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ON EXTENDIBILITY OF ISOMORPHISMS OF CARTAN CONNECTIONS AND BIHOLOMORPHIC MAPPINGS OF BOUNDED DOMAINS

ISAO NARUKI

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Introduction. In 1972 the author proved in his preprint [5] that any biholomorphic mapping of two smooth, strongly pseudo-convex bounded domains is smooth up to the boundary, under somewhat too strong assumption on the boundary behavior of the Bergman kernel. Fortunately Fefferman proved a relevant smoothness theorem in [2]. He also proved the extendibility of biholomorphic mappings mentioned above by analysing directly the behavior of geodesics (of the Bergman geometry) near the boundary. The author's method is first to reduce the problem to the heigher dimensional case in which the biholomorphic map is extended up to almost all part of the boundary, and then to extend it to the remaining part applying an extendibility theorem for Cartan connections. Since this method clarifies the geometric essence from a different point of view and since this last theorem is interesting in its own right, it seems to be worth reproducing here the argument of [5] shortly.

1. Extendibility of Isomorphisms of Cartan Connections. We begin with some elementary study of Riemannian geometry. Throughout this section we assume the differentiability of class C^k, $k \geqq 1$, so that "smooth", "diffeomorphism" mean "C^k smooth", "C^k diffeomorphism". Let M be a Riemannian manifold and C a closed submanifold of M of codimension not less than 2. Given two points $x, y \in M \backslash C$, we can define two distances between x and y, the one with respect to M and the other with respect to $M \backslash C$ which is regarded as a new Riemannian manifold. By the codimension reason, curves in M having end points fixed in $M \backslash C$ can be approximated smoothly by those lying in $M \backslash C$. Thus the two distances above coincide. In particular

LEMMA 1.1. *The metric completions of $M \backslash C$, M coincide.*

As is well known, a one to one correspondence of two Riemannian manifolds is an isometry (metric-tensor-preserving diffeomorphism) if and only if it is distance-preserving. (See e.g. Helgason [3], Theorem 11.1)

On the other hand one can always find an arbitrarily small convex neighborhood at any point of a Riemannian manifold. These facts prove

LEMMA 1.2. *Let U, U' be open subsets of Riemannian manifolds M, M' respectively. Then a one to one correspondence of U, U' is an isometry if and only if it preserves the distances of M, M' restricted to U, U'.*

We shall now study principal bundles with Riemannian metrics which are in a sence compatible with the actions of the structure groups. Let G be a Lie group and P a principal G-bundle (over which G acts on the right). A Riemannian metric on P is said to be *G-admissible* if the right action R_a of each $a \in G$ is uniformly continuous with respect to the distance. If a G-admissible metric is given over P, each right action maps Cauchy sequences into Cauchy sequences, so G acts also over the metric completion $\hat{P}$ of P. ($\hat{P}$ is not a manifold in general. But it should be noted that P is open in $\hat{P}$.)

THEOREM 1.1. *Let P, P' be principal G-bundles with G-admissible metrics and C, C' G-invariant closed submanifolds of P, P' of codimension not less than 2. Assume further that the bases P/G, P'/G are compact. Then a G-equivariant isometry φ of $P \backslash C$ onto $P' \backslash C'$ can be extended to an isometry $\tilde{\varphi}$ of P onto P'.*

PROOF. Without loss of generality we may assume that P, P' are connected. By Lemma 1.1, $\widehat{P \backslash C} = \hat{P}$, $\widehat{P' \backslash C'} = \hat{P}'$; hence φ can be extended to a distance preserving map $\hat{\varphi}$ of $\hat{P}$ onto $\hat{P}'$. Moreover, since $\hat{\varphi}$ is homeomorphic, the sets $U = \hat{\varphi}^{-1}(\hat{\varphi}(P) \cap P')$, $U' = \hat{\varphi}(P) \cap P'$ are open in P, P' respectively. Note $\hat{\varphi}|_U \colon U \xrightarrow{\sim} U'$ preserves the distances of P, P' restricted to U, U'. By Lemma 1.2, $\hat{\varphi}|_U$ is an isometry, in particular it is a diffeomorphism. Therefore $\Omega = \hat{\varphi}(P) \cup P'$ has a unique C^k structure such that both $\hat{\varphi}|_P \colon P \hookrightarrow \Omega$ and $P' \hookrightarrow \Omega$ are C^k. Since $\hat{\varphi}$ is G-equivariant, Ω and V are G-invariant and Ω is a principal G-bundle. Ω is obviously connected, so Ω/G is a connected smooth manifold in which P'/G is open and compact. Thus $\Omega/G = P'/G$, i.e. $\Omega = P'$. Thus $\hat{\varphi}(P) \subseteq P'$. Changing the roles of P, P', we have $\hat{\varphi}(P) = P'$. Thus $\tilde{\varphi} = \hat{\varphi}|_P$ is the desired extension. q.e.d.

Let us now apply this theorem to isomorphisms of Cartan connections. Let $(G, \tilde{G})$ be pair of Lie groups such that $G \subseteq \tilde{G}$ and $\mathscr{G}$, $\tilde{\mathscr{G}}$ the Lie algebras of G, $\tilde{G}$ respectively. Let further P be a principal G-bundle and ω a $\tilde{\mathscr{G}}$-valued 1-form on P. (P, ω) is called a *Cartan connection of type* $(G, \tilde{G})$ if (i) for each $x \in P$, ω_x is an isomorphism of the tangent space $T_x(P)$ onto $\tilde{\mathscr{G}}$ (ii) $R_a^*\omega = \mathrm{ad}\,(a^{-1})\omega$ for $a \in G$ (iii) $\omega_x(A_x^*) = A$ for $x \in P$ and $A \in \mathscr{G}$, where A^* denotes the generating vector field of the 1-parameter group $R_{\exp tA}$, $t \in \boldsymbol{R}$. Let (P, ω), (P', ω') be Cartan connections of the same type. A diffeomorphism φ of P onto P' is called an *isomorphism* of (P, ω) onto (P', ω') if $\varphi^*\omega' = \omega$.

LEMMA 1.3. *Let (P, ω), (P', ω') be Cartan connections of type $(G, \tilde{G})$ and assume that G is connected. Then every isomorphism of these connections is G-equivariant.*

This is an obvious consequence of the condition (iii) above.

Now we fix a positive definite inner product (,) on $\tilde{\mathscr{G}}$ and define Riemannian metric g over P for every Cartan connection (P, ω) of type $(G, \tilde{G})$ by setting

$$g_x(X, Y) = (\omega_x(X), \omega_x(Y)) \qquad X, Y \in T_x(P)\,.$$

This metric is called the canonical metric of (P, ω).

LEMMA 1.4. *The canonical metric above is G-admissible. Every C^1 isomorphism of Cartan connections is an isometry with respect to the canonical metrics.*

The proof is trivial and omitted. In view of Lemma 1.3 and Lemma 1.4 it follows from Theorem 1.1

THEOREM 1.2. *Let (P, ω), (P', ω') be Cartan connections of type $(G, \tilde{G})$ and C, C' G-invariant closed submanifolds of P, P' of codimension not less than 2. Assume further that P/G, P'/G are compact and that G is connected. Then every C^1 isomorphism of $(P\backslash C, \omega|_{P\backslash C})$ to $(P'\backslash C', \omega'|_{P'\backslash C'})$ can be extended to an isomorphism of (P, ω) to (P', ω').*

2. Biholomorphic Mappings of Strongly Pseudo-convex Bounded Domains. The following corollary of Theorem 1.2 plays the key role in the discussion of this section.

THEOREM 2.1. *Let M, M' be compact, strongly pseudo-convex real hypersurfaces of class $C^k (k \geqq 2)$ in a complex manifold and C, C' closed (real) submanifolds of M, M' of codimension not less than 2. Then a C^2 isomorphism of the CR-structures**) *of $M\backslash C$ and $M'\backslash C'$ is extended to a C^k diffeomorphism of M and M'.*

This is proved in the following way: By a theorem of Tanaka [6] we can functorially assign, to the *CR*-structure of a strongly pseudo-

*) For the definition, see e.g. Greenfield, Ann. Scoula norm. Sup. Pisa, 22 (1968), though such an object appears in some papers of Cartan and Tanaka without any proper denomination.

convex real hypersurface, a Cartan connection (P, ω) of a certain definite type $(G, \tilde{G})$ such that $P/G = M$. Since G is connected and the assignment is compatible with the inclusions $M\backslash C \hookrightarrow M$, $M'\backslash C' \hookrightarrow M'$, Theorem 2.1 follows from Theorem 1.2**).

Now let B be a bounded domain in C^n with C^∞ strongly pseudo-convex boundary ∂B. Usually the Bergman kernel function K of B is defined by

$$K(z) = \sup |u(z)|^2/\|u\|^2$$

where the supremum is taken over the set of holomorphic functions $u \neq 0$ such that $\|u\|^2 = \int |u(z)|^2 d\lambda(z) < +\infty$ $(d\lambda(z) = (\sqrt{-1}/2)^n dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n)$. We regard C^{n+1} as the cartesian product $C \times C^n = \{(z_0, z); z_0 \in C, z \in C^n\}$ and set

$$\tilde{B} = \{(z_0, z); |z_0|^{2(n+1)} K(z) < 1, z \in B\} .$$

We call this new domain $\tilde{B}$ of dimension $n+1$ the suspended domain of B. (In [5], we call the boundary $\partial\tilde{B}$ the suspension of B and denote it by $SS(B)$.)

LEMMA 2.1. *$\partial\tilde{B}$ is C^n and strongly pseudo-convex provided $n \geqq 2$.*

PROOF. By [2, p. 45], there is a C^n function φ in C^n such that $\varphi(z)^{n+1}K(z) = 1$. Since, by Hörmander [4], $\rho(z)^{n+1}K(z)$ ($\rho(z)$ is the usual euclidean distance of z from ∂B) has the positive boundary value, the 1-form $d\varphi$ does not vanish anywhere on ∂B, so that the Levi-form of φ at any $z \in \partial B$ is positive definite. This implies that the Levi-form of the function $\psi(z_0, z) = |z_0|^2 - \varphi(z)$ is positive definite at any point $(0, z)$, such that $z \in \partial B$. (Note that ψ is C^n, and is negative exactly in $\tilde{B}$, and $d\psi \neq 0$ everywhere on $\partial\tilde{B}$.) The positive definiteness of the Levi-form at $z \in \partial\tilde{B}\backslash\{0\} \times \partial B$ is also assured by the fact that $\sqrt{-1}\partial\bar{\partial}\log K$ is a Kähler metric (the Bergman metric of B). q.e.d.

Now suppose that $n \geqq 2$ and f is a biholomorphic mapping of B onto another bounded domain B' with C^∞ strongly pseudo-convex boundary $\partial B'$. We want to extend f smoothly up to ∂B. Let J_f be the Jacobian functional determinant of f, that is, $J_f(z)dz_1dz_2 \cdots dz_n = f^*(dz_1dz_2 \cdots dz_n)$. Because of the invariant property of the kernel function [1], we have

$$|J_f(z)|^2 K'(f(z)) = K(z) \qquad z \in B \tag{2.1}$$

where K' is the Bergman kernel function of B'. For simplicity consider

**) (P, ω) is merely C^{k-1}, but the loss smoothness of the extension never occurs since (P, ω) is naturally a G-structure over M [6] and every C^{k-1} isomorphism of the G-structures over C^k base manifolds induces a C^k diffeomorphism of the bases.

first the case B is simply connected. In such a case there is a holomorphic function χ in B such that $\chi^{n+1} = J_f$, so that, if we define a biholomorphic mapping $\hat{f}$ of $C \times B$ onto $C \times B'$ by $\hat{f}(z_0, z) = (\chi(z)z_0, f(z))$, then $\hat{f}(\tilde{B}) = \tilde{B}'$ by (2.1) and by the definition of the suspended domains $\tilde{B}$, $\tilde{B}'$. Applying Theorem 2.1 to the restriction $\hat{f}|_{\partial\tilde{B}\cap(C\times B)}: \partial\tilde{B} \cap (C \times B) \to \partial\tilde{B}' \cap (C \times B')$, it can be extended to a C^2 diffeomorphism of $\partial\tilde{B}$ onto $\partial\tilde{B}'$. In other words, we obtained a homeomorphism g of $\bar{\tilde{B}}$ onto $\bar{\tilde{B}}'$ such that $g|_{\tilde{B}} = \hat{f}|_{\tilde{B}}$ and that $g|_{\partial\tilde{B}}$ is C^2. Now the regularity theorem of the Dirichlet problem for the Laplacian $\sum_{i=0}^{n} \partial^2/(\partial z_i \partial \bar{z}_i)$ ensures at least that g is C^1 up to the boundary. Since $g(0, z) = \hat{f}(0, z) = (0, f(z))$ for $z \in B$, the restriction g to $\bar{\tilde{B}} \cap (0 \times C^n)$ is C^1 extension of f (when we identify C^n with $0 \times C^n$ by map $z \mapsto (0, z)$). The restriction to the boundary of the extension is therefore CR-isomorphism of the boundaries and is actually C^2, so it is even C^∞ by Theorem 2.1 applied to ∂B, $\partial B'$. Thus the extension is C^∞ up to the boundary by the regularity theorem above.

Now let us eliminate the simply-connectedness assumption. Even in case B, B' are not simply connected, we can construct *topological* coverings $\pi: B^* \to B$, $\pi': B'^* \to B'$ of order $n+1$ and *continuous* functions $\chi: B^* \to C$, $\chi': B'^* \to C$ such that $\chi(\pi^{-1}(z))$, $\chi'(\pi'^{-1}(z'))$ are just the set of $(n+1)$-th roots of $J_f(z)$, $J_f(f^{-1}(z'))$ for $z \in B$, $z' \in B'$ respectively. Then there is a unique homeomorphism $f^*: B^* \to B'^*$ such that $\chi' \circ f^* = \chi$, $\pi' \circ f^* = f \circ \pi$. We now choose domains D, D' such that $\bar{B} \subset D$, $\bar{B}' \subset D'$, and that $\pi_1(B) \simeq \pi_1(D)$, $\pi_1(B') \simeq \pi_1(D')$ under $B \hookrightarrow D$, $B' \hookrightarrow D'$. The last condition here implies that there are covering $\rho: D^* \to D$, $\rho': D'^* \to D'$ such that their restrictions $\rho^{-1}(B) \to B$, $\rho^{-1}(B') \to B'$ coincide with π, π' above. We can make D^*, D'^* *complex manifolds* such that ρ, ρ' are *holomorphic*. Then π, π', χ, χ', f^* are *holomorphic*. Moreover, the domains $(1_C \times \rho)^{-1}(\tilde{B})$, $(1_C \times \rho')^{-1}(\tilde{B}')$ are relatively compact in $C \times D$, $C \times D'$ and have smooth, strongly pseudo-convex boundaries. Define a biholomorphic mapping $\hat{f}^*$ of $C \times B^*$ onto $C \times B'^*$ by $\hat{f}^*(z_0, z) = (\chi(z)z_0, f^*(z))$ $(z_0 \in C, z \in B^*)$. This $\hat{f}^*$ plays the same role as of $\hat{f}$ before, and we have a smooth extension $\tilde{f}^*$ of $\bar{B}^*$ onto $\bar{B}'^*$ of f^* where $\bar{B}^*$, $\bar{B}'^*$ are the closures of B^*, B'^* in D^*, D'^*. Since, for $z \in B$, the set $\rho' \circ \tilde{f}^*(\rho^{-1}(z)) = \rho' \circ f^*(\rho^{-1}(z))$ consists of only one point $f(z)$, the same is true for every $z \in \bar{B}$. That is, there is $\tilde{f}: \bar{B} \to \bar{B}'$ such that $\rho(\tilde{f}^*(z)) = \tilde{f}(\rho(z))$ $(z \in \bar{B}^*)$, which is the desired extension of f. We have thus proved

THEOREM 2.2. *Let B, B' be bounded domains with C^∞ strongly pseudo-convex boundary. Then every biholomorphic mapping of B onto B' is extended to a C^∞ diffeomorphism of $\bar{B}$ onto $\bar{B}'$.*

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Research Institute for
Mathematical Sciences
Kyoto University
Kyoto, Japan