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IMAGINARY ABELIAN NUMBER FIELDS WITH CLASS NUMBER ONE

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We have shown that there exist only a finite number of imaginary abelian number fields with relative class number $h_1 = 1$ [12 or 13]. There an upper bound of the conductors of such fields could be effectively determined, except for biquadratic fields of type (2, 2). Now Baker's and Stark's papers [3 and 10] show that an upper bound can be effectively determined also for those fields, because biquadratic fields of type (2, 2) with $h_1 = 1$ are generated by imaginary quadratic fields with $h_1 = 1$ or 2. So it is a problem of finite amount of calculation to determine all the imaginary abelian number fields with $h_1 = 1$. But an upper bound we can now obtain is too large to solve this problem explicitly. In this paper, we restrict ourselves to the class number (not the relative class number) one problem, and we give some remarks and upper bounds for some cases.

1. In this section we give some remarks which will be useful for the class number one problem. They are not essentially new results, but it will be convenient to remark here.

We define a field of type I to be an imaginary abelian number field which is generated by subfields of prime power conductors. When we write as $K = K_1K_2 \cdots K_r$ for a field of type I, we always mean that K_i are subfields of prime power conductors which are relatively prime. First proposition which is a corollary of genus theory shows that an imaginary abelian number field with class number one is of type I.

PROPOSITION 1. *Let K be an abelian number field of finite degree. Let $k = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r}$ be its conductor. If K has strict class number one, K is generated by subfields K_i whose conductors are $p_i^{e_i}$. Every K_i also has strict class number one.*

PROOF. K is contained in the field L of the k-th roots of unity. Let E_1 be the field of the $p_1^{e_1}$-th roots of unity, and let E_2 be the field of the $p_2^{e_2} \cdots p_r^{e_r}$-th roots of unity. Let T be the inertia subfield of K with respect to p_1. Then it holds $T = K \cap E_2$, and the Galois group of

K/T is isomorphic to that of KE_2/E_2. As the Galois group of L/E_2 is isomorphic to that of E_1 over the rationals, there exists a subfield K_1 of E_1 corresponding to KE_2. Then it holds $K_1E_2 = KE_2$, so p_1 is not ramified at K_1K/K. It is clear that any other finite prime is not ramified at K_1K/K. Then K_1 must be contained in K, because K has strict class number one. Hence $K = K_1T$. If K_1 or T has unramified (with respect to the finite primes) extension, K also has such an extension because the conductors of K_1 and T are relatively prime. Therefore both K_1 and T should have strict class numbers one. It is seen by induction that T is generated by subfields $K_2, \cdots, K_r$.

Relative class number formula by Hasse contains a unit index as a factor. Criteria for determining unit indices are complicated in general [4], but it is easy for the fields of type I.

PROPOSITION 2. *Let K be a totally imaginary algebraic number field of finite degree. We assume that K contains a totally real subfield K_0 such that $[K: K_0] = 2$. Let E and E_0 be unit groups of K and K_0, respectively. Let W be a group of the roots of unity in K. Then the unit index*

$$q = (E: E_0W) \leqq 2 .$$

Let G denote the Galois group of K over K_0. Then it is necessary and sufficient for $q = 1$ that $H^1(G, E) \neq 0$.

PROOF. First assertion was proved in [12]. If we assume cohomology theory of finite groups, we have a following easier proof. E/W is a finitely generated free abelian group. $(E/W)^G$ is a subgroup of finite index because it is known that $q < \infty$. Then it should be $(E/W)^G = E/W$. As the sequence

$$1 \longrightarrow W \longrightarrow E \longrightarrow E/W \longrightarrow 1$$

is exact, and as $H^1(G, E/W) = 0$, the sequence

$$(1) \qquad E_0 \longrightarrow E/W \longrightarrow H^1(G, W) \longrightarrow H^1(G, E) \longrightarrow 0$$

is exact. It is easy to show the order of $H^1(G, W)$ is equal to 2. Then $q = (E: E_0W) \leqq 2$ is clear. It is also clear that $q = 1$ if and only if $H^1(G, E) \neq 0$. For the proof of next proposition, we note that $H^1(G, E)$ is of order $\leqq 2$.

