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COMPLEX HYPERSURFACES WITH $RS = 0$ IN C^{n+1}

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Recently P. J. Ryan [2] studied complex hypersurfaces in a complex space form satisfying the condition

$$(*) \qquad R(X, Y)S = 0$$

for any tangent vectors X and Y of the hypersurface, where R is the curvature tensor, S is the Ricci tensor of the hypersurface and $R(X, Y)$ operates on the tensor algebra as a derivation. He proved that these hypersurfaces are Einstein manifolds if the holomorphic sectional curvature of the ambient space does not vanish (Theorem 4).

In the case where the ambient space is a complex Euclidean space C^{n+1}, he obtained the following two results: Let M be a complex hypersurface in C^{n+1}. (1) If M satisfies the condition (*) and the scalar curvature of M is constant, then M is totally geodesic (Proposition 5). (2) If M is complete and satisfies the condition

$$({*\atop *}) \qquad R(X, Y)R = 0$$

for any tangent vectors X and Y of M, then M is cylindrical, that is, the product of C^{n-1} and a complex curve (Theorem 6).

In this paper we shall obtain the following result.

THEOREM. *A complete complex hypersurface in C^{n+1} satisfying the condition (*) is cylindrical.*

1. Hypersurfaces in C^{n+1}.

Throughout this paper it will be agreed that Greek indices have the range $1, 2, \cdots, n$.

Let M be an n dimensional complex manifold immersed holomorphically in C^{n+1}. Let $e_0, e_1, \cdots, e_n$ be a unitary frame field in C^{n+1}, defined in a neighborhood of M such that $e_0(x)$, $x \in M$, is orthogonal to the tangent space of M at x. Its coframe field $\omega^0, \omega^1, \cdots, \omega^n$ consists of complex valued linear differential forms of type (1, 0) on M such that $\omega^0 = 0$ and $\omega^1, \cdots, \omega^n, \bar{\omega}^1, \cdots, \bar{\omega}^n$ are linearly independent. The induced metric of M can be written as

(1.1) $$ds^2 = 2\sum_{\lambda=1}^{n} \omega^\lambda \bar{\omega}^\lambda ,$$

and the $e_1, \cdots, e_n$ is a unitary frame field of M with respect to this metric. The $\omega^1, \cdots, \omega^n$ is a coframe field of $e_1, \cdots, e_n$.

Associated to the frame $e_0, e_1, \cdots, e_n$, there are complex valued linear differential forms ω_B^A $(A, B = 0, 1, \cdots, n)$ such that

(1.2) $$\omega_B^A + \bar{\omega}_A^B = 0 , \qquad (A, B = 0, 1, \cdots, n) ,$$

(1.3) $$d\omega^A + \sum_{B=0}^{n} \omega_B^A \wedge \omega^B = 0 \qquad (A = 0, 1, \cdots, n) ,$$

(1.4) $$d\omega_B^A + \sum_{C=0}^{n} \omega_C^A \wedge \omega_B^C = 0 \qquad (A, B = 0, 1, \cdots, n) .$$

Since $\omega^0 = 0$, (1.3) becomes

(1.5) $$d\omega^\kappa + \sum_{\lambda=1}^{n} \omega_\lambda^\kappa \wedge \omega^\lambda = 0$$

and

$$\sum_{\lambda=1}^{n} \omega_\lambda^0 \wedge \omega^\lambda = 0 .$$

It follows by Cartan's lemma that

(1.6) $$\omega_\lambda^0 = \sum_\mu H_{\lambda\mu}\omega^\mu , \qquad H_{\lambda\mu} = H_{\mu\lambda} .$$

Then from (1.2) we have

(1.7) $$\omega_0^\lambda = -\sum_\mu \bar{H}_{\lambda\mu}\bar{\omega}^\mu .$$

The ω_μ^λ are the connection forms of M associated to the frame $e_1, \cdots, e_n$ and the covariant differential of e_μ is given by

(1.8) $$De_\mu = \sum_\lambda \omega_\mu^\lambda e_\lambda .$$

The curvature forms Ω_λ^κ are defined by

$$\Omega_\lambda^\kappa = d\omega_\lambda^\kappa + \sum_\alpha \omega_\alpha^\kappa \wedge \omega_\lambda^\alpha .$$

Then from (1.4), (1.6), and (1.7) we have

(1.9) $$\Omega_\lambda^\kappa = \sum_{\nu,\mu} \bar{H}_{\kappa\nu} H_{\lambda\mu} \bar{\omega}^\nu \wedge \omega^\mu .$$

We take the exterior derivative of (1.6) and make use of (1.4) and (1.5). Then we have

$$\sum_\mu (dH_{\lambda\mu} - \sum_\alpha (H_{\alpha\mu}\omega_\lambda^\alpha + H_{\lambda\alpha}\omega_\mu^\alpha) + H_{\lambda\mu}\omega_0^0) \wedge \omega^\mu = 0 .$$

It follows that

(1.10) $$dH_{\lambda\mu} - \sum_{\alpha} (H_{\alpha\mu}\omega^{\alpha}_{\lambda} + H_{\lambda\alpha}\omega^{\alpha}_{\mu}) + H_{\lambda\mu}\omega^{0}_{0} = \sum_{\nu} H_{\lambda\mu\nu}\omega^{\nu} ,$$

where $H_{\lambda\mu\nu}$ are symmetric in all indices.

