KINEMATICS AND DIFFERENTIAL GEOMETRY
OF SUBMANIFOLDS

—Rolling a ball with a prescribed locus of contact—

KATSUMI NOMIZU

(Received June 28, 1977)

The simplest and most illustrative of the kinematic models we discuss in this paper is the rolling of a ball on its tangent plane. Suppose a smooth curve \( x_t \) is given on the unit sphere \( S^2 \) (boundary of the unit ball \( B \)). Is it possible to roll (without skidding or spinning) the ball \( B \) on the tangent plane \( \Sigma \) to \( S^2 \) at \( x_0 \) in such a way that at each time instant \( t \) the point \( x_t \) becomes a point of contact with the plane \( \Sigma \)? We shall show that this is possible and that the locus \( y_t \) of points of contact on \( \Sigma \) is indeed the development of the curve \( x_t \), in the sense of E. Cartan.

When we replace \( S^2 \) by an arbitrary smooth surface \( M \), the rolling of \( M \) on its tangent plane gives rise to a kinematic interpretation of the Levi-Civita connection for \( M \). We also find that we must impose a certain condition on the curve \( x_t \) to prevent the rolling from degenerating into an instantaneous standstill at any instant. This condition is that the tangent vector of \( x_t \) is not a principal direction for the zero principal curvature; this condition is satisfied if the curve \( x_t \) does not go through a flat point.

In the end we shall study the model of rolling an \( n \)-dimensional submanifold \( M \) on another \( n \)-dimensional submanifold \( N \) in a Euclidean space \( E^m \) and obtain a kinematic interpretation of the second fundamental form and the normal connection of a submanifold.

The paper is organized as follows. Section 1 is devoted to the basic concepts in kinematics we need. We define the notion of rolling (without skidding or spinning). In Section 2 we discuss the model of rolling a ball and extend it to higher dimensions in Section 3. In Section 4 we treat the rolling of an arbitrary surface on a plane. Section 5 deals with rolling of a surface on another surface. Finally, in Section 6, we discuss the most general question—rolling an \( n \)-dimensional submanifold

Work supported by NSF Grant MCS 76-06324.
M on another n-dimensional submanifold $N$ in $E^m$. The reference for submanifolds is [2, Vol. II].

1. Motion, instantaneous motion, and rolling. By a motion of a Euclidean space $E^m$ we mean an orientation-preserving isometry of $E^m$. If we take an arbitrary Euclidean (i.e., rectangular) coordinate system, a motion $f$ can be expressed by an $(m + 1) \times (m + 1)$ matrix of the form

$$
\begin{bmatrix}
C & c \\
0 & 1
\end{bmatrix}
$$

where $C \in SO(m)$ and $c$ is an $m$-dimensional (column) vector. A point $x$ is mapped by $f$ upon $f(x) = Cx + c$.

By a 1-parametric motion $\{f_t\}$, $t \in J$, where $J$ is an open or closed interval containing 0 in its interior, we mean a differentiable mapping of $J$ into the space of matrices of the form (1), namely,

$$
\begin{bmatrix}
C_t & c_t \\
0 & 1
\end{bmatrix}
$$

where $C_t$ is an $SO(m)$-valued differentiable function of $t$ and $c_t$ is a vector-valued differentiable function of $t$ such that $f_0 = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$ (identity transformation).

We remark once and for all that it does not matter which Euclidean coordinate system we use in expressing motions and related concepts in the following.

Given a 1-parametric motion $\{f_t\}$, we can define a time-dependent vector field $X_t$ on $E^m$ as follows. Fix $t$. Let $y$ be an arbitrary point and let $x = f_t^{-1}(y)$. Let $(X_t)_y$ be the tangent vector $[df_u(x)/du]_{u=t}$ of the orbit $f_u(x)$ at $u = t$, namely, at the point $y = f_t(x)$.

Using the matrix (2) we can obtain the matrix representing the vector field $X_t$ as follows. Write

$$
(df/dt)f_t^{-1} = \begin{bmatrix} S_t & v_t \\ 0 & 0 \end{bmatrix}
$$

where

$$
S_t = (dC/dt)C_t^{-1} \quad \text{and} \quad v_t = -S_t c_t + dc/dt .
$$

Then it is easy to verify that

$$(X_t)_y = S_t y + v_t .$$
For this reason, we call (3) the instantaneous motion at instant $t$. If $x$ is an arbitrary point, we have
\[
d f_t(x)/dt = (dC/dt)x + dc/dt = (dC/dt)C_\i x + dc/dt,
\]
namely,
\[
d f_t(x)/dt = S_t C_\i x + dc/dt.
\]

The instantaneous motion (3) is called an instantaneous standstill if $S_t = 0$ and $v_t = 0$. It is called an instantaneous translation if $S_t = 0$ and $v_t \neq 0$. In this case, $(X_t)_y = v_t$ for all points $y$, namely, all points have the same velocity at instant $t$.

We say that (3) is an instantaneous rotation if there exists a point $y_0$ such that $(X_t)_y = 0$. If $x_0 = f_t^{-1}(y_0)$, then $d f_t(x_0)/dt = 0$ and $y_0$ is called a center of instantaneous rotation. We shall also require that $S_t \neq 0$ to avoid an instantaneous standstill.

In the case where $m = 3$, an instantaneous rotation has an axis, namely, the line consisting of all points $y$ such that $(X_t)_y = 0$. Suppose $(X_t)_y = (X_t)_y = 0$. Then from (4) we obtain $S_t(y - y_0) = 0$. Since the null space of the skew-symmetric transformation $S_t \neq 0$ is a 1-dimensional subspace, the set of $y$ with $(X_t)_y = 0$ forms a straight line. Indeed, for $S_t \neq 0$, there is a uniquely determined vector $\omega_t$ such that $S_t(U)$ is equal to the cross product $\omega_t \times U$ for every vector $U$. The vector $\omega_t$ is called the angular velocity at instant $t$.

If $x$ is an arbitrary point, the velocity $d f_t(x)/dt$ in (4) can be expressed by
\[
d f_t(x)/dt = \omega_t \times f_t(0) f_t(x) + d f_t(0)/dt
\]

since $c_t = f_t(0)$ and $C_\i x$ is equal to the vector $\overrightarrow{f_t(0)f_t(x)}$ from $f_t(0)$ to $f_t(x)$.

We shall now define rolling of a surface $M$ on another surface $N$. Consider a 1-parametric motion $\{f_t\}$ with the property that for each instant $t$ the image $f_t(M)$ is tangent to $N$ at a certain point $y_t$. If $(d f_t/dt) f_t^{-1}$ is an instantaneous translation, we have skidding at instant $t$. Suppose $(d f_t/dt) f_t^{-1}$ is an instantaneous rotation with $y_t$ as center and $S_t \neq 0$. If the angular velocity $\omega_t$ is normal to $N$ at $y_t$, then we have spinning at instant $t$. If $\omega_t$ is tangent to $N$ at $y_t$, then we say that $(d f_t/dt) f_t^{-1}$ is a rolling. Thus the 1-parametric motion $\{f_t\}$ is a rolling of $M$ on $N$ (without skidding or spinning) if, for each instant $t$, $(d f_t/dt) f_t^{-1}$ is a rolling in the above sense. See [1], pp. 78-79; section called Roulement et pivotment d'une surface mobile sur une surface fixe.
Remark. If \( \{ f_t \} \) is a rolling of \( M \) on \( N \), then \( \{ f_t^{-1} \} \) is a rolling of \( N \) on \( M \).

2. Rolling a ball on a plane. Let us consider the unit sphere \( S^2 \) and the tangent plane \( \Sigma \) of \( S^2 \) at \( x_p \). We shall take a rectangular coordinate system in \( E^3 \) such that \( S^2 \) is given by \( (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \), \( x_3 = (0, 0, -1) \) and \( \Sigma \) is given by \( x^3 = -1 \). Let \( e_1, e_2, e_3 \) be the unit vectors \( (1, 0, 0), (0, 1, 0), (0, 0, 1) \), respectively.

Suppose \( x_t \) is a smooth curve (with non-vanishing tangent vector \( dx/dt \)) on \( S^2 \) starting at \( x_0 \). We wish to roll \( S^2 \) on \( \Sigma \) in such a way that at instant \( t \) the point \( x_t \) becomes a point of contact with \( \Sigma \). Let the rolling \( \{ f_t \} \) be given by (2) and let \( y_t = f_t(x_t) \).

Since \( f_t(S^2) \) is tangent to \( \Sigma \) at \( y_t \), we have

\[
C_t x_t = - e_3 .
\]

Thus

\[
y_t = C_t x_t + c_t = c_t - e_3
\]

that is,

\[
c_t = y_t + e_3 .
\]

Since \( y_t \) is a center of instantaneous rotation, we have from (4), (6), and (7)

\[
S_i(e_i) = dy/dt .
\]

Since the angular velocity \( \omega_t \) lies on \( \Sigma \) (by definition of rolling) and since

\[
\omega_t \times e_3 = S_i(e_i) = dy/dt ,
\]

it follows that \( \omega_t \) is perpendicular to the tangent vector \( dy/dt \) of the curve \( y_t \) and \( \{ dy/dt, \omega_t \} \) have the same orientation as \( \{ e_1, e_2 \} \).

We shall now proceed to prove that the curve \( y_t \) is the development of the curve \( x_t \) into the tangent plane \( \Sigma \). First we observe

\[
C_t(dx/dt) = dy/dt .
\]

This can be seen as follows. From \( y_t = C_t x_t + c_t \), we have

\[
dy/dt = (dC/dt)x_t + C_t(dx/dt) + dc/dt .
\]

Since \( y_t \) is a center of instantaneous rotation, (4) gives

\[
(dC/dt)x_t + dc/dt = 0 .
\]

These two equations give rise to (9).

We define vector fields \( b_1 = b_1(t) \) and \( b_3 = b_3(t) \) along the curve \( x_t \) by
\[ b_1(t) = C_t^{-1}(e_1) \quad \text{and} \quad b_2(t) = C_t^{-1}(e_2) . \]

Then
\[ b_1(0) = e_1 , \quad b_2(0) = e_2 \]
\[ \langle b_1(t), -x_t \rangle = \langle C_t^{-1}(e_1), C_t^{-1}(e_2) \rangle = \langle e_1, e_2 \rangle = 0 \]
\[ \langle b_2(t), -x_t \rangle = \langle C_t^{-1}(e_2), C_t^{-1}(e_3) \rangle = \langle e_2, e_3 \rangle = 0 , \]
since \( C_t \) preserves the inner product \( \langle , \rangle \). Thus \( b_1(t) \) and \( b_2(t) \) are tangent to \( S^2 \) at \( x_t \) for each \( t \).

We shall show

(i) \( b_1(t) \) and \( b_2(t) \) are parallel along the curve \( x_t \) on \( S^2 \) (relative to the Levi-Civita connection of \( S^2 \));

(ii) if we write \( dx/dt = k_1(t)b_1 + k_2(t)b_2 \), then we have \( dy/dt = k_1(t)e_1 + k_2(t)e_2 \).

To show (i), we differentiate the relation \( C_t b_1(t) = e_1 \) and obtain
\[ db_1/dt = -C_t^{-1}(dC_t/dt)b_1 = -C_t^{-1}(dC_t/dt)C_t^{-1}(C_t b_1) \]
\[ = -C_t^{-1}S_1(e_1) = -C_t^{-1}(\omega_t \times e_1) . \]

Here \( \omega_t \times e_1 \) is in the direction of \( e_2 \) and hence \( C_t^{-1}(\omega_t \times e_1) \) is in the direction of \( x_t \). This means that \( db_1/dt \) is normal to \( S^2 \) at \( x_t \) and hence \( \nabla b_1 = 0 \). Thus \( b_1(t) \) is parallel along the curve \( x_t \) relative to the Levi-Civita connection \( \nabla \) of \( S^2 \). The proof for \( b_2(t) \) is similar. The assertion (ii) is obvious, because \( C_t \) maps \( dx/dt, b_1(t) \) and \( b_2(t) \) upon \( dy/dt, e_1 \) and \( e_2 \), respectively. Since \( b_1(t) \) and \( b_2(t) \) are parallel along the curve \( x_t \), it follows that the curve \( y_t \) is the development of the curve \( x_t \) into the tangent plane \( \Sigma \) (see [2, Vol. I, Proposition 4.1]).

What we have shown is that if we roll \( S^2 \) on \( \Sigma \) in such a way that the point \( x_t \) becomes a point of contact at instant \( t \), then \( y_t = f_t(x_t) \) is the development of \( x_t \). We shall now prove that indeed such a rolling \( \{f_t\} \) exists uniquely.

Let \( b_1(t) \) and \( b_2(t) \) be the vector fields which are parallel along the curve \( x_t \) such that \( b_1(0) = e_1 \) and \( b_2(0) = e_2 \). They are uniquely determined. Let \( C_t \) be the unique matrix in \( SO(3) \) such that
\[ C_t b_1(t) = e_1 , \quad C_t b_2(t) = e_2 , \quad \text{and} \quad C_t x_t = -e_3 . \]

Let \( y_t \) be the development of the curve \( x_t \) in \( \Sigma \). It is, of course, uniquely determined by \( x_t \). We have
\[ C_t(dx/dt) = dy/dt . \]

We set
\[ c_t = y_t + e_3 \]
and

\[ f_t = \begin{bmatrix} C_t & c_t \\ 0 & 1 \end{bmatrix}. \]

It is now easy to verify that \( \{f_t\} \) is a rolling for which \( y_t = f_t(x_t) \) is the center of instantaneous rotation.

Summarizing the discussions we have

**Theorem 1.** Let \( x_t \) be a smooth curve on the unit sphere \( S^2 \). There exists a unique rolling \( \{f_t\} \) of \( S^2 \) on the tangent plane \( \Sigma \) at \( x_0 \) such that \( y_t = f_t(x_t) \) is the locus of points of contact on \( \Sigma \). The curve \( y_t \) is the development of the curve \( x_t \) into \( \Sigma \) in the sense of E. Cartan.

3. **Rolling an \( n \)-dimensional sphere.** We extend the result in 2 to higher dimensions. Let \( S^n \) be the unit sphere in \((n + 1)\)-dimensional Euclidean space \( E^{n+1} \), say, \((x_1)^2 + \cdots + (x^{n+1})^2 = 1\). Let \( x_t \) be a smooth curve on \( S^n \) starting at \( x_0 = (0, \cdots, 0, -1) \). Let \( \Sigma \) be the tangent hyperplane \( x^{n+1} = -1 \). We shall write \( e_1, \cdots, e_n, e_{n+1} \) for the standard basis in the vector space \( E^{n+1} \).

We consider a 1-parametric motion \( f_t \) as in (2) with \( C_t \in SO(n + 1) \) such that \( y_t = f_t(x_t) \) is a point of contact with \( \Sigma \) at time instant \( t \). We have

\[ (6') \quad C_t x_t = -e_{n+1} \]

\[ (7') \quad c_t = y_t + e_{n+1}. \]

Assuming that \( y_t \) is a center of instantaneous rotation we obtain \( S_t y_t + dc/dt = 0 \) and thus

\[ (8') \quad S_t(e_{n+1}) = dy/dt. \]

For \( n > 2 \), we cannot speak of the angular velocity \( \omega_t \). In order to define \( f_t \) as rolling on \( \Sigma \) we require that \( S_t \) maps every vector on \( \Sigma \) into \( \text{Span} \ e_{n+1} \). Under this condition, \( (8') \) determines \( S_t \) uniquely.

In order to prove that the curve \( y_t \) is the development of the curve \( x_t \), we define \( b_i(t) = C_t^{-1} e_i, 1 \leq i \leq n \). They are vector fields tangent to \( S^n \) along the curve \( x_t \). To show that they are parallel along \( x_t \) with respect to the Levi-Civita connection \( \nabla \) on \( S^n \), we obtain, as before,

\[ db_i/dt = -C_t^{-1}(S_t e_i). \]

Since \( S_t e_i \) is a scalar multiple of \( e_{n+1} \), we see that \( db_i/dt \) is normal to \( S^n \). Thus \( \nabla b_i = 0 \). From \( C_i(b_i) = e_i, 1 \leq i \leq n \), and \( C_i(dx/dt) = dy/dt \), it follows that \( y_t \) is the development of \( x_t \).
It is now clear that Theorem 1 extends to higher dimensions.

4. Rolling a surface on a plane. Let \( M \) be an arbitrary surface in \( E^3 \) and let \( \Sigma \) be the tangent plane to \( M \) at a point \( x_0 \). Let \( x_t \) be a smooth curve on \( M \). We wish to find a rolling \( \{ f_t \} \) of \( M \) on \( \Sigma \) such that \( y_t = f_t(x_t) \) is the locus of points of contact (and centers of instantaneous rotation). We choose a rectangular coordinate system with origin \( x_0 \) such that \( \Sigma \) is given by \( x^3 = 0 \). Let \( e_1, e_2, e_3 \) be the natural basis. Let \( \xi_t \) be the field of unit normal vectors along \( x_t \) such that \( \xi_0 = e_3 \).

For \( f_t \) given as in (2), we obtain as before
\[
C_t(dx/dt) = dy/dt.
\]
Since \( f_t(M) \) is tangent to \( \Sigma \) at \( y_t \), we have
\[
C_t(\xi_t) = e_3.
\]
We define
\[
b_1(t) = C_t^{-1}(e_1) \quad \text{and} \quad b_2(t) = C_t^{-1}(e_2)
\]
as before. They are tangent to \( M \) along the curve \( x_t \).

We may write
\[
d\xi/dt = \lambda_1(t)b_1 + \lambda_2(t)b_2.
\]
Differentiating (10) we obtain
\[(dC/dt)\xi + C_t(d\xi/dt) = 0\]
and hence
\[(dC/dt)\xi = -C_t(\lambda_1b_1 + \lambda_2b_2) = -\lambda_1e_1 - \lambda_2e_2.
\]
Thus we obtain by using (10) again
\[
S_t(e_3) = -\lambda_1e_1 - \lambda_2e_2.
\]
Since \( S_t \) is not to be 0, \( \lambda_1(t) \) and \( \lambda_2(t) \) should not vanish simultaneously.

Let \( \omega_t \) be the angular velocity so that \( S_t(U) = \omega_t \times U \) for every vector \( U \). Since \( \omega_t \) lies on \( \Sigma \) (for \( \{ f_t \} \) is a rolling), we see that both \( S_t(e_1) = \omega_t \times e_1 \) and \( S_t(e_2) = \omega_t \times e_2 \) are in the direction of \( e_3 \). Actually, we have \( \omega_t = \lambda_1e_1 - \lambda_2e_2 \).

From
\[
\frac{db_1}{dt} = -C_t^{-1}(S_t e_1) \quad \text{and} \quad \frac{db_2}{dt} = -C_t^{-1}(S_t e_2),
\]
we see that \( \frac{db_1}{dt} \) and \( \frac{db_2}{dt} \) are in the direction of \( \xi_t \). This means that \( b_1(t) \) and \( b_2(t) \) are parallel along the curve \( x_t \) with respect to the
Levi-Civita connection of $M$. Now the equation (9) implies that the curve $y_t$ is the development of the curve $x_t$.

As we stated, $\lambda_1(t)$ and $\lambda_2(t)$ are not 0 simultaneously. We can interpret this fact as follows. The equation (11) actually defines the second fundamental form $A$ on the vector $dx/dt$, that is, 

$$\frac{d\xi}{dt} = -A(dx/dt).$$

So our condition $S_t \neq 0$ is equivalent to $A(dx/dt) \neq 0$ for each $t$.

Conversely, suppose this condition $A(dx/dt) \neq 0$ is satisfied for each $t$. Then we may take parallel vector fields $b_1(t)$ and $b_2(t)$ along the curve $x_t$ such that $b_1(0) = e_1$, $b_2(0) = e_2$ and define $C_t$ as the matrix in $SO(3)$ such that $C_t b_1(t) = e_1$, $C_t b_2(t) = e_2$ and $C_t \xi = e_3$. Then define $c_t$ by 

$$c_t = y_t - C_t x_t,$$

where the curve $y_t$ is the development of the curve $x_t$ in $\Sigma$. It is then easy to check that 

$$f_t = \begin{bmatrix} C_t & e_t \\ 0 & 1 \end{bmatrix}$$

is the rolling with the locus of contact $y_t = f_t(x_t)$.

The condition $A(dx/dt) \neq 0$ is satisfied if the second fundamental form $A$ does not admit 0 as an eigenvalue, namely, if 0 is not a principal curvature at $x_t$. This is certainly the case if the curve $x_t$ does not go through a flat point of $M$.

**THEOREM 2.** Let $x_t$ be a smooth curve on a surface $M$ which does not go through a flat point of $M$. There exists a unique rolling $\{f_t\}$ of $M$ on the tangent plane $\Sigma$ at $x_0$ such that $y_t = f_t(x_t)$ is the locus of points of contact. The curve $y_t$ is the development of the curve $x_t$ into $\Sigma$.

The extreme opposite of the assumption $A(dx/dt) \neq 0$ is the case where $M$ is locally flat and $A(dx/dt) = 0$ for all $t$, for example, when the curve $x_t$ is a generator on a cone or a cylinder $M$. One can easily see that there is no rolling of the kind in Theorem 2.

We also remark that, in the situation of Theorem 2, a vector field $U(t)$ along the curve $x_t$ is parallel with respect to the Levi-Civita connection of $M$ if and only if $C_t(U(t))$ is a constant vector for all $t$. This is the kinematic interpretation of the Levi-Civita connection for the surface $M$.

5. Rolling a surface on another surface. Let $M$ and $N$ be two orientable surfaces tangent to each other at $x_0$. For a given smooth curve $x_t$ on $M$, we shall find a rolling $\{f_t\}$ of $M$ on $N$ such that $y_t =$
$f_t(x_t) \in N$ is the locus of contact.

We choose a field of unit normal vectors $\xi$ for $M$ and $\eta$ for $N$ such that they coincide at $x_0$. (For example, if two spheres $M$ and $N$ are tangent and outside of each other, then when we choose $\xi$ as an inward unit normal for $M$, $\eta$ will be an outward unit normal for $N$.) We write $\xi_t$ and $\eta_t$ for $\xi$ at $x_t$ and $\eta$ at $y_t$, respectively.

If $\{f_t\}$ is given by (2), then

\begin{equation}
\eta_t = C_t \xi_t .
\end{equation}

Let $a_1 = a_1(t)$ and $a_2 = a_2(t)$ be orthonormal vector fields which are parallel along $x_t$ on $M$. We let

\begin{equation}
b_1(t) = C_t(a_1(t)), \ b_2(t) = C_t(a_2(t)) .
\end{equation}

We have

\begin{equation}
d a_1/dt = \lambda_1 \xi, \ d a_2/dt = \lambda_2 \xi
\end{equation}

\begin{equation}
d \xi/dt = -\lambda_1 a_1 - \lambda_2 a_2
\end{equation}

where $\lambda_1 = \lambda_1(t)$ and $\lambda_2 = \lambda_2(t)$ are suitable functions. On the other hand, we have

\begin{equation}
d b_1/dt = \mu_1 \eta + \kappa b_2
\end{equation}

\begin{equation}
d b_2/dt = \mu_2 \eta - \kappa b_1
\end{equation}

\begin{equation}
d \eta/dt = -\mu_1 b_1 - \mu_2 b_2
\end{equation}

where $\mu_1 = \mu_1(t)$, $\mu_2 = \mu_2(t)$, and $\kappa = \kappa(t)$ are suitable functions. Differentiating (14) and using (15), (16) we obtain for $S_t = (dC/dt)C_t^{-1}$ the following:

\begin{equation}
S_t(b_1) = \kappa b_2 + (\mu_1 - \lambda_1)\eta
\end{equation}

\begin{equation}
S_t(b_2) = -\kappa b_1 + (\mu_2 - \lambda_2)\eta .
\end{equation}

From (13) we obtain

\begin{equation}
S_t(\eta_t) = (\lambda_1 - \mu_1)b_1 + (\lambda_2 - \mu_2)b_2 .
\end{equation}

If $S_t \neq 0$, the angular velocity $\omega_t$ is given by

\begin{equation}
\omega_t = (\mu_2 - \lambda_2)b_1 - (\mu_1 - \lambda_1)b_2 + \kappa \eta .
\end{equation}

Since $\{f_t\}$ is a rolling, $\omega_t$ is tangent to $N$ at $y_t$. Thus $\kappa$ must be 0. This means that $b_1$ and $b_2$ are parallel along the curve $y_t$. We should also require that $S_t \neq 0$, that is,

\begin{equation}(\mu_1 - \lambda_1)^2 + (\mu_2 - \lambda_2)^2 \neq 0 \text{ for any } t .
\end{equation}

To discuss this condition, we introduce the following concept. Let $M, N$ be two oriented surfaces with unit normal vectors $\xi$ and $\eta$, respec-
tively. Let \( x \in M, y \in N, X \in T_x(M), \) and \( Y \in T_y(N). \) We say that \( M \) and \( N \) have the same shape along \( X \) and \( Y \) if there is a linear isometry \( F \) of \( T_x(E^m) \) onto \( T_y(E^m) \) such that

\[
F(\xi) = \eta, \quad F(X) = Y \quad \text{and} \quad F(AX) = B(Y),
\]

where \( A \) and \( B \) are the shape operators of \( M \) and \( N \) relative to \( \xi \) and \( \eta, \) respectively.

Now the third equations of (15) and (16) can be written as

\[
A(dx/dt) = \lambda_1 a_1 + \lambda_2 a_2 \\
B(dy/dt) = \mu_1 b_1 + \mu_2 b_2.
\]

Since \( C_t \) maps \( dx/dt, a_1, a_2 \) upon \( dy/dt, b_1, b_2, \) respectively, the equalities \( \mu_1 = \lambda_1 \) and \( \mu_2 = \lambda_2 \) will mean that \( M \) and \( N \) have the same shape along the vectors \( dx/dt \) and \( dy/dt. \)

In order to find a rolling \( \{ f_t \} \) from the given curve \( x_t \) on \( M, \) we must know how to determine the curve \( y_t \) on \( N. \) This can be done by making use of the development \( z_t \) of \( x_t \) into the tangent plane \( T_{x_0}(M). \)

If we write

(19) \[
dx/dt = k_1 a_1 + k_2 a_2
\]

with suitable functions \( k_1 = k_1(t) \) and \( k_2 = k_2(t), \) then

(20) \[
dy/dt = k_1 b_1 + k_2 b_2
\]

because of (14) and \( C_t(dx/dt) = dy/dt. \)

Let \( e_1 = a_1(0) \) and \( e_2 = a_2(0). \) Since \( a_1(t) \) and \( a_2(t) \) are parallel along the curve \( x_t, \) the development \( z_t \) of \( x_t \) is given by integrating

(21) \[
dx/dt = k_1 e_1 + k_2 e_2.
\]

Similarly, the development of the curve \( y_t \) into the tangent plane \( T_{y_0}(N) = T_{x_0}(M) \) is also given by (21), namely, \( z_t \) is the development of \( y_t. \) This means that \( y_t \) is determined as the unique curve in \( N \) with \( y_0 = x_0 \) whose development into \( T_{y_0}(N) \) is equal to \( z_t. \)

Summarizing the discussions we can state

**Theorem 3.** Let \( M \) and \( N \) be two orientable surfaces which are tangent to each other at \( x_0. \) Let \( x_t \) be a smooth curve on \( M. \) Then we can find a unique rolling \( \{ f_t \} \) of \( M \) onto \( N \) such that \( y_t = f_t(x_t) \) is the locus of centers of instantaneous rotation provided the following condition is satisfied. Let \( z_t \) be the development of the curve \( x_t \) into the tangent plane \( T_{x_0}(M). \) Let \( y_t \) be the unique curve on \( N \) such that its development into \( T_{y_0}(N) = T_{x_0}(M) \) is \( z_t. \) Take the fields of unit normals \( \xi \) and \( \eta \) for \( M \) and \( N \) such that they coincide at \( x_0 = y_0. \) The condition
to be satisfied is that, for each $t$, $M$ and $N$ do not have the same shape along the vectors $dx/dt$ and $dy/dt$ (for the chosen normals $\xi$ and $\eta$).

We remark that in the case of two spheres tangent to, and outside of, each other, the condition in question is satisfied for an arbitrary curve $x_t$ for the choice of inward normals $\xi$ for one sphere $M$ and outward normals $\eta$ for the other sphere $N$.

It is possible to find a number of sufficient conditions under which a rolling (with $S_t \neq 0$) is possible for a given curve $x_t$. For example,

(i) $A(dx/dt) = 0$ for every $t$ and $N$ has no flat point; for example, $x_t$ is a generator of a cylinder and $N$ is a sphere.

(ii) $A(dx/dt) \neq 0$ for each $t$ and $N$ is a plane.

(iii) $dx/dt$ is not a principal vector at any point and $N$ is umbilical (a plane or a sphere).

A rolling is possible for an arbitrary curve on $M$ if $M$ and $N$ satisfy the following condition:

(iv) the principal curvatures of $M$ are greater than those of $N$; here we assume that $\xi$ and $\eta$ are chosen so that $\xi_n = \eta_n$.

If $M$ and $N$ have the same shape along unit vectors $X$ and $Y$, then we have $\langle AX, X \rangle = \langle BY, Y \rangle$. But $\langle AX, X \rangle$ is greater than or equal to the smaller principal curvature of $M$ at the point and $\langle BY, Y \rangle$ is smaller than or equal to the larger principal curvature of $N$ at the corresponding point. Thus condition (iv) is sufficient.

6. The case of submanifolds. Now let $M$ and $N$ be two $n$-dimensional submanifolds in an $m$-dimensional Euclidean space $E^m$ which are tangent to each other at a point $x_0$. We shall first define the notion of rolling $\{f_t\}$ of $M$ and $N$.

Let $\{f_t\}$ be a 1-parametric motion of $E^m$ given by (2). Assume that $f_t(M)$ is tangent to $N$ at a point $y_t$ at each instant $t$. We assume that the instantaneous motion $X_t$ vanishes at $y_t$ (that is, $y_t$ is a center of instantaneous rotation) and that $S_t \neq 0$. We say that $\{f_t\}$ is a rolling if the skew-symmetric transformation $S_t$ maps the tangent space $T_{y_t}(N)$ into the normal space $T_{y_t}^\perp(N)$ and maps $T_{y_t}^\perp(N)$ into $T_{y_t}(N)$, thus,

\[
\langle S_t(X), Y \rangle = 0 \text{ for all } X, Y \in T_{y_t}(N)
\]
\[
\langle S_t(U), V \rangle = 0 \text{ for all } U, V \in T_{y_t}^\perp(N).
\]

This is clearly a generalization of the definition in the case where $n = 2$ and $m = 3$ as well as the case of $S^*$ and its tangent plane.

Suppose that a smooth curve $x_t$ is given on $M$. We wish to find a rolling $\{f_t\}$ of $M$ on $N$ such that $y_t = f_t(x_t)$ is the locus of centers of
instantaneous rotation. Let \( \{a_i(0), \cdots, a_n(0)\} \) be an orthonormal basis in \( T_{x_0}(M) \) and let \( \{a_{n+1}(0), \cdots, a_m(0)\} \) be an orthonormal basis in the normal space \( T_{x_0}^*(M) \). Let \( a_i(t), 1 \leq i \leq n \), be the tangent vector parallel along \( x_t \) on \( M \) with initial condition \( a_i(0) \); thus

\[
F_t a_i = 0
\]

\[
da_i/dt = \alpha(dx/dt, a_i)
\]

for \( 1 \leq i \leq n \), where \( F \) is the Levi-Civita connection of \( M \) and \( \alpha \) denotes the second fundamental form as a bilinear mapping \( T_{x_0}(M) \times T_{x_0}(M) \to T_{x_0}^*(M) \) for each \( x \in M \).

For each \( j, n + 1 \leq j \leq m \), let \( a_j(t) \) be the normal vector field parallel along \( x_t \), with respect to the normal connection of \( M \) with initial vector \( a_j(0) \). Thus

\[
F^\perp_t a_j = 0
\]

\[
da_j/dt = -A_{a_j}(dx/dt)
\]

for \( n + 1 \leq j \leq m \), where \( F^\perp \) denotes covariant differentiation for the normal connection and \( A_{a_j} \) denotes the shape operator corresponding to the normal vector \( a_j \).

Now suppose \( \{f_i\} \) is a rolling given by (2). We define

\[
b_k(t) = C_t(a_k(t))
\]

for \( 1 \leq k \leq m \). Since \( f_i(M) \) is tangent to \( N \) at \( y_t \), it follows that \( b_1(t), \cdots, b_n(t) \) are tangent to \( N \) at \( y_t \) and \( b_{n+1}(t), \cdots, b_m(t) \) are normal to \( N \) at \( y_t \).

Let \( 1 \leq i \leq n \). Differentiating (25) and using (23) we obtain

\[
db_i/dt = (dC/dt)a_i + C_t(da_i/dt)
\]

\[
= (dC/dt)C_t^{-1}b_i + C_t(\alpha(dx/dt, a_i))
\]

On the other hand, we have

\[
db_i/dt = F_t b_i + \beta(dy/dt, b_i)
\]

where \( F \) is the Levi-Civita connection for \( N \) and \( \beta \) is the second fundamental form \( T_y(N) \times T_y(N) \to T_y^*(N) \) for \( N \). Thus we obtain

\[
S_t(b_i) = F_t b_i + \beta(dy/dt, b_i) - C_t(\alpha(dx/dt, a_i))
\]

By definition of a rolling, \( S_t(b_i) \) is normal to \( N \). Since \( \beta(dy/dt, b_i) \) and \( C_t(\alpha(dx/dt, a_i)) \) are normal to \( N \), we must have

\[
F_t b_i = 0
\]

namely, \( b_i \) is parallel along \( y_t \), and
\( S_t(b_i) = \beta(dy/dt, b_i) - C_t(\alpha(dx/dt, a_i)) \).

Now let \( n + 1 \leq j \leq m \). Differentiating (25) and using (24) we obtain

\[
\frac{db_j}{dt} = (dC/dt)a_j + C_t(da_j/dt) = (dC/dt)C_i^i b_j + C_t(-A_s_j(dx/dt)) .
\]

On the other hand, we have

\[
\frac{db_j}{dt} = -B_{xj}(dy/dt) + V^i b_j ,
\]

where \( B_{xj} \) is the shape operator for \( N \) corresponding to the normal vector \( b_j \) and \( V^i \) is covariant differentiation along \( y_t \) for the normal connection of \( N \). Thus we obtain

\[
S_t(b_j) = C_t(A_{a_j}(dx/dt)) - B_{xj}(dy/dt) + V^i b_j .
\]

Since \( S_t(b_j) \) must be tangent to \( N \) at \( y_t \), we conclude that \( V^i b_j = 0 \), namely, \( b_j \) is \( V^i \)-parallel along the curve \( y_t \). We have also

\[
S_t(b_j) = C_t(A_{a_j}(dx/dt)) - B_{xj}(dy/dt) .
\]

The skew-symmetric transformation \( S_t \) given in the form (26) and (27) can be expressed more conveniently if we adopt the following operators \( \rho \) for \( M \) and \( \tau \) for \( N \).

For each point \( x \) of \( M \), define for each \( X \in T_x(M) \) a linear endomorphism of \( T_x(E^m) = T_x(M) + T_x^t(M) \) by

\[
\rho_x(Y) = \alpha(X, Y) \quad \text{for} \quad Y \in T_x(M) \quad \rho_x(U) = -A_v(X) \quad \text{for} \quad U \in T_x^t(M) .
\]

Then \( \rho_x \) is a skew-symmetric endomorphism of \( T_x(E^m) \). Similarly, for each point \( y \) of \( N \), we define \( \tau_x \), \( X \in T_y(N) \), by

\[
\tau_x(Y) = \beta(X, Y) \quad \text{for} \quad Y \in T_y(N) \quad \tau_x(U) = -B_v(X) \quad \text{for} \quad U \in T_y^t(N) .
\]

With these operators we may write (26) in the form

\[
S_t(b_i) = \tau_t(b_i) - C_t(\rho_t(a_i)) = \tau_t(b_i) - C_t\rho_tC_t^{-1}(b_i) ,
\]

where we write \( \rho_t \) for \( \rho_{dx/dt} \) and \( \tau_t \) for \( \tau_{dy/dt} \) for brevity. From (27) we have

\[
S_t(b_j) = -C_t\rho_t(a_j) + \tau_t(b_j) = \tau_t(b_j) - C_t\rho_tC_t^{-1} .
\]

Thus we may simply write
In order to discuss conditions under which $S_t \neq 0$, we extend the notion of same shape to $n$-dimensional submanifolds $M$ and $N$ in $E^m$. Let $X \in T_x(M)$ and $Y \in T_y(N)$. We say that $M$ and $N$ have the same shape along $X$ and $Y$ if there is a linear isometry $F$ of $T_x(E^m)$ onto $T_y(E^m)$ such that

$$F(T_x(M)) = T_y(N), \quad F(T_y(M)) = T_x(N)$$

$$F(X) = Y \quad \text{and} \quad F \circ \rho_x = \tau_y \circ F.$$  

From (28) we see that if $S_t = 0$, then $F = C_t$ satisfies these conditions for $X = dx/dt$ and $Y = dy/dt$, namely, $M$ and $N$ have the same shape along $X$ and $Y$.

We may find a number of sufficient conditions under which a rolling (with $S_t \neq 0$) exists. Recall that the relative nullity space of a submanifold $M$ at $x$ is the subspace of $T_x(M)$ consisting of all $X \in T_x(M)$ with $\rho_x = 0$. Its dimension is called the index of relative nullity. For example, a rolling exists for a curve $x_t$ if

1. $dx/dt$ is in the relative nullity space for every $t$ and $N$ has index of relative nullity 0,
2. $dx/dt$ is not in the relative nullity space for any $t$ and $N$ is a Euclidean $n$-plane,
3. the sectional curvature of any plane containing the tangent vector $dx/dt$ never vanishes and $N$ is a Euclidean $n$-plane.

The last case follows from the second because the sectional curvature of any plane containing a vector $X$ with $\rho_x = 0$ must vanish. We also remark that the so-called Veronese variety (the image of $S^n$ by a certain isometric imbedding into $E^m$, where $m = n(n+3)/2$ has index of relative nullity 0.

We also recall that a non-zero vector $X \in T_x(M)$ is called a principal vector if there is $\xi_t \in T^t_x(M)$ such that $A_t(X) = \langle \xi_t, \xi_t \rangle X$ for all $\xi \in T^t_x(M)$. A rolling exists for a curve $x_t$ on $M$ if

4. $dx/dt$ is not a principal vector for each $t$ and $N$ is an $n$-plane or an $n$-sphere.

Let us remark that the Veronese variety mentioned above has no principal vector.

In any case, the relationship between the curve $x_t$ on $M$ and the curve $y_t$ on $N$ is as before. Their developments into $T_{x_0}(M) = T_{y_0}(N)$ coincide.

We now consider the following special case.

Let $M$ be an $n$-dimensional submanifold in $E^m$ and $x_0$ a point of $M$. 

As the second submanifold $N$ we take the $n$-plane $T_{x_0}(M)$. We are going to obtain a kinematic interpretation of the second fundamental form and the normal connection of $M$.

Let $X \in T_{x_0}(M)$ and take any curve $x_t$ with initial tangent vector $X$. If we have a rolling $\{f_t\}$ of $M$ onto $N$ determined by $x_t$, then (28) implies

$$S_0 = -\rho_X.$$  

This gives the kinematic interpretation of the second fundamental form $\alpha$ of $M$ at $x_0$, because $\rho$ determines $\alpha$ completely.

If $\xi_t$ is a field of normal vectors along $x_t$, we may write

$$\xi_t = \sum_{j=n+1}^{m} q_j(t)a_j,$$

where $a_{n+1}(t), \ldots, a_m(t)$ are $F^1$-parallel normal vector fields along $x_t$ as before. Then

$$C_1(\xi_t) = \sum_{j=n+1}^{m} q_j(t)b_j$$

and, as we have seen before, each $b_j = C_1(a_j)$ is $F^1$-parallel along the curve $y_t = f_t(x_t)$. Since $N$ is an $n$-plane, this means that each $b_j$ is a constant vector in $E^n$. It follows that $\xi_t$ is $F^1$-parallel along the curve $x_t$ if and only if $C_1(\xi_t)$ is a constant vector in $E^n$ (that is, each $q_j(t)$ is constant). This is the kinematic interpretation of the normal connection of $M$.

REFERENCES


DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND
02912 U.S.A.