Tôhoku Math. Journ.
28 (1976), 135-143.

ADJOINT EQUATIONS OF AUTONOMOUS LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE RETARDATIONS

TOSHIKI NAITO

(Received August 25, 1975)

1. Introduction. Let $\rho \geqq r \geqq 0$, $p \geqq 1$ be given real numbers (ρ may be $+\infty$) and $g(\theta)$ be Lebesgue integrable, positive and nondecreasing on $[-\rho, 0]$, where $[-\rho, 0]$ denotes $(-\infty, 0]$ when $\rho = +\infty$. Let $\mathscr{B} = \mathscr{B}([-\rho, 0], \boldsymbol{C}^d)$ be the Banach space of functions ϕ mapping $[-\rho, 0]$ into $\boldsymbol{C}^d$, the complex d-dimensional column vector space, which are Lebesgue measurable on $[-\rho, 0]$, are continuous on $[-r, 0]$ and have the property such that

$$\|\phi\| = \left[\sup_{-r \leqq \theta \leqq 0} |\phi(\theta)|^p + \int_{-\rho}^{0} |\phi(\theta)|^p g(\theta) d\theta\right]^{1/p} < \infty ,$$

where $|v|$ denotes a norm of v in $\boldsymbol{C}^d$. We shall discuss the adjoint equation of a linear functional differential equation

$$\frac{dx}{dt} = f(x_t) , \tag{1.1}$$

where f is a bounded linear operator on $\mathscr{B}$ into $\boldsymbol{C}^d$. Denote by ${}^T v$ the transposed vector of $v \in \boldsymbol{C}^d$ and by ${}^T\boldsymbol{C}^d$ the space $\{{}^T v; v \in \boldsymbol{C}^d\}$. For a given function ϕ mapping $[-\rho, 0]$ into $\boldsymbol{C}^d$, the function ϕ^* mapping $[0, \rho]$ into ${}^T\boldsymbol{C}^d$ is defined by $\phi^*(s) = {}^T\phi(-s)$, $s \in [0, \rho]$. For a family $\mathscr{F}$ of those functions ϕ, set $\mathscr{F}^* = \{\phi^*; \phi \in \mathscr{F}\}$. For a function x defined on $[t-\rho, t]$ (or $[t, t+\rho]$), designate by x_t (or x^t) the function on $[-\rho, 0]$ (or $[0, \rho]$) such that $x_t(\theta) = x(t+\theta)$, $\theta \in [-\rho, 0]$ (or $x^t(s) = x(t+s)$, $s \in [0, \rho]$).

Now consider a linear functional differential equation for a row vector y

$$\frac{dy}{dt} = -(y^t)\overline{f|} . \tag{1.2}$$

The symbol $\overline{f|}$ denotes the operator on $\mathscr{B}^*$ naturally induced by f which operates on $\mathscr{B}^*$ to the right (see (3.6) and (3.7)). However, we restrict the domain of $\overline{f|}$ on a space $\mathscr{X}^*$ such that $\mathscr{X}$ can be imbedded continuously in $\mathscr{B}$ and that for any $\xi \in \mathscr{X}^*$ and any $\phi \in \mathscr{B}$, the convolution

$\xi * \phi$ (see (3.1)) is defined and belongs to $\mathscr{B}$. Then we can define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathscr{X}^* \times \mathscr{B}$ (see (3.9)) which satisfies the following properties;

(i) $\langle \xi, A\phi \rangle = \langle A^*\xi, \phi \rangle$ for $\xi \in \mathscr{D}(A^*)$ and $\phi \in \mathscr{D}(A)$,

(ii) for any $(\xi, \phi) \in \mathscr{X}^* \times \mathscr{B}$,

$$\langle T^*(t - t_2)\xi, T(t - t_1)\phi \rangle = \text{constant for } t \in [t_1, t_2] ,$$

