),$$

that is,

$$(\mathcal{L}_V f)Y = v(Y)(KV-U) + \{u(Y) - k(V, Y)\}V + f(fK-Kf)Y. \tag{3.13}$$

We first assume that our structure is normal, that is, f and K commute, then, by Theorem B, we have (1.12), from which, putting $X=V$,

$$k(V, Y) = u(Y) + k(V, V)v(Y), \tag{3.14}$$

that is,

$$KV = U + k(V, V)V, \tag{3.15}$$

and consequently, we have, from (3.13),

$$(\mathcal{L}_V f)Y = v(Y)k(V, V)V - k(V, V)v(Y)V = 0.$$

We next assume that our structure is antinormal, that is, f and K anticommute, then, by Theorem C, we have (1.13), from which we have (3.14) and (3.15), and consequently, we have, from (3.13),

$$(\mathcal{L}_V f)Y = 2f^2KY$$
$$= 2\{-KY + u(KY)U + v(KY)V\}$$
$$= -2\{KY - v(Y)U - u(Y)V - k(V, V)v(Y)V\}$$

which shows that the vanishing of $\mathcal{L}_V f$ is equivalent to (3.9) or to (3.10).

Thus we have, taking account of Theorem 3.3,

THEOREM 3.5. *For the* (f, g, u, v, λ)*-structure with* $\lambda=0$ *naturally induced on a hypersurface* M^{2n} *of an odd-dimensional unit sphere* S^{2n+1}*, the Lie derivative of the tensor field* f *with respect to* U *vanishes. If the structure is normal, then the Lie derivative of the tensor field* f *with respect to* V *also vanishes. If the structure is antinormal, the Lie derivative of the tensor field* f *with respect to* V *vanishes if and only if* k *has the form* (3.10)*, that is, the vector field* V *is parallel.*

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