Tôhoku Math. Journ.
30 (1978), 471-486.

KNESER'S PROPERTY AND BOUNDARY VALUE PROBLEMS FOR SOME RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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(Received May 16, 1977, revised September 9, 1977)

1. Introduction. Recently, we [4] have extended an existence theorem of Nagumo for boundary value problems in second order ordinary differential equations (cf. [3], [4]). This paper is a further extension of our result to functional differential equations, and the proof given in this paper is simpler than that in [4].

As the phase space for retarded functional differential equations, Hale [1] first considered a Banach space of functions which satisfies some axioms. Recently, Hale and Kato [2] have improved the axioms for the phase space. We shall discuss the theory of functional differential equations in a semi-normed linear space as a phase space, and we shall assume some axioms which are essentially equivalent to those in [2]. Under these axioms, our results contain not only the theory for infinite delay but also the theory for finite delay and ordinary differential equations.

First we shall introduce the axioms for the phase space in Section 2, but our notations are somewhat different from those in [2], and we shall prove Kneser's property in Section 3 and apply this to a boundary value problem for some functional differential equation in Section 4. For a contingent functional differential equation, where the phase space is the class of all bounded and continuous functions, Kikuchi [5] proved the Kneser's property on R^n.

2. Preliminaries. Let B be a linear real vector space of functions mapping $(-\infty, 0]$ into R^n with the semi-norm $|\cdot|$. For any elements φ and ψ in B, $\varphi = \psi$ means $\varphi(\theta) = \psi(\theta)$ for all $\theta \in (-\infty, 0]$. The quotient space of B by the semi-norm $|\cdot|$, which is denoted by $\mathscr{B} = B/|\cdot|$, is a normed linear space with the norm $|\cdot|$ which is induced naturally by the semi-norm and for which we shall use the same notation. We do not assume $\mathscr{B}$ is a Banach space. The topology for B is naturally defined by the semi-norm, that is, the family $\{U(\varphi, \varepsilon): \varphi \in B, \varepsilon > 0\}$ is the open base, where $U(\varphi, \varepsilon) = \{\psi \in B: |\varphi - \psi| < \varepsilon\}$. Generally, B is a pseudo-metric space for this topology, and hence it may not be a Hausdorff space. The natural projection $\pi: B \to \mathscr{B}$ is a continuous, isometric, closed and open mapping.

For an R^n-valued function x defined on an interval $(-\infty, \sigma)$ and for a $t \in (-\infty, \sigma)$, let x_t be a function defined on $(-\infty, 0]$ such that

$$x_t(\theta) = x(t + \theta) , \qquad \theta \in (-\infty, 0] .$$

Given an A, $0 < A \leqq \infty$, and a φ in B, let $\mathscr{F}_A(\varphi)$ be the set of all R^n-valued functions x defined on $(-\infty, A)$ such that $x_0 = \varphi$ and x is continuous on $[0, A)$, and denote

$$\mathscr{F}_A = \bigcup \{\mathscr{F}_A(\varphi): \varphi \in B\} .$$

For a $\beta \geqq 0$ and a φ in B, let φ^β be the restriction of φ to the interval $(-\infty, -\beta]$ and let B^β be the space of such functions φ^β. We can define a semi-norm $|\cdot|_\beta$, $\beta \geqq 0$, in B^β by

$$|\eta|_\beta = \inf \{|\psi|: \psi \in B, \psi^\beta = \eta\} , \qquad \eta \in B^\beta .$$

This semi-norm is also a semi-norm in B by the relation $|\varphi|_\beta = |\varphi^\beta|_\beta$, $\varphi \in B$, that is,

$$|\varphi|_\beta = \inf \{|\psi|: \psi \in B, \psi^\beta = \varphi^\beta\} .$$

We shall assume the following axioms on B.

(A1) If x is in $\mathscr{F}_A$, $0 < A \leqq \infty$ and $t \in [0, A)$, then $x_t \in B$ and x_t is continuous in $t \in [0, A)$.

(A2) There is a positive and continuous function $K(\beta)$ of $\beta \geqq 0$ such that

$$|\varphi| \leqq K(\beta) \sup_{-\beta \leqq \theta \leqq 0} |\varphi(\theta)| + |\varphi|_\beta$$

for any $\varphi \in B$ and any $\beta \geqq 0$, where $|\varphi(\theta)|$ is any norm of $\varphi(\theta)$ in R^n.

(A3) For any φ in B and $\beta \geqq 0$, $|\varphi| = 0$ implies $|T^\beta \varphi|_\beta = 0$, where T^β is a linear operator from B into B^β defined by $T^\beta \varphi(\theta) = \varphi(\beta + \theta)$, $\theta \in (-\infty, -\beta]$. Here notice that axiom (A1) assures $T^\beta \varphi \in B^\beta$.

(A4) If $\varphi \in B$ and $|\varphi| = 0$, then $\varphi(0) = 0$.

The following two axioms which are stronger than (A3) and (A4) will be assumed.

(A3*) There is a positive and continuous function $M(\beta)$ of $\beta \geqq 0$ such that

$$|T^\beta \varphi|_\beta \leqq M(\beta) |\varphi|$$

for any $\varphi \in B$ and $\beta \geqq 0$.

(A4*) There is a positive number K_1 such that

$$|\varphi(0)| \leqq K_1 |\varphi| , \qquad \varphi \in B .$$

EXAMPLE 2.1. Let B be the set of all R^n-valued functions which are continuous on a compact interval $[-r, 0]$, and let

$$|\varphi| = \sup\{|\varphi(\theta)|: -r \leqq \theta \leqq 0\}, \qquad \varphi \in B.$$

Then $|\cdot|$ is a semi-norm in B and $\mathscr{B}$ is the Banach space $C([-r, 0]; R^n)$ of all continuous functions from $[-r, 0]$ into R^n with uniform convergence topology, and in particular, $\mathscr{B} = R^n$ if $r = 0$. This space B satisfies all axioms in the above.

For other examples, see [2].

It is not difficult to prove the following two lemmas.

LEMMA 2.1. *If both spaces B_1 and B_2 satisfy one of the above axioms, then a semi-normed linear space $B = B_1 \times B_2$ also satisfies the same axiom and $\mathscr{B} = \mathscr{B}_1 \times \mathscr{B}_2$.*

LEMMA 2.2. *Let X and Y be any topological spaces. If F is a continuous mapping from a subset $\mathscr{D}$ of $X \times \mathscr{B}$ into Y, then F can be naturally regarded as a continuous mapping from a subset $D = (1_X \times \pi)^{-1}(\mathscr{D})$ of $X \times B$ into Y, where $1_X: X \to X$ is the identity mapping on X. Conversely, if Y is a Hausdorff space and F is a continuous mapping from a subset D of $X \times B$ into Y, then F can be naturally regarded as a continuous mapping from a subset $\mathscr{D} = (1_X \times \pi)(D)$ of $X \times \mathscr{B}$ into Y.*

By Lemma 2.2, if D is a subset of $R \times B$ and $F: D \to R^n$ is a continuous function, then F can be regarded as a continuous function from a subset $\mathscr{D}$ of $R \times \mathscr{B}$ into R^n, and *vice versa*, and so we consider the following functional differential equation;

$$\text{(E)} \qquad x'(t) = F(t, x_t) \qquad (' = d/dt),$$

where F is a continuous function defined on a subset D of $R \times B$.

DEFINITION 2.1. The function x is a solution of (E) on an interval $J \subset R$ if x is a mapping from $\bigcup\{(-\infty, t]: t \in J\}$ into R^n such that $(t, x_t) \in D$ for $t \in J$ and x is continuously differentiable on J and satisfies (E) on J. For a given $(\sigma, \varphi) \in D$, we say $x = x(\sigma, \varphi)$ is a solution of (E) through (σ, φ) if there is an $A > \sigma$ such that x is a solution of (E) on $[\sigma, A)$ and $x_\sigma = \varphi$.

THEOREM 2.1 (Existence). *Suppose* (A1) *and* (A2). *Let Ω be an open subset of $R \times B$ and $F: \Omega \to R^n$ be continuous. Then for any $(\sigma, \varphi) \in \Omega$, there exists a solution of* (E) *through (σ, φ).*

REMARK. This theorem can be proved by the same method as in the proof of Theorem 2.1 in [2], though they further assumed (A3) and (A4*) (which correspond to (α_2) and (α_4) in [2], respectively) since their initial function is an element of $\mathscr{B}$. Under axioms (A1) through (A4), we have the following assertion: If x and y are solutions of (E) on $[\sigma, A)$ such that $|x_\sigma - y_\sigma| = 0$, then the function $z: (-\infty, A) \to R^n$ defined by $z_\sigma = x_\sigma$ and $z(t) = y(t)$ for $t \in [\sigma, A)$ is also a solution of (E). This means that the initial value problems are determined by the elements of $\mathscr{B}$.

3. Kneser's property. Throughout this section, let I be a compact interval $[\sigma, T]$, $\sigma < T$, and let $C = C(I; R^n)$ be the Banach space of all continuous functions from I into R^n with the norm $\|\cdot\|$ defined by

$$\|\xi\| = \sup\{|\xi(t)|: t \in I\}\,, \qquad \xi \in C\,.$$

For an R^n-valued function u defined on $(-\infty, T]$, let $u|_I$ be the restriction of u to the interval I. If $u|_I$ is continuous on I, then we write $\|u|_I\|$ simply by $\|u\|$.

Clearly, if $F: I \times B \to R^n$ is bounded and continuous, then all solutions of (E) through (σ, φ) are continuable to the whole interval I for any $\varphi \in B$ under axioms (A1) and (A2).

THEOREM 3.1. *Suppose* (A1) *and* (A2). *If $F: I \times B \to R^n$ is a bounded and continuous function, then the set*

$$S = S(\varphi) = \{x|_I: x \textit{ is a solution of } (\mathrm{E}) \textit{ through } (\sigma, \varphi)\}$$

is a continuum (i.e., compact and connected) in C for any $\varphi \in B$.

PROOF. Let $\varphi \in B$ be fixed and $M > 0$ be a bound for F, that is, $|F(t, \psi)| \leqq M$ for $(t, \psi) \in I \times B$.

Let L be the set of all functions $u: (-\infty, T] \to R^n$ such that $u_\sigma = \varphi$ and u is $(M+1)$-Lipschitzian on I, that is,

$$|u(t) - u(t')| \leqq (M+1)|t - t'| \quad \text{for} \quad t, t' \in I\,,$$

and let

$$E = \{u_t: u \in L, t \in I\}\,.$$

Then we can regard L as a subset of C, and in this sense, L is clearly compact in C. Therefore, by (A2), it is not difficult to show that E is a compact subset of B, and hence F is uniformly continuous on $I \times E$. This implies that for any ε, $0 < \varepsilon < 1$, there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ such that for any $t, s \in I$ and $\psi, \eta \in E$,

$$(3.1) \qquad |F(t, \psi) - F(s, \eta)| \leqq \varepsilon \quad \text{if} \quad |t - s| \leqq \delta_0,\ |\psi - \eta| \leqq \delta_0\,.$$

Now let ε, $0 < \varepsilon < 1$, be fixed and K be a positive number such that

$K \geqq \max\{K(\beta): 0 \leqq \beta \leqq T - \sigma\}$. For the above $\delta_0 = \delta_0(\varepsilon) > 0$, we can find a number δ, $0 < \delta < \min\{\delta_0, \delta_0/2K(M+1), \varepsilon/(M+1)\}$, such that for any $t, s \in I$ and $u, w \in L$,

$$(3.2) \qquad |u_t - w_s| \leqq \delta_0/2 \quad \text{if} \quad |t - s| \leqq \delta, \|u - w\| \leqq \delta .$$

Let

$$\Delta: \sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_N = T$$

be any division of I such that $\max_{1 \leqq i \leqq N} (\sigma_i - \sigma_{i-1}) < \delta$, and let $v_0, v_1, \cdots, v_{N-1}$ be any vectors in R^n such that $|v_i| \leqq \varepsilon$, $0 \leqq i \leqq N-1$. For this Δ and $v = (v_0, v_1, \cdots, v_{N-1})$, we construct the function $\varphi^\varepsilon(v)(\cdot): (-\infty, T] \to R^n$ in the following way. First, define $\varphi^0: (-\infty, \sigma_0] \to R^n$ and $b_0 \in R^n$ by

$$\varphi^0_{\sigma_0} = \varphi \quad \text{and} \quad b_0 = \varphi(0) .$$

For $k = 0, 1, \cdots, N-1$, we define $\bar{\varphi}^{k+1}: (\sigma_k, \sigma_{k+1}] \to R^n$, $b_{k+1} \in R^n$ and $\varphi^{k+1}: (-\infty, \sigma_{k+1}] \to R^n$ inductively in the following way:

$$\begin{aligned} \bar{\varphi}^{k+1}(t) &= b_k + (t - \sigma_k)\{F(\sigma_k, \varphi^k_{\sigma_k}) + v_k\} \quad \text{for} \quad t \in (\sigma_k, \sigma_{k+1}] , \\ b_{k+1} &= \bar{\varphi}^{k+1}(\sigma_{k+1}) \end{aligned}$$

and

$$\varphi^{k+1}(t) = \begin{cases} \varphi^k(t) , & t \in (-\infty, \sigma_k] \\ \bar{\varphi}^{k+1}(t) , & t \in (\sigma_k, \sigma_{k+1}] . \end{cases}$$

Thus we finally obtain a function φ^N. We denote this function φ^N by $\varphi^\varepsilon(v)$ or simply φ^ε. Since $0 < \varepsilon < 1$, φ^ε belongs to L.

Next we shall show the following inequality concerning φ^ε;

$$(3.3) \qquad |\varphi^\varepsilon(t) - \varphi(0) - \int_\sigma^t F(s, \varphi^\varepsilon_s) ds| \leqq 2\varepsilon(t - \sigma) , \qquad t \in I .$$

For $t \in (\sigma_0, \sigma_1]$, it follows from (3.1) and (3.2) that

$$\begin{aligned} \left|\varphi^\varepsilon(t) - \varphi(0) - \int_\sigma^t F(s, \varphi^\varepsilon_s) ds\right| &\leqq \left|(t - \sigma_0)\{F(\sigma_0, \varphi^\varepsilon_{\sigma_0}) + v_0\} - \int_{\sigma_0}^t F(s, \varphi^\varepsilon_s) ds\right| \\ &\leqq \int_{\sigma_0}^t |F(\sigma_0, \varphi^\varepsilon_{\sigma_0}) - F(s, \varphi^\varepsilon_s)| ds + |v_0|(t - \sigma_0) \\ &\leqq \varepsilon(t - \sigma_0) + \varepsilon(t - \sigma_0) = 2\varepsilon(t - \sigma_0) , \end{aligned}$$

because $|\sigma_0 - s| < \delta < \delta_0$ and $|\varphi^\varepsilon_{\sigma_0} - \varphi^\varepsilon_s| \leqq \delta_0/2 < \delta_0$ for $\sigma_0 \leqq s \leqq t \leqq \sigma_1$. If (3.3) holds for $t \in (\sigma_0, \sigma_k]$, $1 \leqq k \leqq N-1$, then for $t \in (\sigma_k, \sigma_{k+1}]$, it follows from (3.1) and (3.2) that

$$\left|\varphi^\varepsilon(t) - \varphi(0) - \int_\sigma^t F(s, \varphi^\varepsilon_s) ds\right| \leqq \left|\varphi^\varepsilon(t) - \varphi^\varepsilon(\sigma_k) - \int_{\sigma_k}^t F(s, \varphi^\varepsilon_s) ds\right|$$

$$+\left|\varphi^{\varepsilon}(\sigma_k)-\varphi(0)-\int_{\sigma}^{\sigma_k}F(s,\varphi_s^{\varepsilon})ds\right|$$
$$\leqq \left|(t-\sigma_k)\{F(\sigma_k,\varphi_{\sigma_k}^{\varepsilon})+v_k\}-\int_{\sigma_k}^{t}F(s,\varphi_s^{\varepsilon})ds\right|+2\varepsilon(\sigma_k-\sigma)$$
$$\leqq \int_{\sigma_k}^{t}|F(\sigma_k,\varphi_{\sigma_k}^{\varepsilon})-F(s,\varphi_s^{\varepsilon})|ds+|v_k|(t-\sigma_k)+2\varepsilon(\sigma_k-\sigma)$$
$$\leqq \varepsilon(t-\sigma_k)+\varepsilon(t-\sigma_k)+2\varepsilon(\sigma_k-\sigma)=2\varepsilon(t-\sigma)\ ,$$

because $|\sigma_k - s| < \delta < \delta_0$ and $|\varphi_{\sigma_k}^{\varepsilon}-\varphi_s^{\varepsilon}| \leqq \delta_0/2 < \delta_0$ for $\sigma_k \leqq s \leqq t \leqq \sigma_{k+1}$. Therefore (3.3) holds for $t\in I$.

Let

$$K^{\varepsilon}(\Delta)=\{\varphi^{\varepsilon}(v)|_I\colon v=(v_0,v_1,\cdots,v_{N-1}),\ |v_i|\leqq\varepsilon,\ 0\leqq i\leqq N-1\}\ .$$

Then $K^{\varepsilon}(\Delta)$ is a continuum in C, because the mapping $v\mapsto\varphi^{\varepsilon}(v)|_I$ is continuous by (A2) and the set $\{v=(v_0,v_1,\cdots,v_{N-1})\colon |v_i|\leqq\varepsilon,\ 0\leqq i\leqq N-1\}$ is a continuum.

For this set $K^{\varepsilon}(\Delta)$, we shall show that if x is a solution of (E) through (σ,φ), then

$$\text{(3.4)}\qquad \operatorname{dist}(x|_I,K^{\varepsilon}(\Delta))<\varepsilon\ ,$$

where $\operatorname{dist}(x|_I,K^{\varepsilon}(\Delta))=\inf\{\|x|_I-\xi\|\colon \xi\in K^{\varepsilon}(\Delta)\}$. Let x be fixed and let $y\colon(-\infty,T]\to R^n$ be the function satisfying $y_\sigma=\varphi$ and combining the points $(\sigma_0,x(\sigma_0)),(\sigma_1,x(\sigma_1)),\cdots,(\sigma_N,x(\sigma_N))$ linearly on I. Obviously $x\in L$ and $y\in L$. If we show $y|_I\in K^{\varepsilon}(\Delta)$, then (3.4) holds since $\|x-y\|\leqq(M+1)\delta<\varepsilon$. Let

$$v_0=\frac{1}{\sigma_1-\sigma_0}\int_{\sigma_0}^{\sigma_1}\{F(s,x_s)-F(\sigma,\varphi)\}ds\ .$$

Then by (3.1) and (3.2),

$$y(\sigma_1)=x(\sigma_1)=x(\sigma_0)+(\sigma_1-\sigma_0)\{F(\sigma,\varphi)+v_0\}$$

and

$$|v_0|\leqq\frac{1}{\sigma_1-\sigma_0}\int_{\sigma_0}^{\sigma_1}|F(s,x_s)-F(\sigma,\varphi)|ds\leqq\frac{1}{\sigma_1-\sigma_0}\varepsilon(\sigma_1-\sigma_0)=\varepsilon\ ,$$

because $|s-\sigma_0|<\delta<\delta_0$ and $|x_s-\varphi|=|x_s-x_{\sigma_0}|\leqq\delta_0/2<\delta_0$ for $\sigma_0\leqq s\leqq\sigma_1$. Therefore we have

$$y(t)=y(\sigma_0)+(t-\sigma_0)\{F(\sigma_0,y_{\sigma_0})+v_0\}\ ,\qquad t\in(\sigma_0,\sigma_1]\ .$$

Assume that there exist vectors $v_0,v_1,\cdots,v_{k-1}$ such that $|v_i|\leqq\varepsilon$, $0\leqq i\leqq k-1$, $1\leqq k\leqq N-1$, and

$$y(t) = y(\sigma_{i-1}) + (t - \sigma_{i-1})\{F(\sigma_{i-1}, y_{\sigma_{i-1}}) + v_{i-1}\}$$
$$\text{for} \quad t \in (\sigma_{i-1}, \sigma_i] , \qquad i = 1, \cdots, k .$$

If we put

$$v_k = \frac{1}{\sigma_{k+1} - \sigma_k} \int_{\sigma_k}^{\sigma_{k+1}} \{F(s, x_s) - F(\sigma_k, y_{\sigma_k})\} ds ,$$

then by (3.1) and (3.2),

$$y(\sigma_{k+1}) = x(\sigma_{k+1}) = x(\sigma_k) + (\sigma_{k+1} - \sigma_k)\{F(\sigma_k, y_{\sigma_k}) + v_k\}$$

and

$$|v_k| \leqq \frac{1}{\sigma_{k+1} - \sigma_k} \int_{\sigma_k}^{\sigma_{k+1}} |F(s, x_s) - F(\sigma_k, y_{\sigma_k})| ds \leqq \frac{1}{\sigma_{k+1} - \sigma_k} \varepsilon(\sigma_{k+1} - \sigma_k) = \varepsilon ,$$

because $|s - \sigma_k| < \delta < \delta_0$ and

$$\begin{aligned} |x_s - y_{\sigma_k}| &\leqq |x_s - x_{\sigma_k}| + |x_{\sigma_k} - y_{\sigma_k}| \leqq \frac{\delta_0}{2} + K(\sigma_k - \sigma) \sup_{\sigma - \sigma_k \leqq \theta \leqq 0} |x_{\sigma_k}(\theta) - y_{\sigma_k}(\theta)| \\ &\leqq \frac{\delta_0}{2} + K \sup_{t \in I} |x(t) - y(t)| \leqq \frac{\delta_0}{2} + K(M+1)\delta < \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0 . \end{aligned}$$

Thus we have

$$y(t) = y(\sigma_k) + (t - \sigma_k)\{F(\sigma_k, y_{\sigma_k}) + v_k\}$$

for $t \in (\sigma_k, \sigma_{k+1}]$, and hence y can be written as $\varphi^\varepsilon(v)$ for the above $v = (v_0, v_1, \cdots, v_{N-1})$. This implies $y|_I \in K^\varepsilon(\Delta)$.

Finally, we shall show the set S is a continuum in C. Clearly, S is compact in C. Assume that S is not connected. Then there exist two nonempty compact sets S_1 and S_2 such that $S_1 \cap S_2 = \varnothing$ and $S_1 \cup S_2 = S$. Let $\operatorname{dist}(S_1, S_2) = \inf\{\|\xi_1 - \xi_2\| : \xi_1 \in S_1, \xi_2 \in S_2\} = 2\eta > 0$, and let $U = U(S_1, \eta)$ be the open η-neighborhood of S_1 in C. For this $\eta > 0$, we may assume $0 < \varepsilon < \eta$. It follows from (3.4) that both $U \cap K^\varepsilon(\Delta)$ and $U^c \cap K^\varepsilon(\Delta)$ are nonempty, where U^c is the complement of U in C. Since $K^\varepsilon(\Delta)$ is connected, there exists a $v^\varepsilon = (v_0, v_1, \cdots, v_{N-1})$, $|v_i| \leqq \varepsilon$, $0 \leqq i \leqq N-1$, such that

$$\varphi^\varepsilon(v^\varepsilon)|_I \in \partial U \cap K^\varepsilon(\Delta) ,$$

where ∂U is the boundary of U in C.

Now replace ε by $\varepsilon_j > 0$ and $\varphi^{\varepsilon_j}(v^{\varepsilon_j})$ by φ^j, $j = 1, 2, \cdots$, where $\varepsilon_j \to 0$ as $j \to \infty$. Since $\varphi^j \in L$ and L is compact, we may assume that there exists a $z \in L$ such that $\{\varphi^j\}$ converges to z uniformly on I as $j \to \infty$, and hence $|\varphi^j_s - z_s| \to 0$ as $j \to \infty$ for any $s \in I$ by (A2). On the other hand, it follows from (3.3) that φ^j satisfies

$$\left|\varphi^j(t) - \varphi(0) - \int_\sigma^t F(s, \varphi_s^j)ds\right| \leqq 2\varepsilon_j(t-\sigma)\,, \qquad t \in I\,.$$

Therefore we have

$$z(t) - \varphi(0) - \int_\sigma^t F(s, z_s)ds = 0 \qquad \text{for} \quad t \in I\,,$$

and hence z is a solution of (E) through (σ, φ). Since ∂U is closed and $\varphi^j|_I \in \partial U$, we have $z|_I \in \partial U \cap S$. This is a contradiction. q.e.d.

COROLLARY 3.1. *Under the same assumptions as in Theorem* 3.1, *for any* $\varphi \in B$, *the sets*

$$\Sigma = \Sigma(\varphi) = \{x(T): x \textit{ is a solution of } (\mathrm{E}) \textit{ through } (\sigma, \varphi)\}$$

and

$$\mathscr{S} = \mathscr{S}(\varphi) = \{x_T: x \textit{ is a solution of } (\mathrm{E}) \textit{ through } (\sigma, \varphi)\}$$

are continua in R^n *and in* B, *respectively.*

PROOF. The mappings on $S(\varphi)$ defined by $x|_I \mapsto x(T)$ and $x|_I \mapsto x_T$ are continuous onto Σ and $\mathscr{S}$, respectively. Since $S(\varphi)$ is a continuum, Σ and $\mathscr{S}$ are also continua. q.e.d.

COROLLARY 3.2. *In addition to the assumptions as in Theorem* 3.1, *suppose* (A3*) *and* (A4*). *Then for any continuum* Q *in* B, *the sets*

$$S(Q) = \bigcup\{S(\varphi): \varphi \in Q\}\,,$$
$$\Sigma(Q) = \bigcup\{\Sigma(\varphi): \varphi \in Q\}$$

and

$$\mathscr{S}(Q) = \bigcup\{\mathscr{S}(\varphi): \varphi \in Q\}$$

are continua in C, *in* R^n *and in* B, *respectively.*

PROOF. We shall prove this only for the set $S = S(Q)$, since the arguments for the other sets are similar, in particular, $\Sigma(Q)$ is a continuous image of S.

First we shall show that S is compact in C. Let $\{u^k\}$ be any sequence in S. Then for each u^k, there exists a solution x^k of (E) such that $x^k|_I = u^k$ and $x_\sigma^k \in Q$. Since Q is compact and the mapping $x_\sigma^k \mapsto x_\sigma^k(0) = u^k(\sigma)$ is continuous by (A4*), the family $\{u^k\}$ is uniformly bounded and equicontinuous. By taking a subsequence if necessary, we may assume that $x_\sigma^k \to \varphi^0$ in B and $u^k \to u^0$ uniformly on I as $k \to \infty$ for some $\varphi^0 \in Q$ and $u^0 \in C$. Clearly, $u^0(\sigma) = \varphi^0(0)$.

Define $x^0: (-\infty, T] \to R^n$ by $x^0(t) = u^0(t)$ on I and $x_\sigma^0 = \varphi^0$. Since x^k is a solution of (E), it satisfies

$$x^k(t) = x^k(\sigma) + \int_\sigma^t F(s, x_s^k)ds\,, \qquad t \in I\,. \tag{3.5}$$

For any $s \in I$, by (A2) and (A3*),

$$\begin{aligned}|x_s^k - x_s^0| &\leqq K(s-\sigma) \sup_{\sigma - s \leqq \theta \leqq 0} |x_s^k(\theta) - x_s^0(\theta)| + |T^{s-\sigma}(x_\sigma^k - \varphi^0)|_{s-\sigma} \\ &\leqq K(s-\sigma)\|u^k - u^0\| + M(s-\sigma)|x_\sigma^k - \varphi^0|\,,\end{aligned}$$

and hence $|x_s^k - x_s^0| \to 0$ as $k \to \infty$. It follows from this and (3.5) that

$$x^0(t) = x^0(\sigma) + \int_\sigma^t F(s, x_s^0)ds\,, \qquad t \in I\,.$$

Thus x^0 is a solution of (E) and $x_\sigma^0 = \varphi^0 \in Q$, and hence $u^0 = x^0|_I \in S$. This means S is compact in C.

Now we prove the connectedness of S. Assume that S is not connected. Then there exist two nonempty compact sets S_1 and S_2 such that $S_1 \cap S_2 = \varnothing$ and $S_1 \cup S_2 = S$. Define

$$Q_i = \{\varphi \in Q \colon S(\varphi) \cap S_i \text{ is nonempty}\}\,, \qquad i = 1, 2\,.$$

Clearly, Q_i is nonempty, $i = 1, 2$, and $Q = Q_1 \cup Q_2$. The compactness of S_i and the same arguments as in the proof of the compactness of $S = S(Q)$ imply that Q_i is compact in B, $i = 1, 2$. If there is a φ in $Q_1 \cap Q_2$, then $S(\varphi) \cap S_i \equiv X_i$, $i = 1, 2$, are nonempty compact sets, and $X_1 \cup X_2 = S(\varphi)$ while $X_1 \cap X_2 = \varnothing$. This contradicts the connectedness of $S(\varphi)$. Therefore $Q_1 \cap Q_2 = \varnothing$. This contradicts the connectedness of Q. Thus S is connected. q.e.d.

4. Boundary value problems. In this section we assume that the elements of B are R-valued functions defined on $(-\infty, 0]$. For any φ and ψ in B, the notation $\varphi \leqq \psi$ means that $\varphi(\theta) \leqq \psi(\theta)$ for all $\theta \in (-\infty, 0]$, and define a nonnegative real valued function $\lambda(\varphi)$ by

$$\lambda(\varphi)(\theta) = |\varphi(\theta)| \qquad \text{for} \quad \theta \in (-\infty, 0]\,,$$

where $|\varphi(\theta)|$ is the absolute value of $\varphi(\theta)$.

Now we assume the following hypothesis for B.

(A5) λ is a continuous mapping from$B$ into B.

From this hypothesis, we obtain the following lemma.

LEMMA 4.1. *Let φ and ψ be in B. If we assume* (A5), *then the functions $\varphi \vee \psi$ and $\varphi \wedge \psi$ defined by $(\varphi \vee \psi)(\theta) = \max\{\varphi(\theta), \psi(\theta)\}$ and $(\varphi \wedge \psi)(\theta) = \min\{\varphi(\theta), \psi(\theta)\}$ for $\theta \in (-\infty, 0]$ are elements in B, and furthermore $\varphi \vee \psi$ and $\varphi \wedge \psi$ are continuous in $(\varphi, \psi) \in B \times B$.*

PROOF. The equalities $\varphi \vee \psi = \{\varphi + \psi + \lambda(\varphi - \psi)\}/2$ and $\varphi \wedge \psi = \{\varphi + \psi - \lambda(\varphi - \psi)\}/2$ complete the proof.

We consider the following boundary value problem for the second order scalar functional differential equation

$$\text{(E1)} \qquad x''(t) = f(t, x_t, x'(t)) ,$$

$$\text{(4.1)} \qquad x_\sigma = \psi \quad \text{and} \quad x(T) = A ,$$

where f is continuous on a certain subset of $I \times B \times R$, $I = [\sigma, T]$, and $\psi \in B$, $A \in R$. Equation (E1) is equivalent to the system

$$\text{(E2)} \qquad x'(t) = y(t) , \quad y'(t) = f(t, x_t, y(t)) .$$

As was seen in Example 2.1, R can be written as $\mathscr{B}_0$ for some space B_0 which satisfies all axioms (A1), (A2), (A3*) and (A4*). If the space B satisfies one of these axioms, then the product space $B \times B_0$ satisfies the same axiom by Lemma 2.1. By Lemma 2.2, we can assume that the domain of f is a subset of $I \times B \times B_0$. Therefore the results obtained in Sections 2 and 3 are applicable to equations (E2) and (E3) which will appear in Lemma 4.2.

LEMMA 4.2. *Suppose* (A1) *and* (A2). *Let f be a bounded and continuous function on $I \times B \times R$. For any fixed ξ and η in B such that $\xi \leqq \eta$, let $q = q(\xi, \eta)$ be a mapping from $Z = \{\sigma\} \times \{r \in R: \xi(0) \leqq r \leqq \eta(0)\} \times R$ into $B \times R$ such that*

$$q(\sigma, r, y) = (\mu(r), y) ,$$

where

$$\mu(r) = \begin{cases} \dfrac{\eta(0) - r}{\eta(0) - \xi(0)}\xi + \dfrac{r - \xi(0)}{\eta(0) - \xi(0)}\eta & \text{if} \quad \xi(0) < \eta(0) \\ \psi & \text{if} \quad \xi(0) = \eta(0) \end{cases}$$

for any fixed $\psi \in B$ such that $\xi \leqq \psi \leqq \eta$. Then for any continuum H in Z, the set $Q = Q(\xi, \eta; H)$ defined by $q(H)$ is a continuum in $Y = \{\varphi \in B: \xi \leqq \varphi \leqq \eta\} \times R$, and the set

$$\sum (\xi, \eta; H; f) = \{(x(T), y(T)) : (x, y) \text{ is a solution of (E2) such that } (x_\sigma, y(\sigma)) \in Q\}$$

is a continuum in R^2. Here notice that any solution of (E2) *is continuable to $t = T$ by the boundedness of f.*

PROOF. Since we have

$$\text{(4.2)} \qquad \mu(r_1) - \mu(r_2) = \frac{r_1 - r_2}{\eta(0) - \xi(0)}(\eta - \xi) \quad \text{if} \quad \xi(0) < \eta(0) ,$$

q is continuous from Z into Y. Therefore the first part is obvious.

We now prove $\Sigma = \Sigma(\xi, \eta; H; f)$ is a continuum in R^2. Since f is bounded and H is compact, Σ can be written as $\{(x(T), y(T)) : (x, y)$ is a solution of (E3) such that $(x_\sigma, y(\sigma)) \in Q\}$ for a system

$$x'(t) = P_M(y(t)) , \qquad y'(t) = f(t, x_t, y(t)) , \tag{E3}$$

where $P_M : R \to R$ is a continuous function such that

$$P_M(y) = \begin{cases} M & \text{for } \; y > M \\ y & \text{for } \; |y| \leqq M \\ -M & \text{for } \; y < -M \end{cases}$$

and $M > 0$ is so large that any solution (x, y) of (E3) satisfying $(x_\sigma, y(\sigma)) \in Q$ becomes a solution of (E2) and *vice versa*. The right hand sides in (E3) are bounded.

First we show that Σ is compact. Let $\{s^k\}$ be any sequence in Σ. Then there exists a solution (x^k, y^k) of (E3) and $h^k \in H$ such that $s^k = (x^k(T), y^k(T))$ and $(x^k_\sigma, y^k(\sigma)) = q(h^k)$. Here, notice that $h^k = (\sigma, r^k, y^k(\sigma))$ and $\mu(r^k) = x^k_\sigma$ for some $r^k \in [\xi(0), \eta(0)]$. By the compactness of H, we can assume that there exists an $h^0 = (\sigma, r^0, y^0) \in H$ such that $h^k \to h^0$, that is, $r^k \to r^0$ and $y^k(\sigma) \to y^0$ as $k \to \infty$. Since the family of solutions $\{(x^k, y^k)\}$ is uniformly bounded and equicontinuous on I, we can assume that there exist two continuous functions $\hat{x}$ and $\hat{y}$ defined on I such that x^k and y^k converge to $\hat{x}$ and $\hat{y}$ unifomly on I, respectively. Notice that $\mu(r^0)(0) = \lim_{k\to\infty} \mu(r^k)(0) = \lim_{k\to\infty} x^k(\sigma) = \hat{x}(\sigma)$ and $\hat{y}(\sigma) = y^0$.

Let x and y be the functions defined on $(-\infty, T]$ and I, respectively, such that $x_\sigma = \mu(r^0)$ and $x(t) = \hat{x}(t)$ for $t \in I$ and $y(t) = \hat{y}(t)$ for $t \in I$. Then $(x_\sigma, y(\sigma)) = (\mu(r^0), \hat{y}(\sigma)) = q(h^0) \in Q$. Since (x^k, y^k) is a solution of (E3), we obtain

$$x^k(t) = x^k(\sigma) + \int_\sigma^t P_M(y^k(s))ds \qquad \text{for} \quad t \in I ,$$

$$y^k(t) = y^k(\sigma) + \int_\sigma^t f(s, x^k_s, y^k(s))ds \qquad \text{for} \quad t \in I .$$

On the other hand, for any $s \in I$, by $(A2)$,

$$\begin{aligned} |x^k_s - x_s| &\leqq K(s-\sigma) \sup_{\sigma - s \leqq \theta \leqq 0} |x^k_s(\theta) - x_s(\theta)| + |T^{s-\sigma}(x^k_\sigma - x_\sigma)|_{s-\sigma} \\ &\leqq K(s-\sigma) \sup_{t \in I} |x^k(t) - x(t)| + |T^{s-\sigma}(\mu(r^k) - \mu(r^0))|_{s-\sigma} . \end{aligned}$$

It then follows from this, (4.2) and the linearity of $T^{s-\sigma}$ that $|x^k_s - x_s| \to 0$ as $k \to \infty$ for any $s \in I$. Therefore we obtain

$$x(t) = x(\sigma) + \int_\sigma^t P_M(y(s))ds \qquad \text{for} \quad t \in I ,$$

$$y(t) = y(\sigma) + \int_\sigma^t f(s, x_s, y(s))ds \qquad \text{for} \quad t \in I\,,$$

that is, (x, y) is a solution of (E3). Since $(x_\sigma, y(\sigma)) \in Q$, we obtain $(x(T), y(T)) \in \Sigma$. Obviously, $s^k = (x^k(T), y^k(T)) \to (x(T), y(T))$ as $k \to \infty$, which implies that Σ is compact.

The connectedness of Σ can be proved by using the same arguments as in the proof of the connectedness of S in Corollary 3.2 and by the results in Corollary 3.1. q.e.d.

THEOREM 4.1. *Suppose* (A1), (A2), (A4*) *and* (A5). *Let α and β be R-valued functions defined on $(-\infty, T]$ and twice continuously differentiable on $I = [\sigma, T]$ such that $\alpha(s) \leqq \beta(s)$ for $s \in (-\infty, T]$ and $\alpha_\sigma, \beta_\sigma \in B$. Let V and W be R-valued continuously differentiable functions on the domain $\{(t, x) \colon t \in I, \alpha(t) \leqq x \leqq \beta(t)\}$ such that $V(t, x) \leqq W(t, x)$ on this domain. Furthermore, assume that f is a bounded and continuous function on the domain $D = \{(t, \varphi, y) \colon t \in I, \varphi \in B, \alpha_t \leqq \varphi \leqq \beta_t, V(t, \varphi(0)) \leqq y \leqq W(t, \varphi(0))\}$ and that the following inequalities hold;*

$$\begin{cases} \alpha'(t) \geqq V(t, \alpha(t)) & \text{for} \quad t \in I \\ \beta'(t) \leqq W(t, \beta(t)) & \text{for} \quad t \in I\,, \end{cases} \tag{4.3}$$

$$\begin{cases} \alpha''(t) \geqq f(t, \alpha_t, \alpha'(t)) & \text{if} \quad \alpha'(t) \leqq W(t, \alpha(t)) \quad \text{for} \quad t \in I \\ \beta''(t) \leqq f(t, \beta_t, \beta'(t)) & \text{if} \quad \beta'(t) \geqq V(t, \beta(t)) \quad \text{for} \quad t \in I\,, \end{cases} \tag{4.4}$$

$$\begin{cases} f(t, \varphi, V(t, \varphi(0))) - V_t(t, \varphi(0)) - V_x(t, \varphi(0)) V(, t\varphi(0)) \geqq 0 \\ f(t, \varphi, W(t, \varphi(0))) - W_t(t, \varphi(0)) - W_x(t, \varphi(0)) W(t, \varphi(0)) \leqq 0 \\ \qquad\qquad \text{for} \quad \mathrm{t} \in I, \varphi \in B, \alpha_t \leqq \varphi \leqq \beta_t \end{cases} \tag{4.5}$$

and

$$\begin{cases} f(t, \varphi, \alpha'(t)) \leqq f(t, \alpha_t, \alpha'(t)) \text{ if } \alpha(t) = \varphi(0) \text{ and } \alpha'(t) \leqq W(t, \alpha(t)) \\ f(t, \varphi, \beta'(t)) \geqq f(t, \beta_t, \beta'(t)) \text{ if } \beta(t) = \varphi(0) \text{ and } \beta'(t) \geqq V(t, \beta(t)) \\ \qquad\qquad \text{for} \quad t \in I, \varphi \in B, \alpha_t \leqq \varphi \leqq \beta_t\,. \end{cases} \tag{4.6}$$

Then for any number A such that $\alpha(T) \leqq A \leqq \beta(T)$, there exists a $\psi \in B$, $\alpha_\sigma \leqq \psi \leqq \beta_\sigma$, for which (E1) *has at least one solution satisfying* (4.1). *In particular, if $\alpha(\sigma) = \beta(\sigma)$, we can arbitrarily choose ψ such that $\alpha_\sigma \leqq \psi \leqq \beta_\sigma$.*

PROOF. We consider an equivalent system (E2) instead of (E1).

In order to extend f to $I \times B \times R$, first we construct an extension g of f on the domain $t \in I$, $\alpha_t \leqq \varphi \leqq \beta_t$, $y \in R$ so that the following inequalities hold;

$$\alpha''(t) \geqq g(t, \alpha_t, \alpha'(t)) \quad \text{for} \quad t \in I, \tag{4.7}$$

$$\beta''(t) \leqq g(t, \beta_t, \beta'(t)) \quad \text{for} \quad t \in I, \tag{4.8}$$

$$g(t, \varphi, y) \leqq f(t, \varphi, W(t, \varphi(0))) \text{ for } t \in I, \alpha_t \leqq \varphi \leqq \beta_t, y \geqq W(t, \varphi(0)), \tag{4.9}$$

$$g(t, \varphi, y) \geqq f(t, \varphi, V(t, \varphi(0))) \text{ for } t \in I, \alpha_t \leqq \varphi \leqq \beta_t, y \leqq V(t, \varphi(0)), \tag{4.10}$$

$$g(t, \varphi, \alpha'(t)) \leqq g(t, \alpha_t, \alpha'(t)) \text{ for } t \in I, \alpha_t \leqq \varphi \leqq \beta_t, \varphi(0) = \alpha(t) \tag{4.11}$$

and

$$g(t, \varphi, \beta'(t)) \geqq g(t, \beta_t, \beta'(t)) \text{ for } t \in I, \alpha_t \leqq \varphi \leqq \beta_t, \varphi(0) = \beta(t). \tag{4.12}$$

Set $g = f$ on D.

For $t \in I$, $\alpha_t \leqq \varphi \leqq \beta_t$ and $y > W(t, \varphi(0))$, g is constructed in the following way. If $t \in I_\alpha = \{t \in I: \alpha'(t) > W(t, \alpha(t))\}$, $\varphi = \alpha_t$ and $y = \alpha'(t)$, then we define g by

$$g(t, \varphi, y) = \min\{\alpha''(t), f(t, \alpha_t, W(t, \alpha(t)))\}.$$

Then clearly (4.7) holds. For $t \in I_\alpha$, $\varphi = \alpha_t$ and $W(t, \alpha(t)) < y < \alpha'(t)$, define g by joining $f(t, \alpha_t, W(t, \alpha(t)))$ and $g(t, \alpha_t, \alpha'(t))$ linearly in y, that is,

$$g(t, \varphi, y) = \frac{(\alpha'(t) - y)f(t, \alpha_t, W(t, \alpha(t))) + (y - W(t, \alpha(t)))g(t, \alpha_t, \alpha'(t))}{\alpha'(t) - W(t, \alpha(t))}$$

For $t \in I$, $\varphi = \alpha_t$ and $y > \max\{\alpha'(t), W(t, \alpha(t))\} \equiv \gamma(t)$, let

$$g(t, \varphi, y) = g(t, \alpha_t, \gamma(t)).$$

For $t \in I$, $\alpha_t \leqq \varphi \leqq \beta_t$ and $y > W(t, \varphi(0))$, let

$$\begin{aligned} g(t, \varphi, y) = {} & f(t, \varphi, W(t, \varphi(0))) - f(t, \alpha_t, W(t, \alpha(t))) \\ & + g(t, \alpha_t, W(t, \alpha(t)) + y - W(t, \varphi(0))). \end{aligned}$$

Then it is easy to verify that (4.9) and (4.11) hold. Similarly, we can construct g for $t \in I$, $\alpha_t \leqq \varphi \leqq \beta_t$ and $y < V(t, \varphi(0))$ so that (4.8), (4.10) and (4.12) hold. Obviously, g is bounded and continuous under (A1) and (A4*).

For any $t \in I$ and $\varphi \in B$, if we define a function $\Gamma_t\varphi$ by $\Gamma_t\varphi = \alpha_t \vee (\beta_t \wedge \varphi)$, that is,

$$(\Gamma_t\varphi)(\theta) = \max\{\alpha_t(\theta), \min\{\beta_t(\theta), \varphi(\theta)\}\} \quad \text{for} \quad \theta \in (-\infty, 0],$$

then Γ_t is a continuous mapping from B into $\{\varphi \in B: \alpha_t \leqq \varphi \leqq \beta_t\}$ by Lemma 4.1.

Now we define an extension h of g on $I \times B \times R$ by

$$h(t, \varphi, y) = \begin{cases} g(t, \Gamma_t\varphi, y) + \dfrac{\varphi(0) - \beta(t)}{1 + \varphi(0) - \beta(t)} & \text{if} \quad \varphi(0) > \beta(t) \\ g(t, \Gamma_t\varphi, y) \quad \text{if} \quad \alpha(t) \leqq \varphi(0) \leqq \beta(t) & \\ g(t, \Gamma_t\varphi, y) - \dfrac{\alpha(t) - \varphi(0)}{1 + \alpha(t) - \varphi(0)} & \text{if} \quad \varphi(0) < \alpha(t) . \end{cases}$$

By (A4*), h is continuous. Thus we obtain a bounded and continuous extension h of f.

Instead of (E1) or (E2), we now consider the equation

$$\text{(E4)} \qquad x''(t) = h(t, x_t, x'(t))$$

or an equivalent system

$$\text{(E5)} \qquad x'(t) = y(t) , \qquad y'(t) = h(t, x_t, y(t)) .$$

Let $D_0, D_1, \cdots, D_6$ be the sets of points (t, x, y) such that

$$\begin{aligned} &D_0\colon t \in I, \alpha(t) \leqq x \leqq \beta(t) , && V(t, x) \leqq y \leqq W(t, x) , \\ &D_1\colon t \in I, \alpha(t) \leqq x \leqq \beta(t) , && y > W(t, x) , \\ &D_2\colon t \in I , \quad x < \alpha(t) , && y \geqq \alpha'(t) , \\ &D_3\colon t \in I , \quad x < \alpha(t) , && y \leqq \alpha'(t) , \\ &D_4\colon t \in I, \alpha(t) \leqq x \leqq \beta(t) , && y < V(t, x) , \\ &D_5\colon t \in I , \quad x > \beta(t) , && y \leqq \beta'(t) \end{aligned}$$

and

$$D_6\colon t \in I , \quad x > \beta(t) , \quad y \geqq \beta'(t) .$$

We denote the intersection of D_i and the hyperplane $t = T$ by D_i^*, $i = 0, 1, \cdots, 6$.

Consider a solution x of (E4) with initial value $(x_\sigma, x'(\sigma)) \in B \times R$. If

$$x(t_0) < \alpha(t_0) \quad \text{and} \quad x'(t_0) = \alpha'(t_0)$$

for some $t_0 \in I$, then

$$\begin{aligned} x''(t_0) &= h(t_0, x_{t_0}, x'(t_0)) \\ &= g(t_0, \Gamma_{t_0}x_{t_0}, \alpha'(t_0)) - \frac{\alpha(t_0) - x(t_0)}{1 + \alpha(t_0) - x(t_0)} \\ &< g(t_0, \Gamma_{t_0}x_{t_0}, \alpha'(t_0)) \\ &\leqq g(t_0, \alpha_{t_0}, \alpha'(t_0)) \qquad \text{(by (4.11))} \\ &\leqq \alpha''(t_0) \qquad \text{(by 4.7))} . \end{aligned}$$

This means that if $(t, x(t), x'(t))$ is in D_2 at $t = t_1$, then it is in D_2 for $\sigma \leqq t \leqq t_1$, and that if $(t, x(t), x'(t))$ is in D_3 at $t = t_2$, then it is in D_3 for

$t_2 \leqq t \leqq T$. In other words, D_2 is negatively invariant and D_3 is positively invariant. Similarly, we can show that D_5 is negatively invariant and D_6 is positively invariant.

Let H be any continuum in the intersection of D_0 and the hyperplane $t = \sigma$ containing two points $(\sigma, \alpha(\sigma), y^1)$ and $(\sigma, \beta(\sigma), y^2)$, where y^1 and y^2 are any numbers such that

$$V(\sigma, \alpha(\sigma)) \leqq y^1 \leqq \alpha'(\sigma)\ , \quad W(\sigma, \beta(\sigma)) \geqq y^2 \geqq \beta'(\sigma)\ . \tag{4.13}$$

For this set H and the functions α_σ and β_σ, we consider the sets $Q = Q(\alpha_\sigma, \beta_\sigma; H)$ and $\Sigma = \Sigma(\alpha_\sigma, \beta_\sigma; H; h)$ defined in Lemma 4.2 for system (E5). By Lemma 4.2, $\Sigma^* = \{T\} \times \Sigma$ is a continuum.

In order to see that Σ^* is contained in $D_0^* \cup D_3^* \cup D_6^*$, we now consider a solution x of (E4) with initial value $(x_\sigma, x'(\sigma)) \in Q$. By negative invariance of D_2 and D_5, $(t, x(t), x'(t))$ cannot enter $D_2 \cup D_5$. Next we shall show that $(t, x(t), x'(t))$ cannot enter D_1. If it did, then there is a $t_0 \in I$ such that $(t_0, x(t_0), x'(t_0)) \in D_1$, that is,

$$\alpha(t_0) \leqq x(t_0) \leqq \beta(t_0) \quad \text{and} \quad x'(t_0) > W(t_0, x(t_0))\ .$$

Then, by the above arguments and $\alpha(\sigma) \leqq x(\sigma) \leqq \beta(\sigma)$, we have

$$\alpha(t) \leqq x(t) \leqq \beta(t) \qquad \text{for} \quad \sigma \leqq t \leqq t_0\ .$$

Along this solution, set

$$\omega(t) = [x'(t) - W(t, x(t))] \exp \int_{t_0}^{t} W_x(s, x(s))ds\ .$$

Then, as long as $x'(t) > W(t, x(t))$ and $\sigma \leqq t \leqq t_0$,

$$\begin{aligned}
&\omega'(t) \exp\left(-\int_{t_0}^{t} W_x(s, x(s))ds\right) \\
&\quad = x''(t) - W_t(t, x(t)) - W_x(t, x(t))W(t, x(t)) \\
&\quad = g(t, x_t, x'(t)) - W_t(t, x(t)) - W_x(t, x(t))W(t, x(t)) \\
&\quad \leqq f(t, x_t, W(t, x_t(0))) - W_t(t, x_t(0)) - W_x(t, x_t(0))W(t, x_t(0)) \quad \text{(by (4.9))} \\
&\quad \leqq 0 \qquad \text{(by (4.5))}\ .
\end{aligned}$$

From this and $\omega(t_0) > 0$, we obtain $\omega(t) > 0$, that is, $x'(t) > W(t, x(t))$ for $\sigma \leqq t \leqq t_0$. This contradicts the assumption $(x_\sigma, x'(\sigma)) \in Q$ or $x'(\sigma) \leqq W(\sigma, x(\sigma))$, which shows that $(t, x(t), x'(t))$ cannot enter D_1. Similarly, we can show also that $(t, x(t), x'(t))$ cannot enter D_4. Therefore Σ^* is contained in $D_0^* \cup D_3^* \cup D_6^*$.

Now we shall show that both $\Sigma^* \cap \overline{D_3^*}$ and $\Sigma^* \cap \overline{D_6^*}$ are nonempty, where $\overline{D_i^*}$ is the closure of D_i^*, $i = 3, 6$. Let x^k be one of the solution of (E4) such that

$$x_\sigma^k = \alpha_\sigma \quad \text{and} \quad x^{k\prime}(\sigma) = y^1 - 1/k$$

for $k = 1, 2, \cdots$. Then $(t, x^k(t), x^{k\prime}(t)) \in D_3$ for $\sigma < t \leqq T$ by (4.13) and positive invariance of D_3. We may assume that there is a solution x^0 of (E4) such that $x_\sigma^0 = \alpha_\sigma$, $x^{0\prime}(\sigma) = y^1$ and $(x^k, x^{k\prime})$ converges to $(x^0, x^{0\prime})$ uniformly on I as $k \to \infty$ by taking a subsequence if necessary. This solution x^0 satisfies $(x_\sigma^0, x^{0\prime}(\sigma)) \in Q$ and $(T, x^0(T), x^{0\prime}(T)) \in \overline{D_3^*}$, and hence $\Sigma^* \cap \overline{D_3^*}$ is nonempty. Similarly, $\Sigma^* \cap \overline{D_6^*}$ is also nonempty.

For an arbitrary A such that $\alpha(T) \leqq A \leqq \beta(T)$, let N be the set of points (T, A, y) such that $V(T, A) \leqq y \leqq W(T, A)$. Since Σ^* is continuum, it must intersect with the set N, and hence there exists a solution x of (E4) satisfying $(x_\sigma, x'(\sigma)) \in Q$ and $x(T) = A$. Clearly, this solution x satisfies $(t, x(t), x'(t)) \in D_0$ for $t \in I$, and hence $(t, x_t, x'(t)) \in D$ for $t \in I$. Therefore x is a solution of (E1). This completes the proof.

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