where A and $-A^*$ are the infinitesimal generators of the solution semigroups $\{T(t)\}_{t \geqq 0}$ on $\mathscr{B}$ corresponding to (1.1) and $\{T^*(t)\}_{t \leqq 0}$ on $\mathscr{X}^*$ corresponding to (1.2), respectively. As $\mathscr{X}$, we can take the space of continuous functions $C([-\rho, 0], \boldsymbol{C}^d)$ with the supremum norm if $\rho < \infty$ and the space of continuous functions $\mathscr{C}_\gamma = \mathscr{C}_\gamma((-\infty, 0], \boldsymbol{C}^d)$ if $\rho = \infty$, where $\mathscr{C}_\gamma$ is defined to be a Banach space of continuous functions ϕ mapping $(-\infty, 0]$ into $\boldsymbol{C}^d$ such that $\lim_{\theta \to -\infty} |\phi(\theta)| e^{-\gamma\theta} = 0$ with norm

$$\|\phi\|_\gamma = \sup_{\theta \in (-\infty, 0]} |\phi(\theta)| e^{-\gamma\theta} .$$

The parameter γ is greater than a constant β which depends on $g(\theta)$ (see (3.3)), and we assume that $g(\theta)$ satisfies some condition (see (3.4)). Under the above restriction on the domain of $\overline{f}|$, we call equation (1.2) the "adjoint" equation of equation (1.1), which is an extension of the definition in [1]. The operator A^* is said to be "adjoint of A" relative to the bilinear form $\langle \cdot, \cdot \rangle$. Finally, in terms of the "adjoint" A^*, we shall give an explicit representation for the projection operator which corresponds to the direct sum decomposition of $\mathscr{B}$ relative to the point spectrum of A. Since the theory is almost trivial when ρ is finite, we shall prove the theorems when ρ is infinite. For discussions on functional differential equations with infinite retardations, see [2] and [4]. The theory of adjoint equations of functional differential equations with finite retardation is found in [1]. The space $\mathscr{C}_\gamma$ was taken up by Hino in [3] as an example of the phase spaces of functional differential equations with infinite retardations.

2. Linear functional differential equations with phase space $\mathscr{C}_\gamma$. Let $\mathscr{C}_\gamma$, $\gamma \in \boldsymbol{R}$, be the Banach space defined in the above and F be in $\mathscr{L}(\mathscr{C}_\gamma, \boldsymbol{C}^d)$, the family of bounded linear operators on $\mathscr{C}_\gamma$ into $\boldsymbol{C}^d$. Consider a linear functional differential equation

$$\frac{dx}{dt} = F(x_t) . \tag{2.1}$$

Denote by $x(t, \phi)$ the solution of (2.1) such that $x_0(\phi) = \phi$. It is easy to prove that the norm $\|\cdot\|_\gamma$ in $\mathscr{C}_\gamma$ satisfies the conditions similar to (H_1), $\cdots$, (H_4) in [4]. With the aid of these properties we can prove the existence, the uniqueness and the continuation in the future of $x(\phi)$. Furthermore, it holds that $x_t(\phi)\to\phi$ as $t\to 0+$ and that for each fixed $t\geqq 0$ there exists a constant $c(t)$ such that $\|x_t(\phi)\|_\gamma\leqq c(t)\|\phi\|_\gamma$ for $\phi\in\mathscr{C}_\gamma$ (see the proof of Lemma 2.1 in [4]). Namely, the family of operators $\{S(t)\}_{t\geqq 0}$ defined by $S(t)\phi=x_t(\phi)$, $\phi\in\mathscr{C}_\gamma$, is a continuous semi-group on $\mathscr{C}_\gamma$. We call such a semi-group the solution semi-group on $\mathscr{C}_\gamma$ corresponding to equation (2.1). Denote by $\mathscr{C}_\gamma^{(1)}$ the set of functions in $\mathscr{C}_\gamma$ which have continuous first derivatives, and define the function $\tilde{\phi}$ for ϕ in $\mathscr{C}_\gamma^{(1)}$ by

$$\tilde{\phi}(\theta)=\begin{cases} F(\phi) & \text{for } \theta=0 \\ \dfrac{d\phi}{d\theta}(\theta) & \text{for } \theta<0\,. \end{cases}$$

The infinitesimal generator B of $\{S(t)\}$ is given as follow.