PROPOSITION 3. *Let $K = K_1K_2 \cdots K_r$ be a field of type I. Then unit index $q = 1$ if and only if only one of K_i is imaginary.*

PROOF. Let K_1 be imaginary and let $K_2, \cdots, K_r$ be real. Let $p_1^{e_1}$ be

the conductor of K_1. As K_1 is contained in the field of the $p_1^{e_1}$-th roots of unity, the prime divisor $\mathfrak{p}$ of p_1 in K_1 is a principal ideal. We put $\mathfrak{p} = (\pi)$. Let $K^\times$ be the multiplicative group of K, and let P be a group of the principal ideals of K. We obtain an exact sequence

$$K_0^\times \longrightarrow P^G \longrightarrow H^1(G, E) \longrightarrow 0 , \tag{2}$$

from the exact sequence

$$1 \longrightarrow E \longrightarrow K^\times \longrightarrow P \longrightarrow 1 .$$

Now $\mathfrak{p} = (\pi)$ is contained in P^G but is not generated by any element of K_0, as is shown by considering the ramification index of p_1. Hence $H^1(G, E) \neq 0$, i.e., $q = 1$ by Proposition 2. Next we assume that K is a field of the m-th roots of unity, where m is odd or divisible by 4 and is divisible by at least two different primes. Then $\varepsilon = 1 - \zeta$ is a unit in K, where ζ is a primitive m-th root of unity. Let τ be the complex conjugate mapping. Then $\varepsilon^{1-\tau} = -\zeta$ holds. If $q = 1$, ε is equal to some $\varepsilon_0\zeta^j$, $\varepsilon_0 \in E_0$. Then $\varepsilon^{1-\tau} = \zeta^{2j}$ and $\zeta^{2j-1} = -1$ hold. This means $2 \mid m$ and $4 \nmid m$, which is a contradiction. Therefore $q = 2$ in this case. Now let K_1 and K_2 be imaginary. We can assume imaginary K_i, $i \geqq 3$, have odd conductors. Then the index $(W: W')$ is odd, where W' is the group of the roots of unity in the field K_1K_2. If we show $q = 2$ for the field K_1K_2, the same result for K will be obtained by considering exact sequences (1) for K and K_1K_2. So we can assume that $K = K_1K_2$. Let $p_1^{e_1}$ and $p_2^{e_2}$ be conductors of K_1 and K_2, respectively. Let L_1 and L_2 be fields of the $p_1^{e_1}$-th and $p_2^{e_2}$-th roots of unity, respectively. We can assume p_2 is odd. Then the degree $[L_2: K_2]$ is odd. If we consider the exact sequence (2) for the field K_1L_2 and if we take the norm with respect to K_1L_2/K, it will be seen that $q = 2$ holds if it holds for K_1L_2. So we can assume $K_2 = L_2$. If p_1 is also odd, above arguments show that $q = 2$. Finally we consider the case $p_1 = 2$ and $K = K_1L_2$. If $K_1 = L_1$, $q = 2$ as above. If $K_1 \neq L_1$, K_1 does not contain a primitive 4-th root of unity. Also hold that $e_1 > 2$ and $[L_1: K_1] = 2$. Let ζ be a primitive 2^{e_1}-th root of unity. Then the conjugate of ζ over K_1 must be equal to $-\zeta^{-1}$. If we put ξ as a primitive $p_2^{e_2}$-th root of unity,

$$\varepsilon = (1 - \zeta\xi)(1 + \zeta^{-1}\xi)$$

is a unit in K. Then $\varepsilon^{1-\tau} = -\xi^2$. If $q = 1$, $\varepsilon = \varepsilon_0 w$ for some $\varepsilon_0 \in E_0$ and $w \in W$. Then $\varepsilon^{1-\tau} = w^2$ and so $(\xi/w)^2 = -1$ hold. It is a contradiction, for K does not contain any primitive 4-th root of unity. Therefore it must be $q = 2$.

2. We now determine the imaginary abelian number fields with class number one and with Galois groups of type $(2, 2, \cdots, 2)$, with possibly one exception.

LEMMA 1. *Let $K = K_1K_2K_3$ be a field of type I. We assume that K_1 is an imaginary cyclic field of degree 2^m for some integer m, and also assume that K_2 and K_3 are real quadratic fields. Then the class number of K is a multiple of* 2.

