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THE HARDY SPACES ASSOCIATED WITH A PERIODIC FLOW ON A VON NEUMANN ALGEBRA

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0. Introduction. In the study of non-self adjoint subalgebras of von Neumann algebras, several attempts have been made to generalize a theory of function algebras to non-commutative cases. For instance, a theory of subdiagonal algebras was presented by Arveson as an analogue of weak*-Dirichlet algebras in [1]. In this paper we present a method to construct the Hardy spaces associated with a periodic flow on a von Neumann algebra. The method is based on the theory of spectral subspaces for a flow which has been investigated by many authors [2, 3, 9]. Kawamura and Tomiyama [5] studied the Hardy spaces associated with a flow and discussed related situations in operator algebras.

Let T be the unit circle. We define a flow β with period 2π of $L^\infty(T)$ as follows: $\beta_t f(z) = f(e^{-it}z)$, $t \in R$, $z \in T$, $f \in L^\infty(T)$. Let M be a von Neumann algebra acting on a Hilbert spaces H, M_* its predual and α a periodic flow with period 2π on M. Then M, M_*, H and α correspond to $L^\infty(T)$, $L^1(T)$, $L^2(T)$ and β, respectively. Then, in view of the role played by the Hardy spaces H^p in $L^p(T)$, we construct $H^p(\alpha)(p = 1, 2, \infty)$. In particular $H^\infty(\alpha)$ is not only a σ-weakly closed non-self adjoint subalgebra but also turns to be a maximal subdiagonal algebra. If there exists an ergodic, periodic flow on M, then M is generated by a single unitary operator. In this case we use the Cesaro mean defined by a periodic flow on M. If M is σ-finite, we have a decomposition of a von Neumann algebra with respect to a periodic flow and reconsider a part of Takesaki's consequence in [10] for a von Neumann algebra with a homogeneous periodic state.

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1. Preliminaries. Let M be a von Neumann algebra acting on a Hilbert space H, M_* its predual and $\alpha_t (t \in R)$ a flow on M, that is, a one-parameter group of *-automorphisms of M which is weak*-continuous in the sense that, for each $x \in M$ and $\rho \in M_*$, the function $t \to \rho(\alpha_t(x))$ is continuous. Let U_t $(t \in R)$ be a strongly continuous unitary group on H.

We define two representations $U(\cdot)$ and $\alpha(\cdot)$ of $L^1(R)$ into the bounded operators on H and M, respectively, by $U(f)\xi = \int_{-\infty}^{\infty} f(t)\, U_t\xi dt$ $(\xi \in H)$ and $\alpha(f)x = \int_{-\infty}^{\infty} f(t)\alpha_t(x)dt$ $(x \in M)$ where $f \in L^1(R)$. For $f \in L^1(R)$, we put $Z(f) = \{t \in R: \hat{f}(t) = 0\}$. Let $Sp\ \alpha$ be defined as $\cap \{Z(f): f \in L^1(R), \alpha(f) = 0\}$. If $\xi \in H$ and $x \in M$, let $Sp_U(\xi) = \cap \{Z(f): f \in L^1(R), U(f)\xi = 0\}$ and $Sp_\alpha(x) = \cap \{Z(f): f \in L^1(R), \alpha(f)x = 0\}$.

A flow α on M is said to be periodic if there exists $T > 0$ such that α_T is the identity automorphism of M. The smallest such $T > 0$ is called the period of the flow α. We suppose without loss of generality that all the flows treated here have period 2π.

2. The spectral subspaces and the algebra $H^\infty(\alpha)$. Let M be a von Neumann algebra and α a periodic flow on M. Then we put the spectral subspace $H^\infty(\alpha) = \{x \in M: Sp_\alpha(x) \subset [0, \infty)\}$. If $Sp_\alpha(x) \subset [s, \infty)$ and $Sp_\alpha(y) \subset [t, \infty)$, then we have $Sp_\alpha(xy) \subset [s + t, \infty)$ [2, §3, Lemma 1] and $Sp_\alpha(x^*) = -Sp_\alpha(x)$. As $\alpha(f)$ is σ-weakly continuous for each $f \in L^1(R)$ [2, §2, Remarks], $H^\infty(\alpha)$ is a σ-weakly closed, non-self adjoint subalgebra of M.

Now for each $n \in Z$, we consider the integration

$$\varepsilon_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{int}\alpha_t(x)dt\ , \quad x \in M$$

and set $M_n = \{x \in M: \alpha_t(x) = e^{-int}x\}$, $n \in Z$. Then we have the following properties;

$$\varepsilon_n(M) = M_n\ , \quad \varepsilon_n \circ \varepsilon_m = \delta_{nm}\varepsilon_n\ ,$$
$$\varepsilon_n(axb) = a\varepsilon_n(x)b\ , \quad a, b \in M_0\ .$$

Clearly $M_0 = H^\infty(\alpha) \cap H^\infty(\alpha)^*$ is the algebra of all fixed points with respect to α, and ε_0 is a unique, faithful, normal, α_t-invariant projection of norm one from M onto M_0. Thus the von Neumann algebra M is α-finite in the sense that there exists a family F of normal α_t-invariant states of M such that if x is any non-zero positive element in M then for some ρ in F, $\rho(x) \neq 0$ [6]. Then we have the following lemma:

LEMMA 1. *Keep the notation as above.*

(a) *For any $n, m \in Z$ we have $M_nM_m \subset M_{n+m}$ and $M_n^* = M_{-n}$.*

(b) *Let $x, y \in M$. If $\varepsilon_n(x) = \varepsilon_n(y)$ for each $n \in Z$, then $x = y$.*

(c) *For $x \in M$, we have $Sp_\alpha(x) = \{n \in Z: \varepsilon_n(x) \neq 0\}$.*

(d) *For $n \in Z$, $M_n = \{x \in M: Sp_\alpha(x) = \{n\}\}$.*

PROOF. (a) and (b) are clear.

(c) For $f \in L^1(R)$, $n \in Z$ and $x \in M$, we have

$$\varepsilon_n(\alpha(f)x) = \hat{f}(n)\varepsilon_n(x) \cdots\cdots\cdots\cdots (*) .$$

If t is not an integer, then there exists $f \in L^1(R)$ such that $\hat{f}(t) \neq 0$ and $\hat{f}(n) = 0$ for each $n \in Z$ [8, Theorem 2.6.2]. By (b) and $(*)$, $\alpha(f)x = 0$. But $\hat{f}(t) \neq 0$ and so $t \notin Sp_\alpha(x)$. Thus $Sp_\alpha(x) \subset Z$. Suppose $\varepsilon_n(x) \neq 0$. If $f \in L^1(R)$ such that $\alpha(f)x = 0$, then $\hat{f}(n) = 0$ and so $n \in Sp_\alpha(x)$. On the other hand, if $\varepsilon_n(x) = 0$, then there exists $f \in L^1(R)$ such that $\hat{f}(n) \neq 0$ and $\hat{f}(m) = 0$ for each $m(\neq n) \in Z$. By (b) and $(*)$, $\alpha(f)x = 0$. Then $n \notin Sp_\alpha(x)$. Therefore we have $Sp_\alpha(x) = \{n \in Z : \varepsilon_n(x) \neq 0\}$.

(d) is clear from (c). q.e.d.

As $\rho((-1)^n\varepsilon_n(x))$ is the Fourier coefficient for $\rho(\alpha_{-t+\pi}(x))$, $x \in H^\infty(\alpha)$ if and only if $\varepsilon_n(x) = 0 (n \leqq -1)$ iff the function $[-\pi, \pi] \ni t \longrightarrow \rho(\alpha_{-t+\pi}(x))$ belongs to the disk algebra for each $\rho \in M_*$. Therefore, taking the periodic flow β of $L^\infty(T)$ defined in Introduction, we easily note that $H^\infty(\beta) = H^\infty$. So we may consider this algebra $H^\infty(\alpha)$ the generalized notion of H^∞ in $L^\infty(T)$.

Next we define the notion of the Cesaro mean for an element in M. For $x \in M$, we put $\sigma_n(x, t) = (1/n)\sum_{k=0}^{n} S_k(x, t) (n \geqq 1)$ where $S_n(x, t) = \sum_{k=-n}^{n} e^{ikt}(-1)^k\varepsilon_k(x) (n \geqq 0)$. Since $\rho(\sigma_n(x, t))$ is the Cesaro mean for the continuous function $\rho(\alpha_{-t+\pi}(x))$ for each $\rho \in M_*$, we have the following:

THEOREM 1. *For $x \in M$, $t \in R$, we have $\sigma_n(x, t) \longrightarrow \alpha_{-t+\pi}(x)$ in the weak*-topology as $n \longrightarrow \infty$. In particular $\sigma_n(x, \pi) \longrightarrow x$ in the weak*-topology as $n \longrightarrow \infty$. Thus M is linearly spanned by $\bigcup_{n \in Z} M_n$ in the weak*-topology.*

We recall that H^∞ is a maximal weak*-Dirichlet algebra of $L^\infty(T)$. The notion of weak*-Dirichlet algebras is extended in the present case.

DEFINITION 1. Let M be a von Neumann algebra acting on a separable Hilbert space H and Φ a faithful, normal projection of norm one from M into itself. A subalgebra N of M is said to be subdiagonal with respect to Φ if (1) $N + N^*$ is σ-weakly dense in M; (2) $\Phi(xy) = \Phi(x)\Phi(y)$ for $x, y \in N$; (3) $\Phi(M) \subset N \cap N^*$; (4) $(N \cap N^*)^2$ is non-degenerate. A subdiagonal algebra N of M with respect to Φ is said to be maximal if it is contained properly in no larger subdiagonal algebra of M with respect to Φ.

Then $H^\infty(\alpha)$ may be characterized as a maximal subdiagonal algebra with respect to ε_0. We give here a simple proof when the flow α is periodic. Kawamura and Tomiyama [5] proved this fact for any (not necessarily periodic) flow α on M such that M is α-finite.

THEOREM 2. *Let M be a von Neumann algebra acting on a separable Hilbert space H and α a periodic flow. Then $H^\infty(\alpha)$ is a maximal subdiagonal algebra with respect to ε_0.*

PROOF. By Theorem 1, we have $\sigma_n(x, \pi) \longrightarrow x$ in the weak*-topology as $n \longrightarrow \infty$ for each $x \in M$. Thus $H^\infty(\alpha) + H^\infty(\alpha)^*$ is weak*-dense in M and so $H^\infty(\alpha) + H^\infty(\alpha)^*$ is σ-weakly dense in M. Putting

$$H_0^\infty(\alpha) = \{x \in H^\infty(\alpha) \colon \varepsilon_0(x) = 0\} ,$$

$H_0^\infty(\alpha)$ is a two-sided ideal of $H^\infty(\alpha)$. Therefore one may easily show that ε_0 is multiplicative on $H^\infty(\alpha)$. The statements (3) and (4) of Definition 1 are clear. Hence $H^\infty(\alpha)$ is a subdiagonal algebra with respect to ε_0.

Next we show that $H^\infty(\alpha)$ is maximal. Put $A = \{x \in M \colon \varepsilon_0(H_0^\infty(\alpha)x) = 0\}$. Since A is a maximal subdiagonal algebra of M containing $H^\infty(\alpha)$ [1, Theorem 2.2.1], it is sufficient to show that $H^\infty(\alpha) = A$. For any $x \in A$, $\sigma_n(x, \pi)$ converges to x in the weak*-topology by Theorem 1. Let m be a negative integer. Let $\varepsilon_m(x) = u|\varepsilon_m(x)|$ be the canonical polar decomposition of $\varepsilon_m(x)$ with u partial isometric. Note that $u \in M_m$ and $|\varepsilon_m(x)| \in M_0$. By Lemma 1 (a), $u^* \in M_{-m} \subset H_0^\infty(\alpha)$. For $n > -m$, we have

$$\varepsilon_0(u^*\sigma_n(x, \pi)) = \frac{1}{n}\sum_{k=0}^{n-1}\varepsilon_0(u^*S_k(x, \pi)) = \frac{n+m}{n}u^*\varepsilon_m(x) \longrightarrow u^*\varepsilon_m(x) \quad \text{as} \quad n \longrightarrow \infty .$$

On the other hand, as ε_0 is normal and $x \in A$, $\varepsilon_0(u^*\sigma_n(x, \pi))$ converges to $\varepsilon_0(u^*x) = 0$ in the weak*-topology. Thus $u^*\varepsilon_m(x) = 0$ and so $\varepsilon_m(x) = 0$. Hence $x \in H^\infty(\alpha)$. This completes the proof. q.e.d.

DEFINITION 2. A flow α is said to be ergodic if, for $x \in M$, $\alpha_t(x) = x$ for all $t \in R$ implies $x = \omega 1$ for some complex number ω.

THEOREM 3. *Let M be a von Neumann algebra. If there exists an ergodic, periodic flow on M, then M is generated by a single unitary operator.*

PROOF. Let α be an ergodic, periodic flow on M. Note from the proof of Theorem 3.2 (1) in [9] that $Sp\ \alpha = Z$. On the other hand $Sp\ \alpha = \bigcup_{x \in M} Sp_\alpha(x)$ [3, Lemma 2.13]. Therefore $M_1 \neq \{0\}$. There exists $u \in M_1$ such that $||u|| = 1$. As $u^*u, uu^* \in M_0$ and α is ergodic, $u^*u = uu^* = 1$ and so u is a unitary operator. If $x \in M_1$, there exists a complex number ω such that $u^*x = \omega 1$. Thus $x = \omega u$ and so $M_1 = Cu$. By Lemma 1, $M_n = Cu^n$ for each $n \in Z$. Therefore M is generated by a single unitary operator by Theorem 1. q.e.d.

3. **The space $H^1(\alpha)$.** Let M be a von Neumann algebra and α a periodic flow. We define the periodic action α_t' on M_* such that $\alpha_t'(\rho)(a) =$

$\rho(\alpha_{-t}(a))$, $a \in M$, $\rho \in M_*$ and the integration

$$\alpha'(f)\rho(a) = \int_{-\infty}^{\infty} f(t)\alpha'_t(\rho)dt\ , \quad a \in M,\ \rho \in M_*\ .$$

Let the spectrum for ρ in the following: $Sp_{\alpha'}(\rho) = \cap\{Z(f)\colon f \in L^1(R),\ \alpha'(f)\rho = 0\}$ and put the following integration:

$$\varepsilon'_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{int}\alpha'_t(\rho)dt\ , \quad n \in Z\ .$$

Now we define the Hardy space $H^1(\alpha) = \{\rho \in M_*\colon Sp_{\alpha'}(\rho) \subset [0, \infty)\}$. Then $H^1(\alpha)$ is a norm-closed subspace of M_*. As in §2, we define the Cesaro mean for ρ in M_*.

THEOREM 4. (a) *The following statement are equivalent.*

(1) $\rho \in H^1(\alpha)$.

(2) *The function* $[-\pi, \pi] \ni t \longrightarrow \alpha'_{-t+\pi}\rho(x)$ *belongs to the disk algebra for each* $x \in M$.

(3) $\rho(H_0^\infty(\alpha)) = 0$ *where* $H_0^\infty(\alpha) = \{x \in H^\infty(\alpha)\colon \varepsilon_0(x) = 0\}$.

(b) *Let* $\rho \in M_*$ *and* $\sigma_n(\rho, t)$ *the Cesaro mean for* ρ. *Then* $\sigma_n(\rho, t)$ *converges to* $\alpha'_{-t+\pi}(\rho)$ *in the norm on* M_*. *Therefore* $H^1(\alpha) + H^1(\alpha)^*$ *is norm-dense in* M_*.

PROOF. (a) cf. [2, Proposition 5.1].

(b) Since the action $t \longrightarrow \alpha'_{-t}(\rho)$ ($\rho \in M_*$) moves continuously in the norm of M_* [2, Proposition 3.0], we easily note that $\sigma_n(\rho, t)$ converges to $\alpha'_{-t+\pi}(\rho)$ in the norm on M_* (cf. [4, p. 17, Theorem]). q.e.d.

4. A decomposition of von Neumann algebras and the space $H^2(\alpha)$. Suppose that M is σ-finite and α a periodic flow on M. Then there exists a faithful, normal, α_t-invariant state φ of M. Consider the *-representation $\{\pi, H\}$ of M, where π is the representation associated with φ via the Gelfand-Neumark-Segal construction and H is the associated Hilbert space. As φ is faithful, π is a *-isomorphism and so we may identify M with $\pi(M)$ for simplicity. Thus there is a cyclic and separating vector ξ_0 for M such that $\varphi(a) = (a\xi_0, \xi_0)$, $a \in M$. As φ is α_t-invariant, there exists a strongly continuous unitary group u_t such that $\alpha_t(a) = u_t a u_t^*$ and $u_t\xi_0 = \xi_0$. Since the period of α is 2π, that of u is 2π. Hence $u_t = \sum_{n=-\infty}^{\infty} e^{-int}p_n$, $\sum_{n=-\infty}^{\infty} p_n = 1$, where the mutually orthogonal projection p_n are also written as follows:

$$p_n\xi = \frac{1}{2\pi}\int_0^{2\pi} e^{int}u_t\xi dt\ , \quad \xi \in H\ .$$

Then p_n is the projection of H onto the closed subspace

$$H_n = \{\xi \in H: u_t\xi = e^{-int}\xi\} = \{\xi \in H: Sp_u(\xi) = \{n\}\}, n \in Z .$$

LEMMA 2. (1) $\varepsilon_n(x)\xi_0 = p_n x\xi_0,\ x \in M.$

(2) $x\xi_0 = \sum_{n=-\infty}^{\infty} \varepsilon_n(x)\xi_0,\ x \in M.$

(3) *For each* $n \in Z$ *we have* $H_n = [M_n\xi_0]$.

(4) $M_nH_m \subset H_{n+m},\quad n,\quad m \in Z.$

THEOREM 5. *Let M be a σ-finite von Neumann algebra and α a periodic flow on M. Then, in the pre-Hilbert space structure induced by a faithful, normal, α_t-invariant state φ, M is decomposed into an orthogonal direct sum as follows:*

$$M = \cdots \oplus M_{-2} \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus M_2 \oplus \cdots .$$

In this case $H^\infty(\alpha)$ in the previous section has the following form:

$$H^\infty(\alpha) = M_0 \oplus M_1 \oplus M_2 \oplus \cdots\cdots$$

$$M_n = \left\{e^{-int} \sum_{m=-\infty}^{\infty} p_{m+n}xp_m: x \in M\right\},\quad \varepsilon_n(x) = \sum_{m=-\infty}^{\infty} p_{m+n}xp_m .$$

Now we define the Hardy space $H^2(\alpha)$ in the following way:

$$H^2(\alpha) = \{\xi \in H: Sp_u(\xi) \subset [0, \infty)\} .$$

By Lemma 2, we have

$$H^2(\alpha) = H_0 \oplus H_1 \oplus H_2 \oplus \cdots = [H^\infty(\alpha)\xi_0] = \sum_{n=0}^{\infty} p_nH .$$

Next suppose that M has a homogeneous periodic state φ in the sense that $G(\varphi) = \{\sigma \in \mathrm{Aut}(M): \varphi \circ \sigma = \varphi\}$ acts ergodically on M and the modular automorphism group σ_t^φ of M associated with φ is a periodic flow. Since a homogeneous state is faithful, then Takesaki proved that there exists an isometry u of M_1 such that $M_n = M_0u^n (n \geqq 1)$ and $M_n = u^{*-n}M_0 (n \leqq -1)$. But in case M has a faithful, normal α_t-invariant state, we don't know the relation between M_n and M_0. It may happen that there exists $n \in Z$ such that $M_n = \{0\}$ and $H_n = \{0\}$. For instance we take $M = B(H) (\dim H \geqq 2)$ and a strongly continuous unitary group $u_t = p + e^{it}q$ where p and q are non-zero projections of M such that $p + q = 1$. Then we consider a periodic flow $\alpha_t = \mathrm{ad}\, u_t$.

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