THEOREM 2.1. *$\mathscr{D}(B)=\{\phi;\ \phi\in\mathscr{C}_\gamma^{(1)}$ and $\tilde{\phi}\in\mathscr{C}_\gamma\}$, and $B\phi=\tilde{\phi}$ for $\phi\in\mathscr{D}(B)$.*

PROOF. Set $X=\{\phi;\ \phi\in\mathscr{C}_\gamma^{(1)}$ and $\tilde{\phi}\in\mathscr{C}_\gamma\}$. It is easy to see that $\mathscr{D}(B)\subset X$ and $B\phi=\tilde{\phi}$ for $\phi\in\mathscr{D}(B)$. Now suppose that ϕ is in X. Then

$$\lim_{\theta\to-\infty}\left|\frac{d\phi}{d\theta}(\theta)\right|e^{-\gamma\theta}=0\,.$$

Since $S(h)\phi(\theta)=\phi(\theta+h)$ for $\theta+h\leqq 0$, the mean value theorem implies that for any $\varepsilon_1>0$ there exists a $\delta(\varepsilon_1)\leqq 0$ such that for any $h\in(0,1)$

$$\left|\frac{S(h)\phi(\theta)-\phi(\theta)}{h}-\tilde{\phi}(\theta)\right|e^{-\gamma\theta}\leqq\varepsilon_1 \quad \text{for } \theta\in(-\infty,\delta(\varepsilon_1)]\,. \tag{2.2}$$

Furthermore, since $\tilde{\phi}$ is continuous at $\theta=0$, $x(t,\phi)$ is continuously differentiable on $\boldsymbol{R}$. Hence $(dx/dt)(t,\phi)$ is uniformly continuous on $[\delta(\varepsilon_1),1]$ and consequently for any $\varepsilon_2>0$, there exists a $\rho(\delta(\varepsilon_1),\varepsilon_2)$ such that $|(dx/dt)(t_1,\phi)-(dx/dt)(t_2,\phi)|\leqq\varepsilon_2$ if $|t_1-t_2|\leqq\rho$ and $t_1,t_2\in[\delta(\varepsilon_1),1]$. Therefore, if $h<\rho$,

$$\left|\frac{S(h)\phi(\theta)-\phi(\theta)}{h}-\tilde{\phi}(\theta)\right|e^{-\gamma\theta}\leqq\varepsilon_2 c(\gamma,\varepsilon_1) \quad \text{for } \theta\in[\delta(\varepsilon_1),0]\,, \tag{2.3}$$

where $c(\gamma,\varepsilon_1)\leqq e^{\gamma\delta(\varepsilon_1)}$ if $\gamma\geqq 0$ and $c(\gamma,\varepsilon_1)\leqq 1$ if $\gamma<0$. Inequalities (2.2) and (2.3) mean that $\lim_{h\to 0+}(S(h)\phi-\phi)/h=\tilde{\phi}$. Thus we obtain $X\subset\mathscr{D}(B)$.
q.e.d.

3. The adjoint equation. For a given function ϕ mapping an interval I into C^k, denote by $\overline{\phi|}$ and ϕ^* the functions on the interval $J = \{-\theta; \theta \in I\}$ defined by

$$\overline{\phi|}(\theta) = \phi(-\theta) \quad \text{and} \quad \phi^*(\theta) = {}^T\overline{\phi|}(\theta) \quad \text{for} \quad \theta \in J ,$$

respectively. As in Section 1, let $\mathscr{B}^* = \{\phi^*; \phi \in \mathscr{B}\}$ and $\mathscr{C}_\gamma^* = \{\phi^*; \phi \in \mathscr{C}_\gamma\}$. For norms in $\mathscr{B}^*$ and $\mathscr{C}_\gamma^*$, we use the same notations as for norms in $\mathscr{B}$ and $\mathscr{C}_\gamma$, that is, we put $\| \phi^* \| = \| \phi \|$ for $\phi \in \mathscr{B}$ and $\| \phi^* \|_\gamma = \| \phi \|_\gamma$ for $\phi \in \mathscr{C}_\gamma$. For functions ξ: $[0, \infty) \to C$ and ϕ: $(-\infty, 0] \to C^d$, we define the convolution $\xi * \phi$ of ξ and ϕ formally by

$$(3.1) \qquad (\xi * \phi)(\theta) = \int_\theta^0 \xi(u - \theta)\phi(u)du \quad \text{for} \quad \theta \in (-\infty, 0] .$$