PROOF. Let k_1, k_2 and k_3 be conductors of K_1, K_2 and K_3, respectively, Then K contains imaginary cyclic fields E_1 and E_2 of degree 2^m with conductors k_1k_2 and k_1k_3. Both of them have relative class numbers which are multiple of 2, because K_1K_2 and K_1K_3 are their unramified extensions and their maximal real subfields have prime power conductors k_1. Similarly K contains an imaginary cyclic subfield E_3 of degree 2^m with a conductor $k_1k_2k_3$ whose relative class number is a multiple of 4. Hasse's class number formula shows

$$h_{1,K} = 2^{-3} \prod_{i=1}^{3} h_{1,E_i} \cdot h_{1,K_1} ,$$

because unit indices are one for K or for any subfield and

$$w_K = w_{K_1}, w_{E_i} = 2 .$$

Here w_K is the number of the roots of unity in K, and similarly for other fields. Then above argument shows that $h_{1,K}$ is a multiple of 2.

If an imaginary abelian number field L contains a subfield K of type I, the class number of L is a multiple of that of K, as is seen easily from [2, Chap. 8, Th. 9] or [1].

PROPOSITION 4. *Let K be a field of type I of degree 2^m for some integer m. If the Galois group of K over the rationals Q is a direct product of at least four cyclic subgroups, the class number of K is a multiple of* 2.

PROOF. Lemma 1 and the above remark show that it suffices to consider the case K contains no subfield as in Lemma 1. Then we can assume $K = K_1K_2K_3$ or $K = K_1K_2K_3K_4$, by taking a suitable subfield if necessary. In the first case, $K_1 = Q(\sqrt{-1}, \sqrt{2})$, and K_2 and K_3 are imaginary quadratic fields. In the second case, K_i are all quadratic fields and at least three of them are imaginary. As in the proof of Lemma 1, $h_{1,K}$ is a multiple of 2 in both cases.

PROPOSITION 5. *The imaginary abelian fields of type $(2, 2, \cdots, 2)$ with class number one are as follows: We mean (a, b, c) a field $Q(\sqrt{a}$,*

$\sqrt{b}, \sqrt{c}$) *in the below.*

(i) (-1), (-2), (-3), (-7), (-11), (-19), (-43), (-67), (-163).

(ii) $(-1, 5)$, $(-1, 13)$, $(-1, 37)$, $(-2, 5)$, $(-2, 29)$, $(-3, 2)$, $(-3, 5)$, $(-3, 17)$, $(-3, 41)$, $(-3, 89)$, $(-7, 5)$, $(-7, 13)$, $(-7, 61)$, $(-11, 2)$, $(-11, 17)$, *with possibly one more field.*

(iii) $(-1, -2)$, $(-1, -3)$, $(-1, -7)$, $(-1, -11)$, $(-1, -19)$, $(-1, -43)$, $(-1, -67)$, $(-1, -163)$, $(-2, -3)$, $(-2, -7)$, $(-2, -11)$, $(-2, -19)$, $(-2, -43)$, $(-2, -67)$, $(-3, -7)$, $(-3, -11)$, $(-3, -19)$, $(-3, -43)$, $(-3, -67)$, $(-3, -163)$, $(-7, -11)$, $(-7, -19)$, $(-7, -43)$, $(-7, -163)$, $(-11, -19)$, $(-11, -67)$, $(-11, -163)$, $(-19, -67)$, $(-19, -163)$, $(-43, -67)$, $(-43, -163)$, $(-67, -163)$.

(iv) $(-1, -2, 5)$, $(-1, -3, 5)$, $(-1, -7, 5)$, $(-1, -7, 13)$, $(-2, -3, 5)$, $(-2, -7, 5)$, $(-3, -7, 5)$, $(-3, -11, 2)$, $(-3, -11, 17)$.

(v) $(-1, -2, -3)$, $(-1, -2, -11)$, $(-1, -3, -7)$, $(-1, -3, -11)$, $(-1, -3, -19)$, $(-1, -7, -19)$, $(-2, -3, -7)$, $(-3, -11, -19)$.

PROOF. Proposition 4 shows there does not exist such a field whose Galois group has more than 3 generators.

(i) Imaginary quadratic case is well known.