Using (1.4), (1.6), and (1.7) we get

$$d\omega^{0}_{0} = \sum_{\alpha} H_{\lambda\alpha}\bar{H}_{\alpha\mu}\omega^{\lambda} \wedge \bar{\omega}^{\mu} .$$

In our frame field the Ricci tensor S of M can be expressed by

$$S = \sum_{\lambda,\mu} (S_{\lambda\bar{\mu}}\omega^{\lambda} \otimes \bar{\omega}^{\mu} + S_{\bar{\lambda}\mu}\bar{\omega}^{\lambda} \otimes \omega^{\mu})$$

where $S_{\lambda\bar{\mu}} = S_{\bar{\mu}\lambda} = \bar{S}_{\bar{\lambda}\mu}$ which are given by

(1.12) $$S_{\lambda\bar{\mu}} = -\sum_{\alpha} H_{\lambda\alpha}\bar{H}_{\alpha\mu} .$$

In our notations the condition (*) is

$$\sum_{\alpha} (S_{\alpha\bar{\mu}}\Omega^{\alpha}_{\lambda} + S_{\lambda\bar{\alpha}}\bar{\Omega}^{\alpha}_{\mu}) = 0 .$$

Substituting (1.9) and (1.12) into the above equation, we have an expression of the condition (*) as follows:

(*′) $$H_{\kappa\lambda}\sum_{\alpha,\beta} \bar{H}_{\mu\alpha}H_{\alpha\beta}\bar{H}_{\beta\nu} = \sum_{\alpha,\beta} H_{\kappa\alpha}\bar{H}_{\alpha\beta}H_{\beta\lambda}\bar{H}_{\mu\nu} .$$

The scalar curvature k of M is given by

(1.13) $$k = -2\sum_{\lambda,\mu} |H_{\lambda\mu}|^2 ,$$

and k is a real analytic function on M.

Let $e'_0, e'_1, \cdots, e'_n$ be another frame field such that e'_0 is orthogonal to the tangent space of M. Then we have

(1.14) $$e'_0 = U^0_0 e_0 \quad \text{and} \quad e'_\mu = \sum_{\lambda} U^{\lambda}_{\mu} e_{\lambda} ,$$

where U^0_0 is a complex valued function with $|U^0_0| = 1$ and the matrix (U^{λ}_{μ}) is a unitary matrix. Let ω'^{λ}, ω'^{A}_{B} be the differential forms with respect to the frame field $e'_0, e'_1, \cdots, e'_n$. Then we have

(1.15) $$\omega^{\lambda} = \sum_{\mu} U^{\lambda}_{\mu}\omega'^{\mu} ,$$

(1.16) $$\sum_{\alpha} \omega'^{\alpha}_{\mu} U^{\lambda}_{\alpha} = dU^{\lambda}_{\mu} + \sum_{\alpha} U^{\alpha}_{\mu}\omega^{\lambda}_{\alpha} ,$$

(1.17) $$H'_{\lambda\mu} = \bar{U}^0_0 \sum_{\alpha,\beta} U^{\alpha}_{\lambda} U^{\beta}_{\mu} H_{\alpha\beta} ,$$

where $\omega'^{0}_{\lambda} = \sum_{\mu} H'_{\lambda\mu}\omega'^{\mu}$.

2. Proof of the theorem. In this section M is an n dimensional connected complex manifold immersed holomorphically in C^{n+1}. We assume that M is complete with respect to the induced metric and satisfies the condition (*).

From the last formula (1.17) in § 1, it is easily seen that the rank of the matrix $(H_{\lambda\mu}(x))$, $x \in M$, is independent of the choice of the frame field. We shall denote it by $p(x)$.

To prove the theorem it suffices to show that $p(x)$ is smaller than 2 everywhere. In fact, if $p(x) \leqq 1$ everywhere, we see easily that M satisfies the condition ($\overset{*}{*}$). Then we can apply Ryan's result to our situation and we can conclude the theorem.

In the rest of this section we assume that there is a point $x_0 \in M$ such that $p(x_0) \geqq 2$ and we shall induce a contradiction.