For $\lambda \in C$, define the operator $M(\lambda)$ on $\mathscr{B}$ by

$$M(\lambda)\phi = \overline{\omega(\lambda)|} * \phi \quad \text{for} \quad \phi \in \mathscr{B} ,$$

where $\omega(\lambda)$ is the function on $(-\infty, 0]$ such that

$$(3.2) \qquad \omega(\lambda)(\theta) = \exp \lambda\theta \quad \text{for} \quad \theta \in (-\infty, 0] .$$

Set

$$(3.3) \qquad \beta = \inf \left\{ \operatorname{Re} \lambda; \int_{-\infty}^0 | e^{\lambda\theta} |^p g(\theta) d\theta < \infty \right\} .$$

LEMMA 3.1. *Assume that the function g in the definition of $\mathscr{B}$ satisfies the condition*

$$(3.4) \qquad g(u + v) \leqq g(u)g(v) \quad \textit{for} \quad u, v \in (-\infty, 0] .$$

Then the operator function $M(\lambda)$ is a holomorphic function from $C_\beta = \{\lambda \in C; \operatorname{Re} \lambda > \beta\}$ into $\mathscr{L}(\mathscr{B}, \mathscr{B})$.

For a proof of this lemma, see [4].

LEMMA 3.2. *Assume that condition* (3.4) *holds. Then, if $\gamma > \beta$, for any $\xi \in \mathscr{C}_\gamma((-\infty, 0], C)^*$ and $\phi \in \mathscr{B}((-\infty, 0], C^d)$ the convolution $\xi * \phi$ is well defined and is in $\mathscr{B}((-\infty, 0], C^d)$ with norm*

$$\| \xi * \phi \| \leqq c(\gamma) \| \xi \|_\gamma \| \phi \| ,$$

where $c(\gamma)$ is a constant which depends on γ.

PROOF. Suppose that $\gamma > \beta$, $\xi \in \mathscr{C}_\gamma^*$ and $\phi \in \mathscr{B}$. Obviously for any $\theta \in (-\infty, 0]$,

$$\left| \int_\theta^0 \xi(u - \theta)\phi(u)du \right| \leqq \int_\theta^0 | \overline{\xi|}(\theta - u) | \, e^{-\gamma(\theta - u)} e^{\gamma(\theta - u)} \, | \phi(u) | \, du .$$

From the definition of $\|\xi\|_\gamma$ we have

$$(3.5)\qquad |\xi*\phi(\theta)| \leqq \|\xi\|_\gamma \int_\theta^0 e^{\gamma(\theta-u)}|\phi(u)|\,du \quad \text{for } \theta\in(-\infty,0]\,.$$

Using the notation $M(\lambda)$, we can express by $(M(\gamma)|\phi|)(\theta)$ the integral term in (3.5). Since condition (3.4) holds, Lemma 3.1 implies that $\xi*\phi$ is in $\mathscr{B}$ and

$$\|\xi*\phi\| \leqq \|M(\gamma)\|\,\|\xi\|_\gamma\,\|\phi\|\,. \qquad \text{q.e.d.}$$

Put $e_j = \text{column}\,(\delta_{j1}\cdots\delta_{jd})$, $j=1,\cdots,d$, where δ_{jk} are Kronecker's δ. For any f in $\mathscr{L}(\mathscr{B}, \boldsymbol{C}^d)$, we define ${}^Tf\in\mathscr{L}(\mathscr{B}, \boldsymbol{C}^d)$ by the relation

$$({}^Tf)(\phi) = \sum_{k=1}^d \left\{\sum_{j=1}^d f_j(\phi_j e_k)\right\} e_k \quad \text{for } \phi = \sum_{j=1}^d \phi_j e_j \in \mathscr{B}\,,$$

where $f_j(\psi)$ is the j-th component of $f(\psi)$, $\psi\in\mathscr{B}$, and define $\overline{f}|\in\mathscr{L}(\mathscr{B}^*, {}^T\boldsymbol{C}^d)$ by the relation

$$(3.6)\qquad (\xi)\overline{f}| = {}^T(({}^Tf)(\xi^*)) \quad \text{for } \xi\in\mathscr{B}^*\,.$$

In fact, corresponding to f in $\mathscr{L}(\mathscr{B}, \boldsymbol{C}^d)$, there exist $d\times d$ matrix functions $\eta(\theta) = (\eta_{jk}(\theta))$ and $\zeta(\theta) = (\zeta_{jk}(\theta))$ such that $\eta_{jk}(\theta)$ are of bounded variation on $[-r, 0]$ and $\zeta_{jk}(\theta)$ are measurable with

$$\int_{-\infty}^{-r} |\zeta_{jk}(\theta)|^q g(\theta)d(\theta) < \infty\,, \quad \frac{1}{q}+\frac{1}{p} = 1\,, \quad j, k = 1,\cdots,d\,,$$

and that

$$f(\phi) = \int_{-r}^0 d\eta(\theta)\phi(\theta) + \int_{-\infty}^{-r} \zeta(\theta)\phi(\theta)g(\theta)d\theta \quad \text{for } \phi\in\mathscr{B}\,.$$

Then ${}^Tf$ and $\overline{f}|$ are expressed by

$$({}^Tf)(\phi) = \int_{-r}^0 d\,{}^T\eta(\theta)\,\phi(\theta) + \int_{-\infty}^{-r} {}^T\zeta(\theta)\phi(\theta)g(\theta)d\theta \quad \text{for } \phi\in\mathscr{B}\,,$$

$$(3.7)\qquad (\xi)\overline{f}| = \int_{-r}^0 \xi(-\theta)d\eta(\theta) + \int_{-\infty}^{-r} \xi(-\theta)\zeta(\theta)g(\theta)d\theta \quad \text{for } \xi\in\mathscr{B}^*\,.$$

The following lemma can be easily proved.

LEMMA 3.3. *If $\gamma>\beta$, then $\mathscr{C}_\gamma\subset\mathscr{B}$ and*

$$\|\phi\| \leqq c(\gamma)\|\phi\|_\gamma \quad \textit{for} \quad \phi\in\mathscr{C}_\gamma\,,$$

where the constant $c(\gamma)$ is less than $\left\{\sup_{\theta\in[-r,0]}|e^{\gamma\theta}|^p + \int_{-\infty}^0 |e^{\gamma\theta}|^p g(\theta)d\theta\right\}^{1/p}$.

By Lemma 3.3, for any $f\in\mathscr{L}(\mathscr{B}, \boldsymbol{C}^d)$ and any $\gamma>\beta$, the restriction of f on $\mathscr{C}_\gamma$, which we denote by f again, is in $\mathscr{L}(\mathscr{C}_\gamma, \boldsymbol{C}^d)$, and therefore

${}^{T}f$ is in $\mathscr{L}(\mathscr{C}_\gamma, \boldsymbol{C}^d)$ and $\overline{f}|$ is in $\mathscr{L}(\mathscr{C}_\gamma^*, {}^{T}\boldsymbol{C}^d)$.

Now, define an operator A^* on $\mathscr{C}_\gamma^*$, $\gamma > \beta$, in the following way. The domain of A^* consists of all functions ξ in $\mathscr{C}_\gamma^*$ which have continuous first derivatives and satisfy $\hat{\xi} \in \mathscr{C}_\gamma^*$, where $\hat{\xi}(0) = (\xi)\overline{f}|$ and $\hat{\xi}(s) = -(d\xi/ds)(s)$ for $s \in (0, \infty)$. The operation of A^* is defined by

$$A^*\xi = \hat{\xi} \quad \text{for} \quad \xi \in \mathscr{D}(A^*)\,. \tag{3.8}$$

Assume that condition (3.4) holds. By Lemma 3.2, for any $\xi = (\xi_1(s), \cdots, \xi_d(s)) \in \mathscr{C}_\gamma((-\infty, 0], \boldsymbol{C}^d)^*$ and any $\phi \in \mathscr{B}((-\infty, 0], \boldsymbol{C}^d]$ the convolutions $\xi_j * \phi$, $j = 1, \cdots, d$, are in $\mathscr{B}$. Hence we can define a bilinear form $\langle \xi, \phi \rangle$ on $\mathscr{C}_\gamma^* \times \mathscr{B}$ by

$$\langle \xi, \phi \rangle = \xi(0) \cdot \phi(0) + \sum_{j=1}^{d} f_j(\xi_j * \phi)\,, \tag{3.9}$$

where $x \cdot y = x_1 y_1 + \cdots + x_d y_d$ for $x \in {}^{T}\boldsymbol{C}^d$ and $y \in \boldsymbol{C}^d$.

Let $\{T(t)\}_{t \geqq 0}$ be the solution semi-group on $\mathscr{B}$ corresponding to the equation

$$\frac{dx}{dt} = f(x_t) \tag{3.10}$$

and A be the infinitesimal generator of $\{T(t)\}$. It was proved in [4] that the domain of A consists of all functions ϕ in $\mathscr{B}$ which are absolutely continuous on any compact interval of $(-\infty, 0]$ and satisfy $\tilde{\phi} \in \mathscr{B}$, where $\tilde{\phi}(0) = f(\phi)$ and $\tilde{\phi}(\theta) = (d\phi/d\theta)(\theta)$ a.e. in $\theta \in (-\infty, 0)$, and that

$$A\phi = \tilde{\phi} \quad \text{for} \quad \phi \in \mathscr{D}(A)\,.$$

PROPOSITION 3.4. *Assume that condition* (3.4) *holds and that* $\gamma > \beta$. *Then the bilinear form* $\langle \cdot, \cdot \rangle$ *is continuous on* $\mathscr{C}_\gamma^* \times \mathscr{B}$, *and*

$$\langle \xi, A\phi \rangle = \langle A^*\xi, \phi \rangle \ \textit{for} \ \xi \in \mathscr{D}(A^*)\,, \quad \phi \in \mathscr{D}(A)\,. \tag{3.11}$$

PROOF. The continuity of $\langle \cdot, \cdot \rangle$ follows from Lemma 3.2. Suppose that ξ is in $\mathscr{D}(A^*)$ and ϕ is in $\mathscr{D}(A)$. Integration by parts implies that

$$\xi_j * A\phi = \overline{\xi_j}|\phi(0) - \xi_j(0)\phi + (A^*\xi)_j * \phi\,, \quad j = 1, \cdots, d\,. \tag{3.12}$$

By Lemma 3.2, $\xi_j * A\phi$, and $(A^*\xi)_j * \phi$, $j = 1, \cdots, d$, are in $\mathscr{B}$, since ξ is in $\mathscr{D}(A^*)$ and ϕ is in $\mathscr{D}(A)$. By the definition of $\overline{f}|$,

$$\sum_{j=1}^{d} f_j(\overline{\xi_j}|\phi(0)) = \sum_{k=1}^{d} \left\{ \sum_{j=1}^{d} f_j(\overline{\xi_j}|e_k) \right\} \phi_k(0) = (\xi)\overline{f}| \cdot \phi(0)\,,$$

which together with relation (3.12) implies that

$$\sum_{j=1}^{d} f_j(\xi_j * A\phi) = (\xi)\overline{f}| \cdot \phi(0) - \xi(0) \cdot f(\phi) + \sum_{j=1}^{d} f_j((A^*\xi)_j * \phi)\,.$$

Hence we obtain

$$\begin{aligned}\langle \xi, A\phi \rangle &= \xi(0)\cdot f(\phi) + \sum_{j=1}^{d} f_j(\xi_j * A\phi)\\ &= (\xi)\overline{f|}\cdot \phi(0) + \sum_{j=1}^{d} f_j((A^*\xi)_j * \phi)\\ &= \langle A^*\xi, \phi\rangle .\end{aligned}$$

q.e.d.

Now, consider the equation

$$\frac{dy}{dt} = -(y^t)\overline{f|}\,, \tag{3.13}$$

where y is a row vector. It is clear that y is a solution of (3.13) if and only if $z \equiv y^*$ is a solution of

$$\frac{dz}{dt} = {}^Tf(z_t)\,. \tag{3.14}$$

Notice that ${}^Tf$ is in $\mathscr{L}(\mathscr{C}_\gamma, \boldsymbol{C}^d)$. From the discussion in Section 2 it follows that for any $\phi \in \mathscr{C}_\gamma$, the solution $z(\phi)$ of (3.14) such that $z_0 = \phi$ exists uniquely on $[0, \infty)$ and satisfies the condition $z_t \in \mathscr{C}_\gamma$ for $t \geqq 0$. This implies that for any $\xi \in \mathscr{C}_\gamma^*$, the solution $y(\xi)$ of (3.13) such that $y^0(\xi) = \xi$ exists uniquely on $(-\infty, 0]$ and satisfies $y^t(\xi) \in \mathscr{C}_\gamma^*$ for $t \in (-\infty, 0]$. Define the operators $T^*(t)$ on $\mathscr{C}_\gamma^*$, $t \leqq 0$, by

$$T^*(t)\xi = y^t(\xi) \quad \text{for} \quad \xi \in \mathscr{C}_\gamma^*\,.$$

Let $\{S(t)\}_{t\geqq 0}$ be the solution semi-group on $\mathscr{C}_\gamma$ corresponding to equation (3.14) and B be its infinitesimal generator. Clearly, $T^*(t)\xi = (S(-t)\xi^*)^*$ for $\xi \in \mathscr{C}_\gamma^*$. Furthermore, by Theorem 2.1, $\mathscr{D}(A^*) = \mathscr{D}(B)^*$ and $A^*\xi = (B\xi^*)^*$ for ξ in $\mathscr{D}(A^*)$, where A^* is the operator defined by (3.8). Therefore, by easy computations, we have

$$\frac{d}{dt}T^*(t)\xi = -A^*T^*(t)\xi = -T^*(t)A^*\xi \tag{3.15}$$

for $t \leqq 0$ and $\xi \in \mathscr{D}(A^*)$.

THEOREM 3.5. *Let x and y be solutions of* (3.10) *and* (3.13) *defined on $[t_1, \infty)$ and $(-\infty, t_2]$ with initial conditions $x_{t_1} = \phi \in \mathscr{B}$ and $y^{t_2} = \xi \in \mathscr{C}_\gamma^*$, respectively, where $\gamma > \beta$ and $-\infty < t_1 \leqq t_2 < +\infty$. Then $\langle y^t, x_t\rangle$ is a constant on $[t_1, t_2]$, that is,*

$$\langle T^*(t - t_2)\xi, T(t - t_1)\phi\rangle = \textit{constant for } t \in [t_1, t_2]\,. \tag{3.16}$$

PROOF. If ξ is in $\mathscr{D}(A^*)$ and ϕ is in $\mathscr{D}(A)$, by Proposition 3.4 and (3.15) we have

$$\frac{d}{dt}\langle T^*(t-t_2)\xi,\ T(t-t_1)\phi\rangle$$
$$= \langle -A^*T^*(t-t_2)\xi,\ T(t-t_1)\phi\rangle + \langle T^*(t-t_2)\xi,\ AT(t-t_1)\phi\rangle$$
$$= 0 \text{ for } t\in[t_1, t_2]\ ,$$

which implies (3.16). Since $\mathscr{D}(A^*)$ and $\mathscr{D}(A)$ are dense in $\mathscr{C}_\gamma^*$ and $\mathscr{B}$, respectively, relation (3.16) holds for any $(\xi, \phi)\in\mathscr{C}_\gamma^*\times\mathscr{B}$. q.e.d.

4. A representation of projection operators. Let $\omega(\lambda)$ be the function defined by (3.2). Suppose f is in $\mathscr{L}(\mathscr{B}, C^d)$ and denote by $D(\lambda)$ the $d\times d$ matrix $\lambda E - f(\omega(\lambda)E)$, where E is the $d\times d$ unit matrix. We define matrices F_j by

$$F_j = F_j(\lambda) = \frac{1}{j!}\frac{d^j}{d\lambda^j}D(\lambda)\ ,\quad j = 0, 1, 2, \cdots ,$$

and define $md\times md$ matrices $D_m = D_m(\lambda)$ by

$$D_m = \begin{bmatrix} F_0 & F_1 \cdots F_{m-1} \\ 0 & F_0 \cdots F_{m-2} \\ & \cdots \\ 0 & 0 \cdots F_0 \end{bmatrix},\quad m = 1, 2, \cdots .$$

In [4], it has been shown that $D(\lambda)$ is holomorphic on $C_\beta = \{\lambda\in C;\ \mathrm{Re}\,\lambda > \beta\}$ and that $P_\sigma(A) = \{\lambda;\ \det D(\lambda) = 0\}$. Suppose that $\mu\in C_\beta$ is a zero of $\det D(\lambda)$ with order n. Then μ is a pole of $D(\lambda)^{-1}$ with order $m\leqq n$. In this section, (n, m) is said to be the index of μ. From the result in [4] it follows that under condition (3.4) on g, $\mathscr{B}$ can be decomposed into the direct sum $\mathscr{B} = \mathfrak{N}((\mu I - A)^m)\oplus\mathscr{R}((\mu I - A)^m)$, $\mathfrak{N}((\mu I - A)^m)$ is n-dimensional and that for $k = 1, 2, \cdots$, $\mathfrak{N}((\mu I - A)^k)$ coincides with the set of functions ϕ of the form

$$\phi(\theta) = e^{\mu\theta}\sum_{j=0}^{k-1}\frac{1}{j!}\theta^j b_j \quad\text{for}\quad \theta\in(-\infty, 0]\ ,$$

where $\hat{b} = \mathrm{col}\,(b_0, b_1, \cdots, b_{k-1})$ satisfies $D_k(\mu)\hat{b} = 0$.

For any γ, $\beta<\gamma<\mathrm{Re}\,\mu$, the bilinear form $\langle\cdot,\cdot\rangle$ can be defined on $\mathscr{C}_\gamma^*\times\mathscr{B}$ and it satisfies relation (3.11). Therefore we can prove the following propositions and theorem by the same way as in [1].

PROPOSITION 4.1. *Let A and $-A^*$ be the infinitesimal generators of semi-groups $\{T(t)\}_{t\geqq 0}$ on $\mathscr{B}$ and $\{T^*(t)\}_{t\leqq 0}$ on $\mathscr{C}_\gamma^*$ defined in Section 3, where $\gamma>\beta$. Then the point spectrum $P_\sigma(A^*)$ coincides with $P_\sigma(A)\cap C_\gamma$. Let (n, m) be the index of $\mu\in P_\sigma(A^*)$. Then $\mathfrak{N}((\mu I - A^*)^m)$ is n-dimen-*

sional. For $k = 1, 2, \cdots$, $\mathfrak{N}((\mu I - A^*)^k)$ *coincides with the set of functions* ξ *of the form*

$$\xi(s) = e^{-\mu s} \sum_{j=0}^{k-1} (-s)^j \frac{a_{k-1-j}}{j!} \quad for \quad s \in [0, \infty) ,$$

where $\hat{a} = row\,(a_0, a_1, \cdots, a_{k-1})$ *satisfies* $\hat{a} D_k(\mu) = 0$.

PROPOSITION 4.2. *Under the same assumption as in Proposition* 4.1, ψ *is in* $\mathscr{R}((\mu I - A)^k)$ *if and only if* $\langle \xi, \psi \rangle = 0$ *for all* ξ *in* $\mathfrak{N}((\mu I - A^*)^k)$, $k = 1, 2, \cdots$.

THEOREM 4.3. *Let* μ *be in* $P_\sigma(A) \cap C_\beta$ *with index* (n, m) *and* π *be the projection from* $\mathscr{B}$ *onto* $\mathfrak{N}((\mu I - A)^m)$ *which corresponds to the direct sum decomposition* $\mathscr{B} = \mathfrak{N}((\mu I - A)^m) \oplus \mathscr{R}((\mu I - A)^m)$. *Then for any base* $\{\phi_1, \cdots, \phi_n\}$ *of* $\mathfrak{N}((\mu I - A)^m)$, *there exists a base* $\{\psi_1, \cdots, \psi_n\}$ *of* $\mathfrak{N}((\mu I - A^*)^m)$ *such that*

$$\pi\phi = \sum_{j=1}^{n} \langle \psi_j, \phi \rangle \phi_j \quad for \quad \phi \in \mathscr{B} .$$

REFERENCES

[1] J. K. HALE, Functional Differential Equations, Springer, New York-Heidelberg-Berlin, 1971.

[2] J. K. HALE, Functional differential equations with infinite delays, J. Math. Anal. Appl., 48 (1974), 276-283.

[3] Y. HINO, Asymptotic behavior of solutions of some functional differential equations, Tôhoku Math. J., 22 (1970), 98-108.

[4] T. NAITO, On autonomous linear functional differential equations with infinite retardations, to appear in J. Differential Equations.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN.