Tôhoku Math. Journ.
24 (1972), 93-103.

A CLASS OF ALMOST CONTACT RIEMANNIAN MANIFOLDS

KATSUEI KENMOTSU

(Received. Nov. 20, 1971)

1. Introduction. Recently S. Tanno has classified connected almost contact Riemannian manifolds whose automorphism groups have the maximum dimension [9]. In his classification table the almost contact Riemannian manifolds are divided into three classes: (1) homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if the sectional curvature for 2-planes which contain ξ, say $K(X, \xi), > 0$, (2) global Riemannian products of a line or a circle and a Kaehlerian manifold with constant holomorphic sectional curvature, if $K(X, \xi) = 0$ and (3) a warped product space $L \times_f CE^n$, if $K(X, \xi) < 0$. It is known that the manifold of the class (1) in the above statement is characterized by some tensor equations; it has a Sasakian structure.

The purpose of this paper is to characterize the warped product space $L \times_f CE^n$ by tensor equations (§ 2) and study their properties. From the definition by means of the tensor equations it is easily verified that the structure is normal, but not quasi-Sasakian (and is hence not Sasakian). In § 2, we define a structure closely related to the warped product which is studied by Bishop-O'Neill [1] and prove the local structure theorem. In § 3 we study some properties of the structure. § 4 is devoted to a study of η-Einstein manifolds. In the section 5 we show one of the main theorems in this paper. In the last section we study invariant submanifolds.

We follow here the notations and the terminology of the Volume 1 of Kobayashi-Nomizu [4].

2. Definition and examples. It is well-known that the structure tensors (ϕ, ξ, η, g) of the almost contact Riemannian maifold M satisfy

$$\phi\xi = 0\,, \qquad \eta(\phi X) = 0\,, \qquad \eta(\xi) = 1\,, \tag{2.1}$$

$$\phi\phi X = -X + \eta(X)\xi\,, \qquad g(X, \xi) = \eta(X)\,, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \tag{2.3}$$

for any vector fields X and Y on M. It is known that the (ϕ, ξ, η, g)-structure is normal if and only if

$$\phi\nabla_X\phi \cdot Y - \nabla_{\phi X}\phi \cdot Y - (\nabla_X\eta)(Y) \cdot \xi = 0 , \tag{2.4}$$

where ∇ denotes the Riemannian connection for g [8].

Throughout this paper we study a class of almost contact Riemannian manifolds which satisfy the following two conditions, say (*),:

$$(*) \qquad \begin{cases} \nabla_X\phi \cdot Y = -\eta(Y)\phi X - g(X, \phi Y)\,\xi , \\ \quad \nabla_X\xi = X - \eta(X)\,\xi . \end{cases}$$

REMARK TO (*). S. S. Eum studied the integrability of invariant hypersurfaces immersed in an almost contact Riemannian manifold which satisfies

$$g(\nabla_X\phi \cdot Y, Z) = (\nabla_X\eta)(\eta(Y)\phi Z - \eta(Z)\phi Y) . \tag{2.5}$$

If we assume $(*)_2$ in an almost contact Riemannian manifold, then $(*)_1$ is equivalent to (2.5).

From (2.4), (ϕ, ξ, η, g)-structure with (*) is normal and since ξ is not a Killing vector field the structure is not quasi-Sasakian (cf. [2]). Thus we have

PROPOSITION 1. *Let M be an almost contact Riemannian manifold with* (*). *Then M is normal but not quasi-Sasakian and hence not Sasakian.*

Taking the Lie derivative of g, ϕ and η along ξ we see

PROPOSITION 2. *Under the same assumption as Proposition* 1,

$$(\nabla_X\eta)(Y) = g(X, Y) - \eta(X) \cdot \eta(Y) , \tag{2.6}$$

$$L(\xi)g = 2(g - \eta \otimes \eta) , \tag{2.7}$$

$$L(\xi)\phi = 0 , \tag{2.8}$$

$$L(\xi)\eta = 0 , \tag{2.9}$$

where $L(\xi)$ denotes the Lie derivative along ξ.

Since the proof of Proposition 2 follows by a routine calculation, we shall omit it. We give here examples of almost contact Riemannian manifolds which satisfy the condition (*). These examples are closely related to the warped product space defined by Bishop-O'Neill [1]: Let B and F be Riemannian manifolds and $f > 0$ a differentiable function on B. Consider the product manifold $B \times F$ with its projection p: $B \times F \to B$ and π: $B \times F \to F$. The warped product $M = B \times_f F$ is the manifold $B \times F$ furnished with the Riemannian structure such that

$$\| X \|^2 = \| p_* X \|^2 + f^2(px) \| \pi_* X \|^2$$

for every tangent vector $X \in T_x(M)$. We have

PROPOSITION 3. *Let F be a Kaehlerian manifold and c a nonzero constant. Let $f(t) = ce^t$ be a function on a line L. Then the warped product space $M = L \times_f F$ have an almost contact metric structure which satisfies* (*).

PROOF. (G, J) denotes the Kaehlerian structure of F and D denotes the Riemannian connection for the Kaehlerian metric G. Let $(t, x_1, \cdots, x_{2n})$ be a local coordinates of M where t and $(x_1, \cdots, x_{2n})$ denotes the local coordinates of L and F, respectively. We define a Riemannian metric tensor g, a vector field ξ and a 1-form η on M as follows:

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t) G_{(x)} \end{pmatrix}, \tag{2.10}$$

$$\xi = \left(\frac{d}{dt}\right), \quad \eta(X) = g(X, \xi) . \tag{2.11}$$

By a direct calculation or Lemma 7.3 of [1] we have easily $(*)_2$ because of $\xi(f) = f$. By $(*)_2$ we see

$$L(\xi)\eta = 0 . \tag{2.12}$$

A (1, 1)-tensor field ϕ is defined ϕ by $\phi_{(t,x)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\phi}_{(t,x)} \end{pmatrix}$, where

$$\tilde{\phi}_{(t,x)} = (\exp(t\xi))_* J_x (\exp(-t\xi))_* . \tag{2.13}$$

(2.13) is well-defined by (2.12). We can easily verify that (ϕ, ξ, η, g) defines an almost contact metric structure on $M = L \times_f F$ by (2.10) $\sim$ (2.13). By (2.13) we see

$$(\exp s\xi)_* \phi = \phi (\exp s\xi)_* . \tag{2.14}$$

Making use of $(*)_2$ and (2.14), we have easily

$$L(\xi)g = 2(g - \eta \otimes \eta) , \tag{2.15}$$

$$L(\xi)\phi = 0 . \tag{2.16}$$

By virtue of $(*)_2$ and (2.16) we have

$$\nabla_\xi \phi = L(\xi)\phi = 0 . \tag{2.17}$$

By (2.10), we have

$$\nabla_{X_0} Y_0 = D_{X_0} Y_0 - g(X_0, Y_0)\xi , \tag{2.18}$$

where X_0 and Y_0 are vector fields with $\eta(X_0) = 0$ and $\eta(Y_0) = 0$, respectively. We see the almost contact structure in consideration satisfies $(*)_1$. Let X_0 and Y_0 denote the F-components of X and Y. Then we have

$$\begin{aligned}\nabla_X\phi\cdot Y &= \nabla_{X_0+\eta(X)\xi}(\phi Y_0) - \phi\nabla_{X_0+\eta(X)\xi}(Y_0+\eta(Y)\xi)\\ &= D_{X_0}(JY_0) - g(X_0, \phi Y_0)\xi + \eta(X)\nabla_\xi(\phi Y_0)\\ &\quad - \phi\{D_{X_0}Y_0 + \eta(Y)X_0 + \eta(X)\nabla_\xi Y_0\}\\ &\qquad\qquad \text{(because of (2.18) and } (*)_2)\\ &= D_{X_0}(JY_0) - \phi D_{X_0}Y_0 - g(X_0, \phi Y_0)\xi - \eta(Y)\phi X_0\\ &\qquad\qquad \text{(because of (2.17))}\\ &= -g(X, \phi Y)\xi - \eta(Y)\phi X\,,\end{aligned}$$

since $\exp t\xi$ is a homothety with respect to the distribution $\eta = 0$ and $DJ = 0$. q.e.d.

Conversely we have the following structure theorem.

THEOREM 4. *Let M be an almost contact Riemannian manifold with $(*)$. Then, for any $p \in M$, some neighborhood $U(p)$ of $p \in M$ is identified with a warped product space $(-\varepsilon, +\varepsilon) \times_f V$ such that $(-\varepsilon, +\varepsilon)$ is an open interval, $f(t) = ce^t$ and V is a Kaehlerian manifold.*

PROOF. We define a distribution $\mathfrak{d}$ by $\eta = 0$. It is completely integrable by (2.6). Let $M(p)$ be the maximal integral submanifold through p. $M(p)$ is a totally umbilical hypersurface of M because of $(*)_2$. J and G denote the restriction of ϕ and g to $M(p)$ respectively. Then $M(p)$ is an almost Hermitian manifold for (J, G).

Moreover, by $(*)_1$, $M(p)$ is a Kaehlerian manifold. By virtue of Proposition 2, $\exp t\xi$ leaves ϕ and η invariant for each t and $\exp t\xi$ are homotheties on $\mathfrak{d}$, whose propotional factor is monotonously increasing as t. Thus the metric are written by

$$g_{(t,x)} = \begin{pmatrix}1 & 0\\ 0 & f^2(t)G_x\end{pmatrix}.$$

From (2.9) the differential equation for f is $f' - f = 0$. We have $f(t) = ce^t$ and M is locally a warped product space. q.e.d.

3. Some Properties. In the theory of Sasakian manifolds the following result is well-known: $K(X, \xi) = 1$ and if a Sasakian manifold is locally symmetric, then it is of constant positive curvature $+1$. On the other hand an almost contact Riemannian manifold with $(*)$ is not compact because of $\operatorname{div}\xi = 2n$ and we get

PROPOSITION 5. *Under the same assumption as Proposition 1,*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X , \tag{3.1}$$

$$K(X, \xi) = -1 , \tag{3.2}$$

$$(\nabla_Z R)(X, Y; \xi) = g(Z, X)Y - g(Z, Y)X - R(X, Y)Z . \tag{3.3}$$

PROOF. (3.1) follows directly from $(*)_2$, (2.6) and the definition of R. (3.2) is a result of (3.1). By virtue of $(*)_2$, (2.6) and (3.1) we get (3.3):

$$\begin{aligned}(\nabla_Z R)(X, Y; \xi) &= \nabla_Z(R(X, Y)\xi) - R(\nabla_Z X, Y) - R(X, \nabla_Z Y)\\ &\quad -R(X, Y)(\nabla_Z \xi)\\ &= g(Z, X)Y - g(Z, Y)X - R(X, Y)Z . \qquad \text{q.e.d.}\end{aligned}$$

COROLLARY 6. *If M is locally symmetric, then it is of constant negative curvature* -1.

PROOF. Corollary 6 follows from (3.3).

We can generalize Corollary 6 slightly as follows:

PROPOSITION 7. *Under the same assumption as Proposition* 1, *if M satisfies the Nomizu's condition, i.e.,* $R(X, Y)R = 0$, *then it is of constant negative curvature* -1.

Since the proof of this Proposition is done by the same method as M. Okumura proved the Theorem 3.2 in [7], we shall omit it.

4. **η-Einstein manifold.** In an almost contact Riemannian manifold, if the Ricci tensor R_1 satisfies $R_1 = ag + b\,\eta\otimes\eta$, where a and b are scalar functions, then it is called an η-Einstein manifold. If a Sasakian manifold is η-Einsteinian and the dimension > 3, then a and b are constant.

PROPOSITION 8. *Let M be an almost contact Riemannian manifold with* $(*)$ *of dimension* $(2n + 1)$. *If M is η-Einsteinian, we have*

$$a + b = -2n , \tag{4.1}$$

$$Z(b) + 2b\,\eta(Z) = 0 , \quad \textit{if} \quad n > 1 , \quad \textit{for any vector field } Z \textit{ on } M. \tag{4.2}$$

PROOF. (4.1) follows from $R_1(X, \xi) = -2n\,\eta(X)$ which is derived from (3.1) and $R_1(X, Y) =$ the trace of the map $[W \to R(W, X)Y]$. As M is an η-Einstein manifold, the scalar curvature S is $2n(a-1)$. We define a (1, 1)-tensor field R^1 as follows: $g(R^1(X), Y) = R_1(X, Y)$. By the identity $\nabla_Y S = 2$ (trace of the map $[X \to (\nabla_X R^1)Y]$), we have

$$Z(a) + \xi(b)\eta(Z) + 2nb\,\eta(Z) = nZ(a) .$$

Setting $Z = \xi$, we get $\xi(b) = -2b$. Therefore we have $Z(b) + 2b\eta(Z) = 0$ if $n > 1$. q.e.d.

COROLLARY 9. *Under the same assumption as the Proposition* 8, *if* $b=$ *constant* (*or* $a=$ *constant*), *then* M *is an Einstein one.*

PROOF. Corollary 9 is a direct consequence of (4.2).

5. Curvature tensor. At first we shall prove

PROPOSITION 10. *Let* R *be the Riemannian curvature tensor of* M *with* (*). *Then*

$$R(X, Y)\phi Z-\phi R(X, Y)Z=g(Y, Z)\phi X-g(X, Z)\phi Y + g(X, \phi Z)Y-g(Y, \phi Z)X\,, \tag{5.1}$$

$$R(\phi X, \phi Y)Z=R(X, Y)Z+g(Y, Z)X-g(X, Z)Y + g(Y, \phi Z)\phi X-g(X, \phi Z)\phi Y\,. \tag{5.2}$$

PROOF. (5.1) follows from (*) and the Ricci's identity:

$$\nabla_X\nabla_Y\phi-\nabla_Y\nabla_X\phi-\nabla_{[X,Y]}\phi=R(X, Y)\phi-\phi R(X, Y)\,.$$

We verify (5.2): By (5.1), we have

$$\begin{aligned} &g(R(X, Y)\phi Z, \phi W)-g(\phi R(X, Y)Z, \phi W)\\ &=g(Y, Z)\,g(\phi X, \phi W)-g(X, Z)\,g(\phi Y, \phi W)\\ &\quad+g(X, \phi Z)g(Y, \phi W)-g(Y, \phi Z)g(X, \phi W)\,. \end{aligned}$$

Using $\eta(R(X, Y)Z)=\eta(Y)g(X, Z)-\eta(X)g(X, Z)$, the above formula is

$$\begin{aligned} g(R(\phi Z, \phi W)X, Y)&=g(R(Z, W)X, Y)+g(Y, Z)g(X, W)\\ &\quad-g(X, Z)g(Y, W)+g(X, \phi Z)g(Y, \phi W)\\ &\quad-g(Y, \phi Z)g(X, \phi W)\,. \end{aligned}$$

q.e.d.

As an application of Proposition 10, we show

PROPOSITION 11. *Let* M *be an almost contact Riemannian manifold with* (*) *of dimension* >3. *If* M *is conformally flat, then* M *is a space of constant negative curvature* -1.

PROOF. Since M is conformally flat, the Riemannian curvature tensor of M is written by

$$\begin{aligned} &R(X, Y)Z\\ &=\frac{1}{(2n-1)}\{R_1(Y, Z)X-R_1(X, Z)Y+g(Y, Z)R^1(X)-g(X, Z)R^1(Y)\}\\ &\quad+\frac{S}{(2n)(2n-1)}\{g(X, Z)Y-g(Y, Z)X\}\,. \end{aligned} \tag{5.3}$$

Let us calculate $R(\xi, Y)\xi$ by the above formula. Using (3.1) and

$$R_1(X, \xi) = -2n\,\eta(X)\ ,$$

we get

$$2nR_1 = (S + 2n)\,g - (S+4n^2+2n)\eta \otimes \eta\ . \tag{5.4}$$

By virtue of (5.1), (5.3) and (5.4), we have

$$\begin{aligned}(S+4n^2+2n)\{g(Y, \phi Z)X - g(X, \phi Z)Y + g(X, Z)\phi Y \\ - g(Y, Z)\phi X + \eta(X)g(Y, \phi Z)\,\xi - \eta(Y)g(X, \phi Z)\,\xi \\ - \eta(Y)\,\eta(Z)\phi X + \eta(X)\,\eta(Z)\phi Y\} = 0\ .\end{aligned} \tag{5.5}$$

Let $\{\xi, E_1, \phi E_1, \cdots, E_n, \phi E_n\}$ be an orthonormal basis of $T_x(M)$, $x \in M$. Setting $X = E_1$, $Y = E_2$ and $Z = \phi E_2$ in (5.5), we see $S = -2n(2n+1)$. Thus we have $R_1 = -2ng$. Proposition 11 follows from (5.3). q.e.d.

In a Sasakian manifold with constant ϕ-holomorphic sectional curvature, say H, the curvature tensor has a special feature [6]: The necessary and sufficient condition for a Sasakian manifold to have constant ϕ-holomorphic sectional curvature H is

$$\begin{aligned}4R(X, Y)Z = (H+3)\,(g(Y, Z)X - g(X, Z)Y) + (H-1)(\eta(X)\,\eta(Z)Y \\ - \eta(Y)\,\eta(Z)X + \eta(Y)g(X, Z)\,\xi - \eta(X)g(Y, Z)\,\xi \\ + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z)\ .\end{aligned}$$

In our case we have

PROPOSITION 12. *Let M be an almost contact Riemannian manifold with* (*). *The necessary and sufficient condition for M to have constant ϕ-holomorphic sectional curvature H is*

$$\begin{aligned}4R(X, Y)Z = (H-3)(g(Y, Z)X - g(X, Z)Y) + (H+1)(\eta(X)\eta(Z)Y \\ - \eta(Y)\,\eta(Z)X + \eta(Y)g(X, Z)\,\xi - \eta(X)g(Y, Z)\,\xi \\ + g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z)\ .\end{aligned} \tag{5.6}$$

PROOF. For any vector fields X and $Y \in \mathfrak{d}$, we have

$$g(R(X, \phi X)X, \phi X) = -Hg(X, X)^2\ . \tag{5.7}$$

By (5.1) we get

$$\begin{aligned}g(R(X, \phi Y)X, \phi Y) = g(R(X, \phi Y)Y, \phi X) - g(X, \phi Y)^2 - g(X, Y)^2 \\ + g(X, X)g(Y, Y)\ ,\end{aligned} \tag{5.8}$$

$$g(R(X, \phi X)Y, \phi X) = g(R(X, \phi X)X, \phi Y)\ , \qquad \text{for } X, Y \in \mathfrak{d}\ . \tag{5.9}$$

Substituting $X + Y$ in (5.7), we see

$$-H(2g(X,Y)^2 + 2g(X,X)g(X,Y) + 2g(X, Y)g(Y,Y) + g(X,X)g(Y,Y))$$
$$= \frac{1}{2}g(R(X+Y, \phi X+\phi Y)(X+Y), \phi X+\phi Y) + \frac{1}{2}H(g(X, X)^2 + g(Y, Y)^2)$$
$$= g(R(Y, \phi X)X, \phi X) + g(R(X, \phi X)X, \phi Y) + g(R(Y, \phi Y)X, \phi X)$$
$$+ g(R(Y, \phi Y)Y, \phi X) + g(R(X, \phi Y)Y, \phi X) + g(R(X, \phi Y)Y, \phi Y)$$
$$+ g(R(X, \phi Y)X, \phi Y) \qquad \text{(because of (5.1))}$$
$$= 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) - g(R(\phi Y, X)Y, \phi X)$$
$$- g(R(X, Y)\phi Y, \phi X) + g(R(X, \phi Y)Y, \phi X) + g(R(X, \phi Y)X, \phi Y),$$

because of (5.9) and the Bianchi identity. It then turns to

$$= 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) + 2g(R(X, \phi Y)Y, \phi X)$$
$$+ g(R(\phi X, \phi Y)X, Y) + g(R(X, \phi Y)X, \phi Y)$$
$$= 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) + 3g(R(X, \phi Y)Y, \phi X)$$
$$+ g(R(X, Y)X, Y),$$

because of (5.2) and (5.8). Thus we get

$$\begin{aligned} (5.10) \qquad & 2g(R(X, \phi X)X, \phi Y) + 2g(R(Y, \phi Y)Y, \phi X) \\ & + 3g(R(X, \phi Y)Y, \phi X) + g(R(X, Y)X, Y) \\ &= -H(2g(X, Y)^2 + 2g(X, X)g(X, Y) + 2g(X, Y)g(Y, Y) \\ & + g(X, X)g(Y, Y)) . \end{aligned}$$

Replacing Y by $-Y$ in (5.10) and summing it to (5.10) we have

$$\begin{aligned} (5.11) \qquad & 3g(R(X, \phi Y)Y, \phi X) + g(R(X, Y)X, Y) \\ &= -H(2g(X, Y)^2 + g(X, X)g(Y, Y)) . \end{aligned}$$

By virtue of (5.11) we see

$$\begin{aligned} (5.12) \qquad & 4g(R(X, Y)X, Y) \\ &= (H-3)(g(X, Y)^2 - g(X, X)g(Y, Y)) - 3(H+1)g(X, \phi Y)^2 . \end{aligned}$$

We verify (5.12):

$$-H(2g(X, \phi Y)^2 + g(X, X)g(\phi Y, \phi Y))$$
$$= -3g(R(X, Y)\phi Y, \phi X) + g(R(X, \phi Y)X, \phi Y)$$
$$= 3g(R(\phi X, \phi Y)X, Y) + g(R(X, \phi Y)X, \phi Y)$$
$$= 3g(R(X, Y)X, Y) + g(R(X, \phi Y)Y, \phi X) + 2g(X, Y)^2 - 2g(X, X)g(Y, Y)$$
$$+ 2g(X, \phi Y)^2 \qquad \text{(because of (5.2) and (5.8))}$$
$$= 3g(R(X, Y)X, Y) - \frac{1}{3}g(R(X,Y)X,Y) - \frac{H}{3}(2g(X, Y)^2 + g(X,X)g(Y,Y))$$
$$+ 2g(X, Y)^2 - 2g(X, X)g(Y, Y) + 2g(X, \phi Y)^2 \qquad \text{(because of (5.11))} .$$

After simplication (5.12) follows. Therefore by a standard calculation we have

$$(5.13)\quad 4R(X, Y)Z = (H-3)(g(Y, Z)X - g(X, Z)Y) + (H+1)(g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z)\ ,$$

where X, Y and $Z \in \mathfrak{d}$.

We verify (5.13) for completeness: We calculate $g(R(X+Z, Y+W)(X+Z), Y+W)$. Using (5.12) we see

$$\begin{aligned}(5.14)\quad & 4g(R(X, Y)Z, W) + 4g(R(X, W)Z, Y) \\ &= (H-3)(g(X, Y)g(Z, W) + g(X, W)g(Y, Z) - 2g(X, Z)g(Y, W)) \\ &\quad - 3(H+1)(g(X, \phi Y)g(Z, \phi W) + g(X, \phi W)g(Z, \phi Y))\end{aligned}$$

and we have

$$\begin{aligned}(5.14)'\quad & -4g(R(X, Z)Y, W) - 4g(R(X, W)Y, Z) \\ &= -(H-3)(g(X, Z)g(Y, W) + g(X, W)g(Y, Z) - 2g(X, Y)g(Z, W)) \\ &\quad + 3(H+1)(g(X, \phi Z)g(Y, \phi W) + g(X, \phi W)g(Y, \phi Z))\ .\end{aligned}$$

Making (5.14) + (5.14)′ we get by virtue of the Bianchi identity

$$\begin{aligned}4g(R(X, W)Z, Y) = {} & (H-3)(g(X, Y)g(Z, W) - g(X, Z)g(Y, W)) \\ & -(H+1)(g(X, \phi Y)g(Z, \phi W) - g(X, \phi Z)g(Y, \phi W) \\ & + 2g(X, \phi W)g(Z, \phi Y))\ ,\end{aligned}$$

where X, Y, Z and $W \in \mathfrak{d}$. For any vector fields X, Y, Z, using (3.1), we get (5.6). q.e.d.

THEOREM 13. *Let M be an almost contact Riemannian manifold with (*). If M is a space of constant ϕ-holomorphic sectional curvature H, then M is a space of constant curvature and $H = -1$.*

PROOF. By virtue of Proposition 12, M is an η-Einstein space:

$$(5.15)\qquad R_1 = \frac{1}{2}(n(H-3)+H+1)g - \frac{1}{2}(n+1)(H+1)\,\eta\otimes\eta\ .$$

Since the coefficients of R_1 is constant on M, we see $H = -1$ by Corollary 9. q.e.d.

OBSERVATION 14. *Let $F[k]$ be a Kaehlerian manifold with constant holomorphic sectional curvature. Then the curvature tensor of the warped product space $L \times_f F[k]$, where $f(t) = ce^t$, is expressed by*

$$
\begin{aligned}
(5.16)\quad R(X, Y)Z = H_1(t)(g(Y, W)X - g(X, Z)Y) + (H_1(t) + 1)(\eta(X)\eta(Z)Y \\
- \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\
+ g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z) .
\end{aligned}
$$

PROOF. (5.16) follows directly from Lemma 7.4 in [1].

REMARK. From the Tanno's Theorem [9], the maximum dimension of the automorphism group of $L \times {}_f F[k]$, where $F[k]$ is connected, is attained if and only if $F[k] = CE^n$ (and hence $H_1(t) = -1$).

6. Invariant submanifold. Invariant submanifolds in a Sasakian manifold are also Sasakian and minimal. In this section we study invariant submanifolds in an almost contact Riemannian manifold M with (*). Let N be an almost contact manifold and (ϕ_0, η_0, ξ_0) denote its structure tensor. An invariant immersion, say i, of N into M is an immersion which satisfies

$$
(6.1)\qquad i_*\phi_0 = \phi i_* , \qquad i_*\xi_0 = \xi .
$$

Then we can easily see that i is a minimal immersion for the induced metric g_0 and $(\phi_0, \xi_0, \eta_0, g_0)$ is an almost contact metric structure with (*) on N. Moreover by the local structure theorem 4, it is easy to show that

PROPOSITION 15. *Let $F[c]$ be a complex projective space CP^{n+1} with a Fubini-Study metric or a complex Euclidean space CE^{n+1} or an open ball CD^{n+1} with a homogeneous Kaehlerian structure of negative constant holomorphic sectional curvature, and let N be an invariant submanifold of codimension 2 in $M = L \times_f F[c]$. If N is an η-Einstein manifold for the induced metric, then N is totally geodesic or N is locally isometric to $L \times_f Q^n$, where Q^n is a hypersphere in $CP^{n+1} (n \geq 2)$.*

PROOF. Since N is an invariant submanifold of M, the distribution defined by $\eta_0 = 0$ is completely integrable. Let $N(p)$ be the maximal integral submanifold through $p \in N$. By Theorem 4, $N(p)$ is a Kaehlerian hypersurface in M and an Einstein manifold for the restricted metric since N is an η-Einstein one. Therefore $N(p)$ is totally geodesic or locally holomorphically isometric to Q^n(see [5]). Thus N is totally geodesic or locally isometric to $L \times_f Q^n$. q.e.d.

Let $\bar{N}$ be an almost complex manifold with an almost complex structure J. When an immersion j of $\bar{N}$ into M satisfies $j_* J = \phi j_*$ and $j^*\eta = 0$, we call $j(\bar{N})$ an invariant hypersurface. Such an immersion is studied by S. Eum [3], etc. If $j(\bar{N})$ is an invariant hypersurface of M, $j(\bar{N})$ is umbilical by $(*)_2$.

REFERENCES

[1] R. L. BISHOP-B. O' NEILL, Manifolds of negative curvature, Trans. Amer. Math. Soc., 145 (1969). 1-50.

[2] D. E. BLAIR, The theory of quasi-Sasakian structure, J. D. Geo., 1 (1967), 331-345.

[3] S. S. EUM, On Kaehlerian hypersurfaces in almost contact metric spaces, Tensor, N. S., 20 (1969), 37-44.

[4] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry, volume 1, 1963, Interscience Publ..

[5] K. NOMIZU AND B. SMYTH, The differential geometry of complex hypersurfaces II, J. Math. Soc. Japan, 20 (1968), 498-521.

[6] K. OGIUE, On almost contact manifolds admitting axiom of planes or axiom of free mobility, Kōdai Math. Sem. Rep., 16 (1964), 223-232.

[7] M. OKUMURA, Some remarks on space with a certain contact structure, Tôhoku Math. J., 14 (1962), 135-145.

[8] S. TANNO, Almost complex structures in bundle spaces over almost contact manifolds, J. Math. Soc. Japan, 17 (1965), 167-186.

[9] S. TANNO, The automorphism groups of almost contact Riemannian manifolds, Tôhoku Math. J., 21 (1969), 21-38.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN