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KINEMATICS AND DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS

——Rolling a ball with a prescribed locus of contact——

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The simplest and most illustrative of the kinematic models we discuss in this paper is the rolling of a ball on its tangent plane. Suppose a smooth curve x_t is given on the unit sphere S^2 (boundary of the unit ball B). Is it possible to roll (without skidding or spinning) the ball B on the tangent plane Σ to S^2 at x_0 in such a way that at each time instant t the point x_t becomes a point of contact with the plane Σ? We shall show that this is possible and that the locus y_t of points of contact on Σ is indeed the development of the curve x_t in the sense of E. Cartan.

When we replace S^2 by an arbitrary smooth surface M, the rolling of M on its tangent plane gives rise to a kinematic interpretation of the Levi-Civita connection for M. We also find that we must impose a certain condition on the curve x_t to prevent the rolling from degenerating into an instantaneous standstill at any instant. This condition is that the tangent vector of x_t is not a principal direction for the zero principal curvature; this condition is satisfied if the curve x_t does not go through a flat point.

In the end we shall study the model of rolling an n-dimensional submanifold M on another n-dimensional submanifold N in a Euclidean space E^m and obtain a kinematic interpretation of the second fundamental form and the normal connection of a submanifold.

The paper is organized as follows. Section 1 is devoted to the basic concepts in kinematics we need. We define the notion of rolling (without skidding or spinning). In Section 2 we discuss the model of rolling a ball and extend it to higher dimensions in Section 3. In Section 4 we treat the rolling of an arbitrary surface on a plane. Section 5 deals with rolling of a surface on another surface. Finally, in Section 6, we discuss the most general question—rolling an n-dimensional submanifold

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M on another n-dimensional submanifold N in E^m. The reference for submanifolds is [2, Vol. II].

1. Motion, instantaneous motion, and rolling. By a motion of a Euclidean space E^m we mean an orientation-preserving isometry of E^m. If we take an arbitrary Euclidean (i.e., rectangular) coordinate system, a motion f can be expressed by an $(m+1) \times (m+1)$ matrix of the form

$$\begin{bmatrix} C & c \\ 0 & 1 \end{bmatrix} \tag{1}$$

where $C \in SO(m)$ and c is an m-dimensional (column) vector. A point x is mapped by f upon $f(x) = Cx + c$.

By a 1-parametric motion $\{f_t\}$, $t \in J$, where J is an open or closed interval containing 0 in its interior, we mean a differentiable mapping of J into the space of matrices of the form (1), namely,

$$f_t = \begin{bmatrix} C_t & c_t \\ 0 & 1 \end{bmatrix} \tag{2}$$

where C_t is an $SO(m)$-valued differentiable function of t and c_t is a vector-valued differentiable function of t such that

$$f_0 = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \text{ (identity transformation).}$$

We remark once and for all that it does not matter which Euclidean coordinate system we use in expressing motions and related concepts in the following.

Given a 1-parametric motion $\{f_t\}$, we can define a time-dependent vector field X_t on E^m as follows. Fix t. Let y be an arbitrary point and let $x = f_t^{-1}(y)$. Let $(X_t)_y$ be the tangent vector $[df_u(x)/du]_{u=t}$ of the orbit $f_u(x)$ at $u = t$, namely, at the point $y = f_t(x)$.

Using the matrix (2) we can obtain the matrix representing the vector field X_t as follows. Write

$$(df/dt)f_t^{-1} = \begin{bmatrix} S_t & v_t \\ 0 & 0 \end{bmatrix} \tag{3}$$

where

$$S_t = (dC/dt)C_t^{-1} \text{ and } v_t = -S_t c_t + dc/dt .$$

Then it is easy to verify that

$$(X_t)_y = S_t y + v_t .$$

For this reason, we call (3) the *instantaneous motion* at instant t. If x is an arbitrary point, we have

$$df_t(x)/dt = (dC/dt)x + dc/dt = (dC/dt)C_t^{-1}C_t x + dc/dt \ ,$$

namely,

$$df_t(x)/dt = S_t C_t x + dc/dt \ . \tag{4}$$

The instantaneous motion (3) is called an *instantaneous* standstill if $S_t = 0$ and $v_t = 0$. It is called an *instantaneous translation* if $S_t = 0$ and $v_t \neq 0$. In this case, $(X_t)_y = v_t$ for all points y, namely, all points have the same velocity at instant t.

We say that (3) is an *instantaneous rotation* if there exists a point y_0 such that $(X_t)_{y_0} = 0$. If $x_0 = f_t^{-1}(y_0)$, then $df_t(x_0)/dt = 0$ and y_0 is called a *center of instantaneous rotation*. We shall also require that $S_t \neq 0$ to avoid an instantaneous standstill.

In the case where $m = 3$, an instantaneous rotation has an axis, namely, the line consisting of all points y such that $(X_t)_y = 0$. Suppose $(X_t)_y = (X_t)_{y_0} = 0$. Then from (4) we obtain $S_t(y - y_0) = 0$. Since the null space of the skew-symmetric transformation $S_t \neq 0$ is a 1-dimensional subspace, the set of y with $(X_t)_y = 0$ forms a straight line. Indeed, for $S_t \neq 0$, there is a uniquely determined vector ω_t such that $S_t(U)$ is equal to the cross product $\omega_t \times U$ for every vector U. The vector ω_t is called the *angular velocity* at instant t.

If x is an arbitrary point, the velocity $df_t(x)/dt$ in (4) can be expressed by

$$df_t(x)/dt = \omega_t \times \overrightarrow{f_t(0)f_t(x)} + df_t(0)/dt \tag{5}$$

since $c_t = f_t(0)$ and $C_t x$ is equal to the vector $\overrightarrow{f_t(0)f_t(x)}$ from $f_t(0)$ to $f_t(x)$.

We shall now define rolling of a surface M on another surface N. Consider a 1-parametric motion $\{f_t\}$ with the property that for each instant t the image $f_t(M)$ is tangent to N at a certain point y_t. If $(df/dt)f_t^{-1}$ is an instantaneous translation, we have *skidding* at instant t. Suppose $(df/dt)f_t^{-1}$ is an instantaneous rotation with y_t as center and $S_t \neq 0$. If the angular velocity ω_t is normal to N at y_t, then we have *spinning* at instant t. If ω_t is tangent to N at y_t, then we say that $(df/dt)f_t^{-1}$ is a rolling. Thus the 1-parametric motion $\{f_t\}$ is a rolling of M on N (without skidding or spinning) if, for each instant t, $(df/dt)f_t^{-1}$ is a rolling in the above sense. See [1], pp. 78-79; section called *Roulement et pivotment d'une surface mobile sur une surface fixe*.

REMARK. If $\{f_t\}$ is a rolling of M on N, then $\{f_t^{-1}\}$ is a rolling of N on M.

2. Rolling a ball on a plane. Let us consider the unit sphere S^2 and the tangent plane Σ of S^2 at x_0. We shall take a rectangular coordinate system in E^3 such that S^2 is given by $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$, $x_0 = (0, 0, -1)$ and Σ is given by $x^3 = -1$. Let e_1, e_2, e_3 be the unit vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, respectively.

Suppose x_t is a smooth curve (with non-vanishing tangent vector dx/dt) on S^2 starting at x_0. We wish to roll S^2 on Σ in such a way that at instant t the point x_t becomes a point of contact with Σ. Let the rolling $\{f_t\}$ be given by (2) and let $y_t = f_t(x_t)$.

Since $f_t(S^2)$ is tangent to Σ at y_t, we have

$$C_t x_t = -e_3 . \tag{6}$$

Thus

$$y_t = C_t x_t + c_t = c_t - e_3$$

that is,

$$c_t = y_t + e_3 . \tag{7}$$

Since y_t is a center of instantaneous rotation, we have from (4), (6), and (7)

$$S_t(e_3) = dy/dt . \tag{8}$$

Since the angular velocity ω_t lies on Σ (by definition of rolling) and since

$$\omega_t \times e_3 = S_t(e_3) = dy/dt ,$$

it follows that ω_t is perpendicular to the tangent vector dy/dt of the curve y_t and $\{dy/dt, \omega_t\}$ have the same orientation as $\{e_1, e_2\}$.

We shall now proceed to prove that the curve y_t is the development of the curve x_t into the tangent plane Σ. First we observe

$$C_t(dx/dt) = dy/dt . \tag{9}$$

This can be seen as follows. From $y_t = C_t x_t + c_t$, we have

$$dy/dt = (dC/dt)x_t + C_t(dx/dt) + dc/dt .$$

Since y_t is a center of instantaneous rotation, (4) gives

$$(dC/dt)x_t + dc/dt = 0 .$$

These two equations give rise to (9).

We define vector fields $b_1 = b_1(t)$ and $b_2 = b_2(t)$ along the curve x_t by

$$b_1(t) = C_t^{-1}(e_1) \quad \text{and} \quad b_2(t) = C_t^{-1}(e_2) \,.$$

Then

$$b_1(0) = e_1 \,, \qquad b_2(0) = e_2$$
$$\langle b_1(t), -x_t \rangle = \langle C_t^{-1}(e_1), C_t^{-1}(e_3) \rangle = \langle e_1, e_3 \rangle = 0$$
$$\langle b_2(t), -x_t \rangle = \langle C_t^{-1}(e_2), C_t^{-1}(e_3) \rangle = \langle e_2, e_3 \rangle = 0 \,,$$

since C_t preserves the inner product $\langle \; , \; \rangle$. Thus $b_1(t)$ and $b_2(t)$ are tangent to S^2 at x_t for each t.

We shall show

(i) $b_1(t)$ and $b_2(t)$ are parallel along the curve x_t on S^2 (relative to the Levi-Civita connection of S^2);

(ii) if we write $dx/dt = k_1(t)b_1 + k_2(t)b_2$, then we have $dy/dt = k_1(t)e_1 + k_2(t)e_2$.

To show (i), we differentiate the relation $C_t b_1(t) = e_1$ and obtain

$$db_1/dt = -C_t^{-1}(dC/dt)b_1 = -C_t^{-1}(dC/dt)C_t^{-1}(C_t b_1)$$
$$= -C_t^{-1}S_t(e_1) = -C_t^{-1}(\omega_t \times e_1) \,.$$

Here $\omega_t \times e_1$ is in the direction of e_3 and hence $C_t^{-1}(\omega_t \times e_1)$ is in the direction of x_t. This means that db_1/dt is normal to S^2 at x_t and hence $\nabla_t b_1 = 0$. Thus $b_1(t)$ is parallel along the curve x_t relative to the Levi-Civita connection ∇ of S^2. The proof for $b_2(t)$ is similar. The assertion (ii) is obvious, because C_t maps dx/dt, $b_1(t)$ and $b_2(t)$ upon dy/dt, e_1 and e_2, respectively. Since $b_1(t)$ and $b_2(t)$ are parallel along the curve x_t, it follows that the curve y_t is the development of the curve x_t into the tangent plane Σ (see [2, Vol. I, Proposition 4.1]).

What we have shown is that if we roll S^2 on Σ in such a way that the point x_t becomes a point of contact at instant t, then $y_t = f_t(x_t)$ is the development of x_t. We shall now prove that indeed such a rolling $\{f_t\}$ exists uniquely.

Let $b_1(t)$ and $b_2(t)$ be the vector fields which are parallel along the curve x_t such that $b_1(0) = e_1$ and $b_2(0) = e_2$. They are uniquely determined. Let C_t be the unique matrix in $SO(3)$ such that

$$C_t b_1(t) = e_1, \; C_t b_2(t) = e_2, \; \text{and} \; C_t x_t = -e_3 \,.$$

Let y_t be the development of the curve x_t in Σ. It is, of course, uniquely determined by x_t. We have

$$C_t(dx/dt) = dy/dt \,.$$

We set

$$c_t = y_t + e_3$$

and

$$f_t = \begin{bmatrix} C_t & c_t \\ 0 & 1 \end{bmatrix}.$$

It is now easy to verify that $\{f_t\}$ is a rolling for which $y_t = f_t(x_t)$ is the center of instantaneous rotation.

Summarizing the discussions we have

THEOREM 1. *Let x_t be a smooth curve on the unit sphere S^2. There exists a unique rolling $\{f_t\}$ of S^2 on the tangent plane Σ at x_0 such that $y_t = f_t(x_t)$ is the locus of points of contact on Σ. The curve y_t is the development of the curve x_t into Σ in the sense of E. Cartan.*

3. **Rolling an n-dimensional sphere.** We extend the result in 2 to higher dimensions. Let S^n be the unit sphere in $(n+1)$-dimensional Euclidean space E^{n+1}, say, $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$. Let x_t be a smooth curve on S^n starting at $x_0 = (0, \cdots, 0, -1)$. Let Σ be the tangent hyperplane $x^{n+1} = -1$. We shall write $e_1, \cdots, e_n, e_{n+1}$ for the standard basis in the vector space E^{n+1}.

We consider a 1-parametric motion f_t as in (2) with $C_t \in SO(n+1)$ such that $y_t = f_t(x_t)$ is a point of contact with Σ at time instant t. We have

$$C_t x_t = -e_{n+1} \tag{6'}$$

$$c_t = y_t + e_{n+1} . \tag{7'}$$

Assuming that y_t is a center of instantaneous rotation we obtain $S_t y_t + dc/dt = 0$ and thus

$$S_t(e_{n+1}) = dy/dt . \tag{8'}$$

For $n > 2$, we cannot speak of the angular velocity ω_t. In order to define f_t as rolling on Σ we require that S_t maps every vector on Σ into Span e_{n+1}. Under this condition, (8′) determines S_t uniquely.

In order to prove that the curve y_t is the development of the curve x_t, we define $b_i(t) = C_t^{-1} e_i$, $1 \leqq i \leqq n$. They are vector fields tangent to S^n along the curve x_t. To show that they are parallel along x_t with respect to the Levi-Civita connection ∇ on S^n, we obtain, as before,

$$db_i/dt = -C_t^{-1}(S_t e_i) .$$

Since $S_t e_i$ is a scalar multiple of e_{n+1}, we see that db_1/dt is normal to S^n. Thus $\nabla_t b_i = 0$. From $C_t(b_i) = e_i$, $1 \leqq i \leqq n$, and $C_t(dx/dt) = dy/dt$, it follows that y_t is the development of x_t.

It is now clear that Theorem 1 extends to higher dimensions.

4. **Rolling a surface on a plane.** Let M be an arbitrary surface in E^3 and let Σ be the tangent plane to M at a point x_0. Let x_t be a smooth curve on M. We wish to find a rolling $\{f_t\}$ of M on Σ such that $y_t = f_t(x_t)$ is the locus of points of contact (and centers of instantaneous rotation). We choose a rectangular coordinate system with origin x_0 such that Σ is given by $x^3 = 0$. Let e_1, e_2, e_3 be the natural basis. Let ξ_t be the field of unit normal vectors along x_t such that $\xi_0 = e_3$.

For f_t given as in (2), we obtain as before

$$C_t(dx/dt) = dy/dt \,. \tag{9}$$

Since $f_t(M)$ is tangent to Σ at y_t, we have

$$C_t(\xi_t) = e_3 \,. \tag{10}$$

We define

$$b_1(t) = C_t^{-1}(e_1) \quad \text{and} \quad b_2(t) = C_t^{-1}(e_2)$$

as before. They are tangent to M along the curve x_t.

We may write

$$d\xi/dt = \lambda_1(t)b_1 + \lambda_2(t)b_2 \,. \tag{11}$$

Differentiating (10) we obtain

$$(dC/dt)\xi_t + C_t(d\xi/dt) = 0$$

and hence

$$(dC/dt)\xi_t = -\,C_t(\lambda_1 b_1 + \lambda_2 b_2) = -\lambda_1 e_1 - \lambda_2 e_2 \,.$$

Thus we obtain by using (10) again

$$S_t(e_3) = -\lambda_1 e_1 - \lambda_2 e_2 \,. \tag{12}$$

Since S_t is not to be 0, $\lambda_1(t)$ and $\lambda_2(t)$ should not vanish simultaneously. Let ω_t be the angular velocity so that $S_t(U) = \omega_t \times U$ for every vector U. Since ω_t lies on Σ (for $\{f_t\}$ is a rolling), we see that both $S_t(e_1) = \omega_t \times e_1$ and $S_t(e_2) = \omega_t \times e_2$ are in the direction of e_3. Actually, we have $\omega_t = \lambda_2 e_1 - \lambda_1 e_2$.

From

$$db_1/dt = -C_t^{-1}(S_t e_1) \quad \text{and} \quad db_2/dt = -C_t^{-1}(S_t e_2) \,,$$

we see that db_1/dt and db_2/dt are in the direction of ξ_t. This means that $b_1(t)$ and $b_2(t)$ are parallel along the curve x_t with respect to the

Levi-Civita connection of M. Now the equation (9) implies that the curve y_t is the development of the curve x_t.

As we stated, $\lambda_1(t)$ and $\lambda_2(t)$ are not 0 simultaneously. We can interpret this fact as follows. The equation (11) actually defines the second fundamental form A on the vector dx/dt, that is,

$$d\xi/dt = -A(dx/dt) .$$

So our condition $S_t \neq 0$ is equivalent to $A(dx/dt) \neq 0$ for each t.

Conversely, suppose this condition $A(dx/dt) \neq 0$ is satisfied for each t. Then we may take parallel vector fields $b_1(t)$ and $b_2(t)$ along the curve x_t such that $b_1(0) = e_1$, $b_2(0) = e_2$ and define C_t as the matrix in $SO(3)$ such that $C_t b_1(t) = e_1$, $C_t b_2(t) = e_2$ and $C_t \xi_t = e_3$. Then define c_t by

$$c_t = y_t - C_t x_t ,$$

where the curve y_t is the development of the curve x_t in Σ. It is then easy to check that

$$f_t = \begin{bmatrix} C_t & c_t \\ 0 & 1 \end{bmatrix}$$

is the rolling with the locus of contact $y_t = f_t(x_t)$.

The condition $A(dx/dt) \neq 0$ is satisfied if the second fundamental form A does not admit 0 as an eigenvalue, namely, if 0 is not a principal curvature at x_t. This is certainly the case if the curve x_t does not go through a flat point of M.

THEOREM 2. *Let x_t be a smooth curve on a surface M which does not go through a flat point of M. There exists a unique rolling $\{f_t\}$ of M on the tangent plane Σ at x_0 such that $y_t = f_t(x_t)$ is the locus of points of contact. The curve y_t is the development of the curve x_t into Σ.*

The extreme opposite of the assumption $A(dx/dt) \neq 0$ is the case where M is locally flat and $A(dx/dt) = 0$ for all t, for example, when the curve x_t is a generator on a cone or a cylinder M. One can easily see that there is no rolling of the kind in Theorem 2.

We also remark that, in the situation of Theorem 2, a vector field $U(t)$ along the curve x_t is parallel with respect to the Levi-Civita connection of M if and only if $C_t(U(t))$ is a constant vector for all t. *This is the kinematic interpretation of the Levi-Civita connection for the surface M.*

5. Rolling a surface on another surface. Let M and N be two orientable surfaces tangent to each other at x_0. For a given smooth curve x_t on M, we shall find a rolling $\{f_t\}$ of M on N such that $y_t =$

$f_t(x_t) \in N$ is the locus of contact.

We choose a field of unit normal vectors ξ for M and η for N such that they coincide at x_0. (For example, if two spheres M and N are tangent and outside of each other, then when we choose ξ as an inward unit normal for M, η will be an outward unit normal for N.) We write ξ_t and η_t for ξ at x_t and η at y_t, respectively.

If $\{f_t\}$ is given by (2), then

$$\eta_t = C_t \xi_t \,. \tag{13}$$

Let $a_1 = a_1(t)$ and $a_2 = a_2(t)$ be orthonormal vector fields which are parallel along x_t on M. We let

$$b_1(t) = C_t(a_1(t)) \,, \quad b_2(t) = C_t(a_2(t)) \,. \tag{14}$$

We have

$$\begin{aligned} da_1/dt &= \lambda_1 \xi \,, \quad da_2/dt = \lambda_2 \xi \\ d\xi/dt &= -\lambda_1 a_1 - \lambda_2 a_2 \end{aligned} \tag{15}$$

where $\lambda_1 = \lambda_1(t)$ and $\lambda_2 = \lambda_2(t)$ are suitable functions. On the other hand, we have

$$\begin{aligned} db_1/dt &= \mu_1 \eta + \kappa b_2 \\ db_2/dt &= \mu_2 \eta - \kappa b_1 \\ d\eta/dt &= -\mu_1 b_1 - \mu_2 b_2 \end{aligned} \tag{16}$$

where $\mu_1 = \mu_1(t)$, $\mu_2 = \mu_2(t)$, and $\kappa = \kappa(t)$ are suitable functions. Differentiating (14) and using (15), (16) we obtain for $S_t = (dC/dt)C_t^{-1}$ the following:

$$\begin{aligned} S_t(b_1) &= \kappa b_2 + (\mu_1 - \lambda_1)\eta \\ S_t(b_2) &= -\kappa b_1 + (\mu_2 - \lambda_2)\eta \,. \end{aligned}$$

From (13) we obtain

$$S_t(\eta_t) = (\lambda_1 - \mu_1)b_1 + (\lambda_2 - \mu_2)b_2 \,.$$

If $S_t \neq 0$, the angular velocity ω_t is given by

$$\omega_t = (\mu_2 - \lambda_2)b_1 - (\mu_1 - \lambda_1)b_2 + \kappa\eta \,.$$

Since $\{f_t\}$ is a rolling, ω_t is tangent to N at y_t. Thus κ must be 0. This means that b_1 and b_2 are parallel along the curve y_t. We should also require that $S_t \neq 0$, that is,

$$(\mu_1 - \lambda_1)^2 + (\mu_2 - \lambda_2)^2 \neq 0 \quad \text{for any} \quad t \,.$$

To discuss this condition, we introduce the following concept. Let M, N be two oriented surfaces with unit normal vectors ξ and η, respectively. Let $x \in M$, $y \in N$, $X \in T_x(M)$, and $Y \in T_y(N)$. We say that *M and N have the same shape along X and Y* if there is a linear isometry F of $T_x(E^m)$ onto $T_y(E^m)$ such that

$$F(\xi) = \eta\ , \qquad F(X) = Y \quad \text{and} \quad F(AX) = B(Y)\ ,$$

where A and B are the shape operators of M and N relative to ξ and η, respectively.

Now the third equations of (15) and (16) can be written as

$$A(dx/dt) = \lambda_1 a_1 + \lambda_2 a_2$$
$$B(dy/dt) = \mu_1 b_1 + \mu_2 b_2\ .$$

Since C_t maps dx/dt, a_1, a_2 upon dy/dt, b_1, b_2, respectively, the equalities $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_2$ will mean that M and N have the same shape along the vectors dx/dt and dy/dt.

In order to find a rolling $\{f_t\}$ from the given curve x_t on M, we must know how to determine the curve y_t on N. This can be done by making use of the development z_t of x_t into the tangent plane $T_{x_0}(M)$. If we write

$$dx/dt = k_1 a_1 + k_2 a_2 \tag{19}$$

with suitable functions $k_1 = k_1(t)$ and $k_2 = k_2(t)$, then

$$dy/dt = k_1 b_1 + k_2 b_2 \tag{20}$$

because of (14) and $C_t(dx/dt) = dy/dt$.
Let $e_1 = a_1(0)$ and $e_2 = a_2(0)$. Since $a_1(t)$ and $a_2(t)$ are parallel along the curve x_t, the development z_t of x_t is given by integrating

$$dz/dt = k_1 e_1 + k_2 e_2\ . \tag{21}$$

Similarly, the development of the curve y_t into the tangent plane $T_{y_0}(N) = T_{x_0}(M)$ is also given by (21), namely, z_t is the development of y_t. This means that y_t is determined as the unique curve in N with $y_0 = x_0$ whose development into $T_{y_0}(N)$ is equal to z_t.

Summarizing the discussions we can state

THEOREM 3. *Let M and N be two orientable surfaces which are tangent to each other at x_0. Let x_t be a smooth curve on M. Then we can find a unique rolling $\{f_t\}$ of M onto N such that $y_t = f_t(x_t)$ is the locus of centers of instantaneous rotation provided the following condition is satisfied. Let z_t be the development of the curve x_t into the tangent plane $T_{x_0}(M)$. Let y_t be the unique curve on N such that its development into $T_{y_0}(N) = T_{x_0}(M)$ is z_t. Take the fields of unit normals ξ and η for M and N such that they coincide at $x_0 = y_0$. The condition* *to be satisfied is that, for each t, M and N do not have the same shape along the vectors dx/dt and dy/dt (for the chosen normals ξ and η).*

We remark that in the case of two spheres tangent to, and outside of, each other, the condition in question is satisfied for an arbitrary curve x_t for the choice of inward normals ξ for one sphere M and outward normals η for the other sphere N.

It is possible to find a number of sufficient conditions under which a rolling (with $S_t \neq 0$) is possible for a given curve x_t. For example,

(i) $A(dx/dt) = 0$ *for every t and N has no flat point*; for example, x_t is a generator of a cylinder and N is a sphere.

(ii) $A(dx/dt) \neq 0$ *for each t and N is a plane.*

(iii) *dx/dt is not a principal vector at any point and N is umbilical (a plane or a sphere).*

A rolling is possible for an arbitrary curve on M if M and N satisfy the following condition:

(iv) *the principal curvatures of M are greater than those of N*; here we assume that ξ and η are chosen so that $\xi_{x_0} = \eta_{x_0}$.

If M and N have the same shape along unit vectors X and Y, then we have $\langle AX, X\rangle = \langle BY, Y\rangle$. But $\langle AX, X\rangle$ is greater than or equal to the smaller principal curvature of M at the point and $\langle BY, Y\rangle$ is smaller than or equal to the larger principal curvature of N at the corresponding point. Thus condition (iv) is sufficient.

6. The case of submanifolds. Now let M and N be two n-dimensional submanifolds in an m-dimensional Euclidean space E^m which are tangent to each other at a point x_0. We shall first define the notion of rolling $\{f_t\}$ of M and N.

Let $\{f_t\}$ be a 1-parametric motion of E^m given by (2). Assume that $f_t(M)$ is tangent to N at a point y_t at each instant t. We assume that the instantaneous motion X_t vanishes at y_t (that is, y_t is a center of instantaneous rotation) and that $S_t \neq 0$. We say that $\{f_t\}$ is a rolling if the skew-symmetric transformation S_t maps the tangent space $T_{y_t}(N)$ into the normal space $T^{\perp}_{y_t}(N)$ and maps $T^{\perp}_{y_t}(N)$ into $T_{y_t}(N)$, thus,

$$\begin{aligned} &\langle S_t(X), Y\rangle = 0 \quad \text{for all} \quad X, Y \in T_{y_t}(N) \\ &\langle S_t(U), V\rangle = 0 \quad \text{for all} \quad U, V \in T^{\perp}_{y_t}(N)\,. \end{aligned} \tag{22}$$

This is clearly a generalization of the definition in the case where $n = 2$ and $m = 3$ as well as the case of S^n and its tangent plane.

Suppose that a smooth curve x_t is given on M. We wish to find a rolling $\{f_t\}$ of M on N such that $y_t = f_t(x_t)$ is the locus of centers of instantaneous rotation. Let $\{a_1(0), \cdots, a_n(0)\}$ be an orthonormal basis in $T_{x_0}(M)$ and let $\{a_{n+1}(0), \cdots, a_m(0)\}$ be an orthonormal basis in the normal space $T^\perp_{x_0}(M)$. Let $a_i(t)$, $1 \leqq i \leqq n$, be the tangent vector parallel along x_t on M with initial condition $a_i(0)$; thus

$$\begin{aligned} &\nabla_t a_i = 0 \\ &da_i/dt = \alpha(dx/dt, a_i) \end{aligned} \tag{23}$$

for $1 \leqq i \leqq n$, where ∇ is the Levi-Civita connection of M and α denotes the second fundamental form as a bilinear mapping $T_x(M) \times T_x(M) \rightarrow T_x^\perp(M)$ for each $x \in M$.

For each j, $n+1 \leqq j \leqq m$, let $a_j(t)$ be the normal vector field parallel along x_t with respect to the normal connection of M with initial vector $a_j(0)$. Thus

$$\begin{aligned} &\nabla_t^\perp a_j = 0 \\ &da_j/dt = -A_{a_j}(dx/dt) \end{aligned} \tag{24}$$

for $n+1 \leqq j \leqq m$, where $\nabla^\perp$ denotes covariant differentiation for the normal connection and A_{a_j} denotes the shape operator corresponding to the normal vector a_j.

Now suppose $\{f_t\}$ is a rolling given by (2). We define

$$b_k(t) = C_t(a_k(t)) \tag{25}$$

for $1 \leqq k \leqq m$. Since $f_t(M)$ is tangent to N at y_t, it follows that $b_1(t), \cdots, b_n(t)$ are tangent to N at y_t and $b_{n+1}(t), \cdots, b_m(t)$ are normal to N at y_t.

Let $1 \leqq i \leqq n$. Differentiating (25) and using (23) we obtain

$$\begin{aligned} db_i/dt &= (dC/dt)a_i + C_t(da_i/dt) \\ &= (dC/dt)C_t^{-1}b_i + C_t(\alpha(dx/dt, a_i)) \,. \end{aligned}$$

On the other hand, we have

$$db_i/dt = \nabla_t b_i + \beta(dy/dt, b_i) \,,$$

where ∇ is the Levi-Civita connection for N and β is the second fundamental form $T_y(N) \times T_y(N) \rightarrow T_y^\perp(N)$ for N. Thus we obtain

$$S_t(b_i) = \nabla_t b_i + \beta(dy/dt, b_i) - C_t(\alpha(dx/dt, a_i)) \,.$$

By definition of a rolling, $S_t(b_i)$ is normal to N. Since $\beta(dy/dt, b_i)$ and $C_t(\alpha(dx/dt, a_i))$ are normal to N, we must have

$$\nabla_t b_i = 0 \,, \quad \text{namely, } b_i \text{ is parallel along } y_t \,,$$

and

(26) $$S_t(b_i) = \beta(dy/dt, b_i) - C_t(\alpha(dx/dt, a_i)) .$$

Now let $n+1 \leqq j \leqq m$. Differentiating (25) and using (24) we obtain

$$\begin{aligned} db_j/dt &= (dC/dt)a_j + C_t(da_j/dt) \\ &= (dC/dt)C_t^{-1}b_j + C_t(-A_{a_j}(dx/dt)) . \end{aligned}$$

On the other hand, we have

$$db_j/dt = -B_{b_j}(dy/dt) + \nabla_t^{\perp} b_j ,$$

where B_{b_j} is the shape operator for N corresponding to the normal vector b_j and $\nabla_t^{\perp}$ is covariant differentiation along y_t for the normal connection of N. Thus we obtain

$$S_t(b_j) = C_t(A_{a_j}(dx/dt)) - B_{b_j}(dy/dt) + \nabla_t^{\perp} b_j .$$

Since $S_t(b_j)$ must be tangent to N at y_t, we conclude that $\nabla_t^{\perp} b_j = 0$, namely, b_j is $\nabla^{\perp}$-parallel along the curve y_t. We have also

(27) $$S_t(b_j) = C_t(A_{a_j}(dx/dt)) - B_{b_j}(dy/dt) .$$

The skew-symmetric transformation S_t given in the form (26) and (27) can be expressed more conveniently if we adopt the following operators ρ for M and τ for N.

For each point x of M, define for each $X \in T_x(M)$ a linear endomorphism of $T_x(E^m) = T_x(M) + T_x^{\perp}(M)$ by

$$\begin{aligned} \rho_X(Y) &= \alpha(X, Y) \quad \text{for} \quad Y \in T_x(M) \\ \rho_X(U) &= -A_U(X) \quad \text{for} \quad U \in T_x^{\perp}(M) . \end{aligned}$$

Then ρ_X is a skew-symmetric endomorphism of $T_x(E^m)$. Similarly, for each point y of N, we define τ_X, $X \in T_y(N)$, by

$$\begin{aligned} \tau_X(Y) &= \beta(X, Y) \quad \text{for} \quad Y \in T_y(N) \\ \tau_X(U) &= -B_U(X) \quad \text{for} \quad U \in T_y^{\perp}(N) . \end{aligned}$$

With these operators we may write (26) in the form

$$\begin{aligned} S_t(b_i) &= \tau_t(b_i) - C_t(\rho_t(a_i)) \\ &= \tau_t(b_i) - C_t \rho_t C_t^{-1}(b_i) , \end{aligned}$$

where we write ρ_t for $\rho_{dx/dt}$ and τ_t for $\tau_{dy/dt}$ for brevity. From (27) we have

$$\begin{aligned} S_t(b_j) &= -C_t \rho_t(a_j) + \tau_t(b_j) \\ &= \tau_t(b_j) - C_t \rho_t C_t^{-1} . \end{aligned}$$

Thus we may simply write

(28) $$S_t = \tau_t - C_t\rho_t C_t^{-1} .$$

In order to discuss conditions under which $S_t \neq 0$, we extend the notion of same shape to n-dimensional submanifolds M and N in E^m. Let $X \in T_x(M)$ and $Y \in T_y(N)$. We say that *M and N have the same shape along X and Y* if there is a linear isometry F of $T_x(E^m)$ onto $T_y(E^m)$ such that

(29) $$F(T_x(M)) = T_y(N) , \qquad F(T_x^{\perp}(M)) = T_y^{\perp}(N)$$
$$F(X) = Y \quad \text{and} \quad F \circ \rho_X = \tau_Y \circ F .$$

From (28) we see that if $S_t = 0$, then $F = C_t$ satisfies these conditions for $X = dx/dt$ and $Y = dy/dt$, namely, M and N have the same shape along X and Y.

We may find a number of sufficient conditions under which a rolling (with $S_t \neq 0$) exists. Recall that the relative nullity space of a submanifold M at x is the subspace of $T_x(M)$ consisting of all $X \in T_x(M)$ with $\rho_X = 0$. Its dimension is called the index of relative nullity. For example, a rolling exists for a curve x_t if

(i) *dx/dt is in the relative nullity space for every t and N has index of relative nullity* 0,

or (ii) *dx/dt is not in the relative nullity space for any t and N is a Euclidean n-plane,*

or (iii) *the sectional curvature of any plane containing the tangent vector dx/dt never vanishes and N is a Euclidean n-plane.*

The last case follows from the second because the sectional curvature of any plane containing a vector X with $\rho_X = 0$ must vanish. We also remark that the so-called Veronese variety (the image of S^n by a certain isometric imbedding into E^m, where $m = n(n+3)/2$ has index of relative nullity 0.

We also recall that a non-zero vector $X \in T_x(M)$ is called a principal vector if there is $\xi_1 \in T_x^{\perp}(M)$ such that $A_\xi(X) = \langle \xi, \xi_1 \rangle X$ for all $\xi \in T_x^{\perp}(M)$. A rolling exists for a curve x_t on M if

(iv) *dx/dt is not a principal vector for each t and N is an n-plane or an n-sphere.*

Let us remark that the Veronese variety mentioned above has no principal vector.

In any case, the relationship between the curve x_t on M and the curve y_t on N is as before. Their developments into $T_{x_0}(M) = T_{y_0}(N)$ coincide.

We now consider the following special case.

Let M be an n-dimensional submanifold in E^m and x_0 a point of M.

As the second submanifold N we take the n-plane $T_{x_0}(M)$. We are going to obtain a *kinematic interpretation of the second fundamental form and the normal connection of M.*

Let $X \in T_{x_0}(M)$ and take any curve x_t with initial tangent vector X. If we have a rolling $\{f_t\}$ of M onto N determined by x_t, then (28) implies

$$S_0 = -\rho_X .$$

This gives the kinematic interpretation of the second fundamental form α of M at x_0, because ρ determines α completely.

If ξ_t is a field of normal vectors along x_t, we may write

$$\xi_t = \sum_{j=n+1}^{m} q_j(t) a_j ,$$

where $a_{n+1}(t), \cdots, a_m(t)$ are $\nabla^\perp$-parallel normal vector fields along x_t as before. Then

$$C_t(\xi_t) = \sum_{j=n+1}^{m} q_j(t) b_j$$

and, as we have seen before, each $b_j = C_t(a_j)$ is $\nabla^\perp$-parallel along the curve $y_t = f_t(x_t)$. Since N is an n-plane, this means that each b_j is a constant vector in E^m. It follows that ξ_t is $\nabla^\perp$-parallel along the curve x_t if and only if $C_t(\xi_t)$ is a constant vector in E^m (that is, each $q_j(t)$ is constant). This is the kinematic interpretation of the normal connection of M.

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