It is clear that $p(x) \geqq 2$ at a point x in a neighborhood of x_0. Take a unitary frame field $e_0, e_1, \cdots, e_n$ as in § 1. Then in our assumption $H_{\lambda\mu}$ are satisfying (*′). Let $U = (U^{\lambda}_{\mu})$ be a unitary matrix and put

$$H'_{\lambda\mu} = \sum_{\alpha,\beta} U^{\alpha}_{\lambda} U^{\beta}_{\mu} H_{\alpha\beta} .$$

Then $H'_{\lambda\mu}$ also satisfy (*′). By a slight modification of Chern's lemma ([1], page 28) we can choose U so that

$$H'_{\lambda\lambda} = a_\lambda \geqq 0 \quad \text{and} \quad H'_{\lambda\mu} = 0 \ (\lambda \neq \mu)$$

at a point x in a neighborhood of x_0. It follows by (*′) that

$$a_\lambda a^3_\mu = a^3_\lambda a_\mu .$$

Thus we have $a_\lambda = a_\mu$, if $a_\lambda a_\mu \neq 0$ and $\lambda \neq \mu$. Therefore, $p(x)$ is constant in a neighborhood of x_0.

Let $m = p(x_0)$. We can take a frame field $e_0, e_1, \cdots, e_n$ in a neighborhood W of x_0 such that the matrix $(H_{\lambda\mu})$ is diagonal and

$$H_{11} = \cdots = H_{mm} > 0 \quad \text{and} \quad H_{m+1\,m+1} = \cdots = H_{nn} = 0 .$$

Since the scalar curvature k is non-positive on M, a continuous function c on M is defined by

$$c = \sqrt{-k/2m} .$$

Let $M' = \{x \in M; k(x) \neq 0\}$. Then M' is an open subset of M and c is analytic on M'.

From (1.13) we have

$$k = -2mH^2_{11} = \cdots = -2mH^2_{mm} \neq 0$$

on the neighborhood W of x_0. Therefore, W is contained in M' and we have

$$H_{11} = \cdots = H_{mm} = c$$

on W.

In the following we agree that the indices have the following ranges:

$$1 \leqq i, j, k \leqq m \quad \text{and} \quad m+1 \leqq r, s, t \leqq n .$$

If we put $\lambda = r$ and $\mu = s$ in the formula (1.10), we have

$$H_{rs\nu} = 0 \qquad \nu = 1, \cdots, n . \tag{2.1}$$

Also if we put $\lambda = i$, $\mu = j$ and $i \neq j$ in (1.10), we have

$$H_{ii}\omega^i_j + H_{jj}\omega^j_i = -\sum_{\nu=1}^{n} H_{ij\nu}\omega^\nu ,$$

that is,

$$c(\omega^i_j + \omega^j_i) = -\sum_{\nu=1}^{n} H_{ij\nu}\omega^\nu . \tag{2.2}$$

Since c is real and $(\omega^i_j + \omega^j_i) + (\bar{\omega}^i_j + \bar{\omega}^j_i) = 0$, we get

$$\sum_{\nu=1}^{n} H_{ij\nu}\omega^\nu + \sum_{\nu=1}^{n} \bar{H}_{ij\nu}\bar{\omega}^\nu = 0 .$$

It follows that

$$H_{ij\nu} = 0 \qquad i \neq j,\ \nu = 1, \cdots, n , \tag{2.3}$$

$$\omega^i_j + \omega^j_i = 0 \qquad i \neq j . \tag{2.4}$$

If we put $\lambda = \mu = i$ in (1.10), we have

$$dc + c\omega^0_0 - 2c\omega^i_i = \sum_\nu H_{ii\nu}\omega^\nu .$$

If $i \neq j$, we know from (2.3) that $H_{iij} = H_{iji} = 0$. Thus we get

$$dc + c\omega^0_0 - 2c\omega^i_i = H_{iii}\omega^i + \sum_r H_{iir}\omega^r .$$

If we take the real part of the above equation, we have

$$2dc = H_{iii}\omega^i + \sum_r H_{iir}\omega^r + \bar{H}_{iii}\bar{\omega}^i + \sum_r \bar{H}_{iir}\bar{\omega}^r .$$

Since the left-hand side of this equation does not depend on the indices $i = 1, \cdots, m$, we get

$$H_{111} = \cdots = H_{mmm} = 0 \quad \text{and} \quad H_{11r} = \cdots = H_{mmr} .$$

Thus we obtain

$$dc + c\omega^0_0 - 2c\omega^i_i = \sum_r h_r\omega^r \qquad i = 1, \cdots, m , \tag{2.5}$$

$$2dc = \sum_r (h_r\omega^r + \bar{h}_r\bar{\omega}^r) , \tag{2.6}$$

where we put

$$h_r = H_{11r} = \cdots = H_{mmr} .$$

If we put $\lambda = i$ and $\mu = r$ in (1.10), we have

$$-c\omega_r^i = \sum_{\nu=1}^{n} H_{ir\nu}\omega^\nu .$$

The right-hand side is

$$\sum_j H_{irj}\omega^j + \sum_s H_{irs}\omega^s = H_{iri}\omega^i = H_{iir}\omega^i = h_r\omega^i .$$

Thus we get

$$c\omega_r^i = -h_r\omega^i , \tag{2.7}$$

$$c\omega_i^r = \bar{h}_r\bar{\omega}^i . \tag{2.8}$$

From (2.4) we see easily that

$$\omega_j^i \wedge \omega_i^j = 0 \qquad i \neq j . \tag{2.9}$$

Using (2.7), (2.8), (2.9), and $\Omega_i^i = c^2\bar{\omega}^i \wedge \omega^i$, we have

$$d\omega_i^i = c^{-2}\Big(c^4 - \sum_r |h_r|^2\Big)\bar{\omega}^i \wedge \omega^i .$$

Since $-2c\omega_i^i = \sum_r h_r\omega^r - dc - c\omega_0^0$, we have $\omega_1^1 = \cdots = \omega_m^m$. It follows that

$$\Big(c^4 - \sum_r |h_r|^2\Big)\bar{\omega}^1 \wedge \omega^1 = \cdots = \Big(c^4 - \sum_r |h_r|^2\Big)\bar{\omega}^m \wedge \omega^m .$$

Therefore, we obtain

$$c^4 = \sum_r |h_r|^2 . \tag{2.10}$$

We take the exterior derivative of (2.7) and make use of (1.5), (2.7), and $\Omega_r^i = 0$. Then we have for $i = 1, \cdots, m$,

$$\Big(dh_r - \sum_s h_s\omega_r^s\Big) \wedge \omega^i = c^{-1}h_r\Big(dc + \sum_s h_s\omega^s\Big) \wedge \omega^i .$$

From this we obtain

$$dh_r - \sum_s h_s\omega_r^s = c^{-1}h_r(dc + \sum_s h_s\omega^s) . \tag{2.11}$$

If $m = n$, the formula (2.10) becomes $c^4 = 0$ on W, that is, the matrix $(H_{\lambda\mu}) = 0$ on W, which contradicts our assumption. Thus we can assume that $m < n$. Then we can define a real vector field X on W by

$$X = c^{-2}\sum_r (\bar{h}_r e_r + h_r\bar{e}_r) .$$

The covariant differential of X is

$$DX = -c^{-1}dcX + c^{-3}\sum_{r,s}(\bar{h}_r\bar{h}_s\bar{\omega}^s e_r + h_r h_s \omega^s \bar{e}_r) - c\sum_j (\omega^j e_j + \bar{\omega}^j \bar{e}_j) .$$

Thus the covariant derivative of X by itself is

$$D_X X = 0 ,$$

which means that the trajectories of X are geodesics of M.

By the completeness of M, there exists a geodesic $\gamma(t)$ $(-\infty < t < \infty)$ and $\varepsilon > 0$ such that

$$\gamma(0) = x_0, \quad \gamma(t) \in W \quad \text{and} \quad \gamma'(t) = X_{\gamma(t)} \quad \text{for} \quad |t| < \varepsilon .$$

Since M' is open in M, there exists an open interval I of real numbers such that $\gamma(t) \in M'$ for $t \in I$. We take a maximal interval with this property.

From (2.6) and (2.10) we have

$$dc(X) = c^2$$

which implies that c satisfies the differential equation

$$\frac{dc\circ\gamma}{dt} = (c\circ\gamma)^2 \tag{2.12}$$

along the geodesic γ within an interval $-\varepsilon < t < \varepsilon$. But c is analytic on M' and γ is also analytic, (2.12) is also satisfied for $t \in I$.

Then we have

$$c(\gamma(t)) = c_0/(1 - c_0 t) \quad \text{for} \quad t \in I , \tag{2.13}$$

where $c_0 = c(x_0)$.

From (2.13) we see that $1/c_0$ is not contained in I. So I is upper bounded. Let t_0 be the right limit of I. Then $0 < t_0 < 1/c_0$. Since (2.13) is satisfied for t, $0 < t < t_0$, we have

$$\lim_{t\to t_0} c(\gamma(t)) = c_0/(1 - c_0 t_0) \neq 0 .$$

On the other hand, $c(\gamma(t))$ is defined for all real numbers and continuous. Since $\gamma(t_0) \notin M'$, $c(\gamma(t_0)) = 0$. Thus we have

$$\lim_{t\to t_0} c(\gamma(t)) = c(\gamma(t_0)) = 0 .$$

This is a contradiction.

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