(ii) $Q(\sqrt{-1}, \sqrt{2})$ is included in (iii). Except this, a field of type I of type $(2, 2)$ is of type $Q(\sqrt{-p}, \sqrt{q})$ or $Q(\sqrt{-p}, \sqrt{-q})$, where p and q are different primes (or 1). Now $Q(\sqrt{-p})$ and $Q(\sqrt{q})$ (in the second case $Q(\sqrt{-q})$) are quadratic fields with strict class number one. Moreover, a quadratic subfield $Q(\sqrt{-pq})$ must have class number 2 in the first case. Iseki [5] or Tatuzawa [11] has shown that imaginary quadratic fields with class number 2 have discriminants d greater than $-90{,}000$ with possibly one exception. It is known $d \geqq -427$ if $d > -90{,}000$.

(iii) In the second case, the class number of $Q(\sqrt{pq})$ is equal to one. Y. Yamamoto calculated class numbers of such fields and obtained this case.

(iv) When it is of type $(2, 2, 2)$, it is of type $Q(\sqrt{-p}, \sqrt{-q}, \sqrt{r})$ or of type $Q(\sqrt{-p}, \sqrt{-q}, \sqrt{-r})$. Here p, q and r are primes (or 1), and a field $Q(\sqrt{-1}, \sqrt{2}, \sqrt{r})$ is counted as $Q(\sqrt{-1}, \sqrt{-2}, \sqrt{r})$. In the former case subfields $Q(\sqrt{-p}, \sqrt{-q})$, $Q(\sqrt{-p}, \sqrt{r})$ and $Q(\sqrt{-q}, \sqrt{r})$ must be in (ii) or (iii). And quadratic subfield $Q(\sqrt{pqr})$ must have class number 2.

(v) In the latter case, subfields $Q(\sqrt{-p}, \sqrt{-q})$, $Q(\sqrt{-p}, \sqrt{-r})$ and $Q(\sqrt{-q}, \sqrt{-r})$ are all in (iii). And quadratic subfield $Q(\sqrt{-pqr})$ must have class number 2 or 4 according as $Q(\sqrt{-p}, \sqrt{-q}, \sqrt{-r})$ contains $Q(\sqrt{-1}, \sqrt{-2})$ or not.

3. We now give an upper bound of conductors of imaginary abelian number fields with class number one, assuming that they do not contain the exceptional field in Proposition 5. Let K be a field of type I, and let k be its conductor. Let $L(s, \chi)$ be an L-function corresponding to K. Let $L_1(s)$ be the product of $L(s, \chi)$ such that $\chi(-1) = -1$. Let $L_2(s)$ be the product of $L(s, \chi)$ for non-trivial χ such that $\chi(-1) = 1$. We now estimate $L_1(1)$. We put

$$s_0 = 1 + \frac{1}{a \log k}$$

for some $a > 0$. By Lemmas 5(1) and 8 of [13], we have

$$\begin{aligned}(3) \qquad -\log L_1(1) &= -\log L_1(s_0) + \int_1^{s_0} \frac{L_1'}{L_1}(s)ds \\ &< \log \zeta(s_0) + \log L_2(s_0) + \int_1^{s_0} \frac{L_1'}{L_1}(s)ds \\ &< \frac{1}{a \log k} + \log a + \log \log k \\ &\quad + \log L_2(s_0) + \int_1^{s_0} \frac{L_1'}{L_1}(s)ds .\end{aligned}$$

Hence it suffices to estimate $L_2(s_0)$ and $(L_1'/L_1)(s)$, choosing a suitable value of a.

LEMMA 2. (i) (*Tatuzawa* [11]) *We put* $S_{m,n} = \sum \chi(r)$, *where the sum is taken for all integers* r *such that* $m \leqq r \leqq n$. *If* c *is an integer such that* $|S_{m,n}| \leqq c$ *for any* m *and* n, *it follows*

$$|L(s, \chi)| \leqq \frac{|s|}{\sigma} \sum_{n=1}^{c} n^{-\sigma} \qquad \text{for } s = \sigma + it,\ \sigma > 0 .$$

(ii) *If* $k \geqq 50{,}000$, *we can take* $c \leqq 0.58\sqrt{k} \log k + 1$. *Then it follows*

$$|L(s, \chi)| < 1.41\, k^{1/3}(\log k)^{2/3} \qquad \text{for } |s - s_0| \leqq 2/3$$

for any non-trivial χ.

PROOF. (i) Tatuzawa states this in his Lemma 5 for real χ, but his proof is valid in general.

(ii) Polya's inequality [9 or 6] show that

$$\begin{aligned}|S_{m,n}| &< 1 + \frac{2}{\pi}\sqrt{k}\left(\frac{1}{2}\log k + \log\log k + 1\right) + \frac{2}{\pi}\frac{k \log k}{\sqrt{k} \log k - 1} \\ &< 0.58\sqrt{k} \log k ,\end{aligned}$$

by making use of inequalities $\log k > 10.8$ and $\log\log k < 0.221 \log k$ for $k \geqq 50{,}000$. Then we have

$$| L(s, \chi) | \leqq \frac{3}{\sqrt{5}} \sum_{n=1}^{c} n^{-1/3} < 1.41\, k^{1/3} (\log k)^{2/3}$$

for any s such that $| s - s_0 | \leqq 2/3$.

LEMMA 3. (i) *Let s and s_1 be real numbers such that $1 < s < s_1$. Let ρ be a zero point of $L(s, \chi)$. We put $C = \Re\,(1/(s_1 - \rho))$. Then it holds*

$$\Re \frac{1}{s - \rho} \leqq \frac{C}{C(s - s_1) + 1} .$$

(ii) *We assume that $a \log k > 237$ and that $| L_1(z)/L_1(s_0) | \leqq e^M$ for some constant $M > 0$ in the circle $| z - s_0 | \leqq 2/3$. If we put*

$$s = 1 + \frac{x}{a \log k} , \qquad 0 \leqq x \leqq 1 ,$$

we have

$$\frac{L_1'}{L_1}(s) < \sum_{\rho} \Re \frac{1}{s - \rho} + 6.24M ,$$

where ρ runs over the zeros of L_1 such that $| \rho - s_0 | \leqq 1/3$.

PROOF. (i) See [13, p. 342].

(ii) See the proof of [13, Lemma 4]. Only change occurs on the coefficient of M by a change of the condition on the value of $a \log k$.

From now on we assume that $k \geqq 50{,}000$. We make use of inequalities $\log k > 10.8$ and $\log\log k < 0.221 \log k$ frequently. First we assume that K is cyclic of degree 4. Then $L_2(s)$ consists of only $L(s, \chi)$ such that $\chi^2 = 1$. As K is of type I, k is a prime number such that $k \equiv 5 \pmod 8$, hence $\chi(2) = -1$ holds. For any real $s > 1$, it follows

$$\begin{aligned} (4) \qquad L_2(s) &= \left(1 + \frac{1}{2^s}\right)^{-1} \prod_{p \neq 2} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \\ &\leqq \frac{2}{3} s \sum_{n=1}^{\infty} \frac{\chi(2n-1)}{(2n-1)^s} < \frac{2}{3} s \sum_{n=1}^{c} \frac{1}{(2n-1)^s} \\ &\leqq \frac{2}{3} s \left(\frac{1}{2} \log c + 1.35\right) \leqq 0.308\, s \log k \end{aligned}$$

by similar method as in the proof of Lemma 2 (i), because

$$\left| \sum_{r=1}^{m} \chi(2r-1) \right| = \left| \sum_{r=1}^{m} \chi(k - 2r + 1) \right| = | \chi(2) \sum \chi(r) | < c ,$$

where the last sum is taken over integers between $(k+1-2m)/2$ and $(k-1)/2$. We now estimate M in Lemma 3. By Lemmas 2 and [13, Lemma 5], we have

$$\begin{aligned} M &< 2\log 1.41 + \frac{2}{3}\log k + \frac{4}{3}\log\log k + \frac{2}{a\log k} \\ &\quad + \log a + 2\log\log k + \log 0.308 \\ &< 1.404\log k - 0.49 + \log a + \frac{2}{a\log k}\,. \end{aligned}$$

Metsänkylä [8] or rather his argument shows

$$\sum_{\rho} \Re\frac{1}{s_1-\rho} < \frac{1}{s_1-1} + \frac{3}{2}\log k$$

for any real s_1 such that $2 > s_1 > 1$, where ρ runs over the zeros of $L_1(s)$ such that $|\rho - s_0| \leqq 1/3$. If we put $s_1 = 1 + 1/(4\log k)$, we have

$$\sum \Re\frac{1}{s_1-\rho} < 5.5\log k$$

and also

$$\Re\frac{1}{s_1-\rho} < 2.75\log k$$

because $\bar{\rho}$ is a zero of $L(s,\bar{\chi})$ if ρ is a zero of $L(s,\chi)$. If we put $s = 1 + x/(a\log k)$, $0 \leqq x \leqq 1$, Lemma 3 shows

$$\begin{aligned} \sum \Re\frac{1}{s-\rho} &\leqq \frac{1}{2.75\log k\,(s-s_1)+1}\sum \Re\frac{1}{s_1-\rho} \\ &< \frac{88a\log k}{5a+44x}\,. \end{aligned}$$

If we put $a = 22$,

$$M < 1.646\log k$$

and

$$\frac{L_1'}{L_1}(s) < \frac{88\log k}{5+2x} + 10.3\log k\,, \qquad s = 1 + \frac{x}{22\log k}$$

hold by Lemma 3. Then we have

$$\int_1^{s_0} \frac{L_1'}{L_1}(s)ds < 2\int_0^1 \frac{dx}{x+2.5} + \frac{10.3}{22} < 1.142\,.$$

Hence we get from (3) and (4)

$$-\log L_1(1) < 2\log\log k + \log 21.5\ .$$

Now we have

$$h_1 = \frac{k}{2\pi^2} L_1(1) > \frac{k}{2\times 9.87\times 21.5(\log k)^2}\ .$$

The right hand side is greater than one for $k \geqq 50{,}000$. Therefore

PROPOSITION 6. *Let K be an imaginary cyclic field of degree 4. Its conductor k is less than 50,000, if it has class number one.*

If K contains a subfield of type I of type $(2, 2, \cdots, 2)$, we also assume that this subfield is one of the fields in Proposition 5. It K has class number one, this condition is satisfied. Any field in Proposition 5 has conductor smaller than 593,000, if it is not an exceptional one. So corresponding L-functions $L(s, \chi)$ such that $\chi(-1) = -1$ have no real zeros such that $0 < \rho < 1$ [7]. Hence if ρ is a zero point of $L_1(s)$ such that $0 < \Re\,\rho < 1$, $\bar{\rho}$ is also a zero of $L_1(s)$. And if ρ is real, ρ is a multiple zero of $L_1(s)$. When K is of degree 6, K contains an imaginary quadratic subfield $Q(\sqrt{-m})$ with class number one. As we assume $k \geqq 50{,}000$, it holds $k = mp$ where p is a prime number such that $p \equiv 1 \pmod 6$. $L_1(s)$ consists of two L-functions defined mod k and L_m mod m corresponding to $Q(\sqrt{-m})$. $L_2(s)$ consists of two L-functions defined mod p. We now assume that $p > 50{,}000$. Then it follows from Lemma 2 that

$$\begin{aligned} L_2(s) &< (\log(0.58\sqrt{p}\log p + 1) + 1)^2 \\ &< 0.583\,(\log p)^2 \end{aligned}$$

for any real $s > 1$. As we can always take $c \leqq k/2$ in Lemma 2, it follows

$$|L_m(s, \chi)| \leqq 1.27\, m^{2/3}$$

for any s such that $|s - s_0| \leqq 2/3$. Then we have

$$M < 1.98\log k + \frac{1}{a\log k} + \log a\ ,$$

by considering $m \leqq 163$ and $p < k$. Now Metsänkylä [8] and Lemma 5 (2) of [13] give

$$\begin{aligned} \sum_{\rho} \Re\frac{1}{s-\rho} &< \log k + \frac{1}{2}\log m + \log p + \frac{1}{s-1} \\ &< 2\log k + \frac{1}{s-1} \end{aligned}$$

for real s such $1 < s < 2$. If we put $s_1 = 1 + 1/(5\log k)$,

$$\sum \Re \frac{1}{s_1 - \rho} < 7 \log k$$

and

$$\Re \frac{1}{s_1 - \rho} < 3.5 \log k$$

hold. The second inequality comes from the first inequality and the fact that L_m has no real zero such that $0 < \rho < 1$. If we put $a = 35$ and $s = 1 + x/(a \log k)$, same arguments as in the case of degree 4 show that

$$\int_1^{s_0} \frac{L_1'}{L_1}(s)ds < 0.988$$

and

$$-\log L_1(1) < \log 55.1 + \log \log k + 2 \log \log p .$$

Now we have

$$h_1 > \frac{w\sqrt{m}}{2^3\pi^3} \frac{k}{55.1 \log k (\log p)^2}$$

where w is the number of the roots of unity in K. As

$$\frac{w\sqrt{m}}{(\log p)^2} = \frac{w\sqrt{m}}{(\log k - \log m)^2}$$

takes the smallest value for $m = 7$, it follows

$$h_1 > \frac{k}{2591 \log k (\log k - 1.94)^2} .$$

If $k > 8.15 \times 10^6$, the right hand side is greater than one. For such k, p is certainly greater than 50,000. Hence

PROPOSITION 7. *Let K be an imaginary abelian field of degree 6. Then its conductor $k < 8.15 \times 10^6$ if K has class number one.*

We made use of the fact that $\Re(1/(s_1 - \rho)) \leqq (1/2) \sum \Re(1/(s_1 - \rho))$ in the above. If the degree n is larger than 6, it is better to apply Metsänkylä's estimate

$$\Re \frac{1}{s_1 - \rho} < \frac{47}{80} \frac{1}{s_1 - 1} + \frac{76.5}{80} \log k$$

and

$$\sum \Re \frac{1}{s_1 - \rho} < \frac{1}{s_1 - 1} + \frac{n-1}{2} \log k$$

which comes from his arguments and Lemma 5(2) of [13]. Now assume that $n \geqq 8$, and we put

$$s_1 = 1 + \frac{2}{(n+1) \log k} .$$

Then we have

$$\Re \frac{1}{s_1 - \rho} < \frac{1}{160} (47n + 200) \log k$$

and

$$\sum \Re \frac{1}{s_1 - \rho} < n \log k .$$

Hence Lemma 3 shows

$$\sum \Re \frac{1}{s - p} \leqq \frac{160na \log k}{(47n + 200)x + 66a - 306a/(n+1)}$$
$$\leqq \frac{160na \log k}{(47n + 200)x + 32a} , \qquad s = 1 + \frac{x}{a \log k}$$

for $a \geqq (n+1)/2$. If we put $a = 5n$,

$$M < 0.401n \log k$$

and

$$\int_1^{s_0} \sum \Re \frac{1}{s - \rho} ds < \frac{160n}{47n + 200} \log\left(\frac{207}{160} + \frac{1.25}{n}\right)$$
$$< \frac{160}{47} \log \frac{207}{160} < 0.878$$

hold. Hence we obtain

$$\int_1^{s_0} \frac{L_1'}{L_1}(s) ds < 1.379 .$$

Lemma 2 shows that

$$| L(s_0, \chi) | \leqq 0.764 \log k$$

for any non-trivial χ. Then it follows from (3) that

$$L_1(1)^{-1} < 26.1n(0.764 \log k)^{n/2} .$$

Hence

$$h_1 \geqq \frac{2}{(2\pi)^{n/2}} \sqrt{d/d_0} \frac{1}{26.1n(0.764 \log k)^{n/2}}$$
$$\geqq \frac{\sqrt{d/d_0}}{13.1n(4.801 \log k)^{n/2}}$$

where d and d_0 are absolute values of discriminants of K and of maximal real subfield, respectively. As $\sqrt{d/d_0} > d^{1/4}$, Lemma 1 of [13] shows

$$h_1 \geqq \frac{1}{13.1n} \left(\frac{\sqrt[4]{k}}{4.801 \log k} \right)^{n/2} \geqq \left(\frac{\sqrt[4]{k}}{15.37 \log k} \right)^{n/2}.$$

If we put $k \geqq 2 \times 10^{10}$, the right hand side is greater than one. Therefore

PROPOSITION 8. *Let K be a field of type I. If K has class number one, and if it does not contain the exceptional field in Proposition* 5, *its conductor is less than* 2×10^{10}.

REMARK. We will be able to obtain a better upper bound, if we consider some types of fields separately and give better estimate of $\sqrt{d/d_0}$. For example, if k is a power of a prime, $\sqrt{d/d_0} \geqq k^{n/4}$. Then $h_1 > 1$ if $k > 50{,}000$. To obtain better upper bound in this case, we have to start with another estimate of Lemma 2.

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