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INDEX OF SOME GAUSS-CRITICAL SUBMANIFOLDS

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Introduction. As is well-known, the Gauss map is an excellent device in classical differential geometry where curves and surfaces in a Euclidean three-space are studied. The same is true when the Gauss map is applied to an m-dimensional submanifold in a Euclidean n-space. In this case the image lies in the Grassmann manifold $G(m, n-m)$ which is not a sphere nor a projective space if m satisfies $1 < m < n-1$.

Let M be an m-dimensional compact orientable C^∞ submanifold in a Euclidean n-space E^n such that the Gauss map $\Gamma: M \to G(m, n-m)$ is regular. We consider only the case $1 < m < n-1$, for, if $m = n-1$, then Γ maps every closed hypersurface onto the $(n-1)$-sphere, whereas, if $m = 1$, a simpler method may be available. Nevertheless, one of the motives of the present study lies in the fact:

Let C be a closed curve with positive curvature in a Euclidean three-space. Then the Gauss image of C in the standard sphere has the least length when and only when C lies in a plane.

Assuming the standard Riemannian metric on $G(m, n-m)$, we get a volume form on $\Gamma(M)$. From the pull back of this volume form to M we get an integral $\mathrm{Vol}^*(\Gamma(M))$. As the Gauss image $\Gamma(M)$ is immersed in the Grassmann manifold, $\mathrm{Vol}^*(\Gamma(M))$ is not always the volume of $\Gamma(M)$. When M moves smoothly in E^n, $\mathrm{Vol}^*(\Gamma(M))$ moves in $\boldsymbol{R}$. Thus we can consider a submanifold M_0 which is a critical point of $\mathrm{Vol}^*(\Gamma(M))$. M_0 is called a Gauss-critical submanifold and is denoted by GCS. As it is always the case with critical points, there arises the problem of finding the index. The purpose of the present paper is to prove that a submanifold M which lies in a linear subspace E^{m+1} as a closed hypersurface with positive second fundamental form is a GCS whose index is zero.

A theorem related to this result has been obtained by Chern and Lashof [1], namely,

THEOREM OF CHERN AND LASHOF. *Let $i: M \to E^n$ be an immersion of an m-dimensional compact manifold M into a Euclidean n-space E^n. Then the total absolute curvature $\tau(M, i, E^n)$ is equal to 2 if and only* *if the immersion i is an imbedding and iM is a convex hypersurface in a linear subspace E^{m+1} of E^n.*

See also N. H. Kuiper [2] for related topics.

In the integral $\mathrm{Vol}^*(\Gamma(M))$ and also in the total absolute curvature the second fundamental form plays an essential role, but the relation between them is difficult to find. An obvious difference is that $\mathrm{Vol}^*(\Gamma(M))$ is defined only when the Gauss map Γ is regular in the present study.

In §1 we recall the Gauss map together with some symbols used in the present paper. $\mathrm{Vol}^*(\Gamma(M))$ is defined. In §2 a general formula of the second variation of $\mathrm{Vol}^*(\Gamma(M))$ is given. In §3 we consider an infinitesimal deformation of a submanifold of E^n starting from a closed hypersurface M_0 of E^{m+1}. In §4 the formula of the second variation is obtained. In §5 an integral inequality on a closed hypersurface of E^{m+1} with positive second fundamental form is obtained and with the use of this inequality the Main Theorem (Theorem 5.6) is proved.

1. Gauss map and Gauss-critical submanifolds. Let (M, g) be an m-dimensional closed C^∞ submanifold of E^n, where g is the Riemannian metric induced by immersion. Let $x^h(h, i, j, \cdots = 1, \cdots, n)$ be the rectangular coordinates in E^n and $y^\kappa(\kappa, \lambda, \mu, \cdots = 1, \cdots, m)$ the local coordinates in any coordinate neighborhood of M such that the immersion $M \to E^n$ is given locally by C^∞ functions

$$x^h = x^h(y^1, \cdots, y^m) .$$

M is assumed to be covered by a set of such coordinate neighborhoods. We define

$$B_\lambda{}^h = B_\lambda^h = \partial x^h/\partial y^\lambda,\ g_{\mu\lambda} = \sum_h B_\mu^h B_\lambda^h . \tag{1.1}$$

Then $g_{\mu\lambda}$ are the covariant components of the Riemannian metric g. The contravariant components are $g^{\mu\lambda}$ which satisfy $g^{\mu\alpha}g_{\lambda\alpha} = \delta_\lambda^\mu$ $(\alpha, \beta, \gamma, \cdots = 1, \cdots, m)$. We use them for lowering and raising the indices in Greek letters. Let $\left\{{\kappa \atop \mu\lambda}\right\}$ be the Christoffel symbols derived from $g_{\mu\lambda}$ and ∇_λ the symbol of Van der Waerden-Bortolotti's covariant differentiation. Then

$$H_{\mu\lambda}{}^h = \nabla_\mu B_\lambda^h = \partial_\mu\partial_\lambda x^h - \left\{{\kappa \atop \mu\lambda}\right\}\partial_\kappa x^h , \tag{1.2}$$

where $\partial_\lambda x^h = \partial x^h/\partial y^\lambda$, is the second fundamental tensor.

We define

$$G_{\mu\lambda} = \sum_h H_{\mu\alpha}{}^h H_\lambda{}^{\alpha h} . \tag{1.3}$$

Then $G_{\mu\lambda}$ is non-negative and we can define an m-form ω on M whose local expression is

$$\omega = [\det(G_{\mu\lambda})]^{1/2} dy^1 \wedge \cdots \wedge dy^m . \tag{1.4}$$

From ω we define

$$\gamma = \int_M \omega . \tag{1.5}$$

Now the Grassmann manifold $G(m, n-m)$ is the space of m-planes Π of E^n passing the origin and the Gauss map Γ carries a point p of M into an element Π of $G(m, n-m)$ which is an m-plane parallel to M_p. On the other hand the Grassmann manifold bears the standard Riemannian metric $\tilde{g}$ of K. Leichtweiss [3], [5] and the ratio of any line element in (M, g) to its Gauss image is given by

$$[g_{\mu\lambda} dy^\mu dy^\lambda]^{1/2} : [G_{\mu\lambda} dy^\mu dy^\lambda]^{1/2} .$$

Thus, if Γ is regular, namely $\det(G_{\mu\lambda}) > 0$, ω is the volume form of the Gauss image [4]. But, as Γ is an immersion, γ differs in general from $\mathrm{Vol}(\Gamma(M))$ and is denoted by $\mathrm{Vol}^*(\Gamma(M))$.

Let M_0 be one of such submanifolds and $\{M(t), t \in \boldsymbol{R}\}$ a set of m-dimensional closed submanifolds such that $M(0) = M_0$. Moreover, we assume that, if $I = [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, there exists a C^∞ map $\varphi: M_0 \times I \to E^n$ where $\varphi(M_0, t) = M(t)$ for $t \in I$. For sufficiently small ε, Γ is regular for all $M(t)$, $-\varepsilon \leqq t \leqq \varepsilon$, and we put $\gamma(t) = \mathrm{Vol}^*(\Gamma(M(t)))$. If $t = 0$ is a critical point of $\gamma(t)$ for every such set $\{M(t), t \in I\}$, we say that M_0 is a Gauss-critical submanifold and denote it by GCS.

The equation of GCS was obtained in [4]. A submanifold of E^n which lies in a subspace E^{m+1} of E^n as a closed hypersurface with positive second fundamental form is a GCS. The purpose of the present paper is to prove that the index is zero for such a submanifold.

2. **Second variation of** $\mathrm{Vol}^*(\Gamma(M))$**.** If v is the volume form of (M, g), we get from (1.4) and (1.5)

$$\gamma(t) = \int_{M(t)} [\det(G_{\mu\lambda})/\det(g_{\mu\lambda})]^{1/2} v \tag{2.1}$$

for $\mathrm{Vol}^*(\Gamma(M(t)))$. As, in the second member, $G_{\mu\lambda}$, $g_{\mu\lambda}$, and v depend on t, in order to get the derivatives $d\gamma/dt$ and $d^2\gamma/dt^2$ we prefer for the present the following expression,

$$\gamma(t) = \int_{M(t)} [\det(G_{\mu\lambda})]^{1/2} dy \tag{2.2}$$

where $dy = [\det(g_{\mu\lambda})]^{-1/2}v$ does not depend on t. Notice that we take coordinate neighborhoods such that $p \in M_0$ and $\varphi(p, t) \in M(t)$ have always the same local coordinates and the immersion $M(t) \to E^n$ is expressed locally by

$$x^h = x^h(y^1, \cdots, y^m; t) . \tag{2.3}$$

From (2.2) we get

$$\frac{d\gamma}{dt} = \int_{M(t)} \frac{\partial}{\partial t}[\det(G_{\mu\lambda})]^{1/2} dy , \tag{2.4}$$

$$\frac{d^2\gamma}{dt^2} = \int_{M(t)} \frac{\partial^2}{\partial t^2}[\det(G_{\mu\lambda})]^{1/2} dy . \tag{2.5}$$

Let us define $(G^{-1})^{\mu\lambda}$ by

$$(G^{-1})^{\mu\alpha} G_{\lambda\alpha} = \delta^\mu_\lambda .$$

Then we have

$$\frac{\partial}{\partial t}[\det(G_{\mu\lambda})]^{1/2} = \frac{1}{2}[\det(G_{\mu\lambda})]^{1/2}(G^{-1})^{\beta\alpha}\frac{\partial}{\partial t}G_{\beta\alpha} , \tag{2.6}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}[\det(G_{\mu\lambda})]^{1/2} &= \frac{1}{4}[\det(G_{\mu\lambda})]^{1/2}\left[(G^{-1})^{\beta\alpha}\frac{\partial}{\partial t}G_{\beta\alpha}\right]^2 \\ &\quad - \frac{1}{2}[\det(G_{\mu\lambda})]^{1/2}(G^{-1})^{\beta\sigma}(G^{-1})^{\alpha\rho}\frac{\partial}{\partial t}G_{\beta\alpha}\frac{\partial}{\partial t}G_{\sigma\rho} \\ &\quad + \frac{1}{2}[\det(G_{\mu\lambda})]^{1/2}(G^{-1})^{\beta\alpha}\frac{\partial^2}{\partial t^2}G_{\beta\alpha} .\end{aligned} \tag{2.7}$$

If we use vector fields ξ^h, η^h defined by

$$\xi^h = \frac{\partial x^h}{\partial t}, \quad \eta^h = \frac{\partial^2 x^h}{\partial t^2} ,$$

we get from (1.1) and (1.2)

$$\frac{\partial}{\partial t}B^h_\lambda = \partial_\lambda \xi^h = \nabla_\lambda \xi^h, \quad \frac{\partial}{\partial t}g_{\mu\lambda} = \sum_h B^h_\mu \nabla_\lambda \xi^h + \sum_h B^h_\lambda \nabla_\mu \xi^h ,$$

$$\frac{\partial}{\partial t}g^{\mu\lambda} = -g^{\mu\sigma}g^{\lambda\rho}\frac{\partial}{\partial t}g_{\sigma\rho} = -\sum_h B^{\mu h}\nabla^\lambda \xi^h - \sum_h B^{\lambda h}\nabla^\mu \xi^h ,$$

$$\frac{\partial}{\partial t}\left\{ {\gamma \atop \beta\alpha} \right\} = \sum_h B^{\gamma h}\nabla_\beta \nabla_\alpha \xi^h + \sum_h H_{\beta\alpha}{}^h \nabla^\gamma \xi^h ,$$

$$\frac{\partial}{\partial t}H_{\beta\alpha}{}^h = \nabla_\beta \nabla_\alpha \xi^h - B^{\gamma h}\sum_i B^i_\gamma \nabla_\beta \nabla_\alpha \xi^i - B^{\gamma h}\sum_i H_{\beta\alpha}{}^i \nabla_\gamma \xi^i ,$$

where $B^{\lambda h} = g^{\lambda\alpha}B^h_\alpha$. We also obtain from (1.3)

$$(2.8)\quad \frac{\partial}{\partial t}G_{\beta\alpha} = \Big(\sum_h H_{\beta\rho}{}^h\nabla_\alpha\nabla_\sigma\xi^h + \sum_h H_{\alpha\rho}{}^h\nabla_\beta\nabla_\sigma\xi^h\Big)g^{\sigma\rho}$$
$$- \sum_i H_{\beta\sigma}{}^i H_{\alpha\rho}{}^i g^{\sigma\mu}g^{\rho\lambda}\Big(\sum_h B^h_\mu\nabla_\lambda\xi^h + \sum_h B^h_\lambda\nabla_\mu\xi^h\Big),$$

$$(2.9)\quad \frac{\partial^2}{\partial t^2}G_{\beta\alpha} = \Big(\sum_h H_{\beta\rho}{}^h\nabla_\alpha\nabla_\sigma\eta^h + \sum_h H_{\alpha\rho}{}^h\nabla_\beta\nabla_\sigma\eta^h\Big)g^{\sigma\rho}$$
$$- \sum_i H_{\beta\sigma}{}^i H_{\alpha\rho}{}^i g^{\sigma\mu}g^{\rho\lambda}\Big(\sum_h B^h_\mu\nabla_\lambda\eta^h + \sum_h B^h_\lambda\nabla_\mu\eta^h\Big)$$
$$- \left[\frac{\partial}{\partial t}\left\{ {\gamma \atop \beta\sigma} \right\}\sum_h H_{\alpha\rho}{}^h\nabla_\gamma\xi^h + \frac{\partial}{\partial t}\left\{ {\gamma \atop \alpha\sigma} \right\}\sum_h H_{\beta\rho}{}^h\nabla_\gamma\xi^h\right]g^{\sigma\rho}$$
$$+ \left[\sum_h \frac{\partial}{\partial t}H_{\beta\rho}{}^h\nabla_\alpha\nabla_\sigma\xi^h + \sum_h \frac{\partial}{\partial t}H_{\alpha\rho}{}^h\nabla_\beta\nabla_\sigma\xi^h\right]g^{\sigma\rho}$$
$$+ \left[\sum_h H_{\beta\rho}{}^h\nabla_\alpha\nabla_\sigma\xi^h + \sum_h H_{\alpha\rho}{}^h\nabla_\beta\nabla_\sigma\xi^h\right]\frac{\partial}{\partial t}g^{\sigma\rho}$$
$$- \left[\sum_i \frac{\partial}{\partial t}H_{\beta\sigma}{}^i H_{\alpha\rho}{}^i + \sum_i H_{\beta\sigma}{}^i\frac{\partial}{\partial t}H_{\alpha\rho}{}^i\right]g^{\sigma\mu}g^{\rho\lambda}\left[\sum_h B^h_\mu\nabla_\lambda\xi^h\right.$$
$$\left.+ \sum_h B^h_\lambda\nabla_\mu\xi^h\right] - \sum_i H_{\beta\sigma}{}^i H_{\alpha\rho}{}^i\left[\frac{\partial}{\partial t}g^{\sigma\mu}g^{\rho\lambda} + g^{\sigma\mu}\frac{\partial}{\partial t}g^{\rho\lambda}\right]$$
$$\times\left[\sum_h B^h_\mu\nabla_\lambda\xi^h + \sum_h B^h_\lambda\nabla_\mu\xi^h\right]$$
$$- 2\sum_i H_{\beta\sigma}{}^i H_{\alpha\rho}{}^i g^{\sigma\mu}g^{\rho\lambda}\sum_h \nabla_\mu\xi^h\nabla_\lambda\xi^h .$$

3. A submanifold M_0 of E^n which lies in E^{m+1} as a closed hypersurface with positive second fundamental form. Now let us consider the case where $M(0) = M_0$ is imbedded in E^n as a closed hypersurface with positive second fundamental form in a subspace E^{m+1} of E^n. As $M(t)$, $t \neq 0$, need not be confined to that E^{m+1}, the vector fields ξ^h and η^h are arbitrary for $t = 0$. Let $\underset{X}{C}$ $(X = m+2, \cdots, n)$ be orthonormal normal vectors of E^{m+1} which are constant vectors. Let us take a rectangular coordinate system of E^n such that the E^{m+1} under consideration is given by $x^{m+2} = \cdots = x^n = 0$ and the components $\underset{X}{C}{}^h$ of $\underset{X}{C}$ satisfy $\underset{X}{C}{}^1 = \cdots = \underset{X}{C}{}^{m+1} = 0$. We also obtain

$$(3.1)\qquad H_{\mu\lambda}{}^h = h_{\mu\lambda}N^h$$

for M_0 where N^h is the unit normal vector field lying in E^{m+1} and $h_{\mu\lambda}$ the second fundamental form of M_0 as a hypersurface of E^{m+1}.

We also get

$$\nabla_\mu N^h = -h_\mu{}^\alpha B^h_\alpha \,, \tag{3.2}$$

$$\nabla_\nu h_{\mu\lambda} = \nabla_\mu h_{\nu\lambda} \tag{3.3}$$

and the equation of Gauss,

$$K_{\nu\mu\lambda}{}^\kappa = h_{\mu\lambda} h_\nu{}^\kappa - h_{\nu\lambda} h_\mu{}^\kappa \,. \tag{3.4}$$

With the use of these equations we want to get formulas for $d\gamma/dt$ and $d^2\gamma/dt^2$ at $t = 0$. In the following calculation all quantities are to be evaluated at $t = 0$.

Then we get

$$\frac{\partial}{\partial t}\begin{Bmatrix} \gamma \\ \beta\alpha \end{Bmatrix} = \sum_h B^{\gamma h} \nabla_\beta \nabla_\alpha \xi^h + h_{\beta\alpha} \sum_h N^h \nabla^\gamma \xi^h \,, \tag{3.5}$$

$$\frac{\partial}{\partial t} H_{\beta\alpha}{}^h = \nabla_\beta \nabla_\alpha \xi^h - B^{\gamma h} \sum_i B^i_\gamma \nabla_\beta \nabla_\alpha \xi^i - h_{\beta\alpha} B^{\gamma h} \sum_i N^i \nabla_\gamma \xi^i \,. \tag{3.6}$$

We also get from (1.3) and (3.1)

$$G_{\mu\lambda} = h_{\mu\alpha} h_\lambda{}^\alpha \,, \quad (G^{-1})^{\mu\lambda} = k^\mu{}_\gamma k^{\gamma\lambda} \,, \tag{3.7}$$

where $k^{\mu\lambda}$ are defined by $k^{\mu\alpha} h_{\lambda\alpha} = \delta^\mu_\lambda$.

In order to apply Green's theorem in the subsequent calculation we rewrite (2.4) and (2.5) in the following form where v is the volume form of M_0,

$$\frac{d\gamma}{dt} = \int_{M_0} Av \,, \tag{3.8}$$

$$\frac{d^2\gamma}{dt^2} = \int_{M_0} Bv \,. \tag{3.9}$$

From (3.7) we get

$$[\det(G_{\mu\lambda})/\det(g_{\mu\lambda})]^{1/2} = \det(h_\mu{}^\lambda) \,.$$

Hence, in view of (2.6), (2.8), and (3.1), we can calculate the integrand A of (3.8) as follows,

$$\begin{aligned} A &= \frac{1}{2}\det(h_\mu{}^\lambda) k^\beta{}_\gamma k^{\gamma\alpha} \frac{\partial}{\partial t} G_{\beta\alpha} \\ &= \det(h_\mu{}^\lambda) k^\beta{}_\gamma k^{\gamma\alpha}\Big[h_\alpha{}^\sigma \sum_h N^h \nabla_\beta \nabla_\sigma \xi^h - h_\beta{}^\sigma h_\alpha{}^\rho \sum_h B^h_\sigma \nabla_\rho \xi^h \Big] \\ &= \det(h_\mu{}^\lambda)\Big[k^{\beta\sigma} \sum_h N^h \nabla_\beta \nabla_\sigma \xi^h - \sum_h B^h_\sigma \nabla^\sigma \xi^h \Big] \end{aligned}$$

$$= \nabla_\beta\Big[\det(h_\mu{}^\lambda)k^{\beta\sigma}\sum_h N^h\nabla_\sigma\xi^h\Big]$$
$$- \sum_h \nabla_\beta[\det(h_\mu{}^\lambda)k^{\beta\sigma}N^h]\nabla_\sigma\xi^h - \det(h_\mu{}^\lambda)\sum_h B_\sigma^h\nabla^\sigma\xi^h .$$

But the first term in the last member vanishes on integration by Green's theorem. We use the symbol $\cong$ if two members are equal except such a divergence term, hence

$$A \cong - \sum_h \nabla_\beta[\det(h_\mu{}^\lambda)k^{\beta\sigma}N^h]\nabla_\sigma\xi^h - \det(h_\mu{}^\lambda)\sum_h B_\sigma^h\nabla^\sigma\xi^h .$$

On the other hand we have

$$\nabla_\beta[\det(h_u{}^\lambda)k^{\beta\sigma}N^h]$$
$$= \det(h_\mu{}^\lambda)[(k_\kappa{}^\nu\nabla_\beta h_\nu{}^\kappa k^{\beta\sigma} - k^{\beta\nu}k^{\sigma\kappa}\nabla_\beta h_{\nu\kappa})N^h + k^{\beta\sigma}\nabla_\beta N^h] .$$

Hence we get $A \cong 0$ because of (3.2) and (3.3). This proves $d\gamma/dt = 0$ at $t = 0$ and the following theorem [4].

THEOREM 3.1. *A C^∞ submanifold M_0 of E^n which lies in a linear subspace E^{m+1} as a closed hypersurface with positive second fundamental form is a Gauss-critical submanifold.*

REMARK. Our assumption is that the Gauss map of M is regular. Then it is an inevitable consequence that we consider only the case of positive second fundamental form. As a consequence there exists the matrix $(k_\mu{}^\lambda)$.

4. The second derivative $(d^2\gamma/dt^2)_0$ when M_0 is a hypersurface of an E^{m+1} with positive second fundamental form. We have proved in §3 that $(d\gamma/dt)_0$ vanishes for the submanifold M_0 under consideration, and this implies that the terms of B involving η^h do not contribute to the integral $(d^2\gamma/dt^2)_0$. In order to get an expression for $(d^2\gamma/dt^2)_0$ which is convenient for us in finding the index, we need some lengthy calculation as is predictable from (2.7), (2.8), (2.9).

We first obtain after some straightforward calculation

$$(G^{-1})^{\beta\alpha}\frac{\partial}{\partial t}G_{\beta\alpha} = 2k^{\beta\sigma}\sum_h N^h\nabla_\beta\nabla_\sigma\xi^h - 2\sum_h B_\alpha^h\nabla^\alpha\xi^h ,$$

$$(G^{-1})^{\beta\sigma}(G^{-1})^{\alpha\rho}\frac{\partial}{\partial t}G_{\beta\alpha}\frac{\partial}{\partial t}G_{\sigma\rho}$$
$$= 2\Big[k^\beta{}_\gamma k^{\gamma\alpha}\sum_i N^i\nabla_\beta\nabla_\sigma\xi^i\sum_h N^h\nabla_\alpha\nabla^\sigma\xi^h$$
$$+ k^{\beta\sigma}k^{\alpha\rho}\sum_i N^i\nabla_\beta\nabla_\alpha\xi^i\sum_h N^h\nabla_\sigma\nabla_\rho\xi^h$$

$$- 2k^{\beta\alpha} \sum_i N^i \nabla_\beta \nabla^\sigma \xi^i \sum_h (B^h_\sigma \nabla_\alpha \xi^h + B^h_\alpha \nabla_\sigma \xi^h)$$

$$+ \sum_i B^{\beta i} \nabla^\alpha \xi^i \sum_h (B^h_\beta \nabla_\alpha \xi^h + B^h_\alpha \nabla_\beta \xi^h) \Big] ,$$

$$(G^{-1})^{\beta\alpha} \frac{\partial^2}{\partial t^2} G_{\beta\alpha}$$

$$= [\text{terms involving } \eta]$$

$$+ 2k^\beta{}_\gamma k^{\gamma\alpha} \sum_h \nabla_\beta \nabla_\sigma \xi^h \nabla_\alpha \nabla^\sigma \xi^h$$

$$- 2k^\beta{}_\gamma k^{\gamma\alpha} \sum_{h,i} B^i_\rho B^{\rho h} \nabla_\beta \nabla_\sigma \xi^h \nabla_\alpha \nabla^\sigma \xi^i$$

$$- 4k^{\beta\alpha} \sum_i B^i_\gamma \nabla_\beta \nabla_\alpha \xi^i \sum_h N^h \nabla^\gamma \xi^h$$

$$- 4k^{\beta\sigma} \sum_i N^i \nabla_\beta \nabla^\rho \xi^i \sum_h (B^h_\sigma \nabla_\rho \xi^h + B^h_\rho \nabla_\sigma \xi^h)$$

$$- 2m \sum_i N^i \nabla_\beta \xi^i \sum_h N^h \nabla^\beta \xi^h$$

$$+ 2 \sum_i (B^{\beta i} \nabla^\alpha \xi^i + B^{\alpha i} \nabla^\beta \xi^i) \sum_h (B^h_\beta \nabla_\alpha \xi^h + B^h_\alpha \nabla_\beta \xi^h)$$

$$- 2 \sum_h \nabla_\alpha \xi^h \nabla^\alpha \xi^h .$$

Substituting these formulas into (2.7) we obtain

$$B \cong \det(h_u{}^\lambda) \Big[\Big(k^{\beta\alpha} \sum_h N^h \nabla_\beta \nabla_\alpha \xi^h \Big)^2 \tag{4.1}$$

$$- k^\beta{}_\gamma k^{\gamma\alpha} \sum_i N^i \nabla_\beta \nabla_\sigma \xi^i \sum_h N^h \nabla_\alpha \nabla^\sigma \xi^h$$

$$- k^{\beta\sigma} k^{\alpha\rho} \sum_i N^i \nabla_\beta \nabla_\alpha \xi^i \sum_h N^h \nabla_\sigma \nabla_\rho \xi^h$$

$$+ k^\beta{}_\gamma k^{\gamma\alpha} \sum_h \nabla_\beta \nabla_\sigma \xi^h \nabla_\alpha \nabla^\sigma \xi^h$$

$$- k^\beta{}_\gamma k^{\gamma\alpha} \sum_{h,i} B^i_\rho B^{\rho h} \nabla_\beta \nabla_\sigma \xi^h \nabla_\alpha \nabla^\sigma \xi^i$$

$$- 2k^{\beta\alpha} \sum_i N^i \nabla_\beta \nabla_\alpha \xi^i \sum_h B^h_\sigma \nabla^\sigma \xi^h$$

$$- 2k^{\beta\alpha} \sum_i B^i_\gamma \nabla_\beta \nabla_\alpha \xi^i \sum_h N^h \nabla^\gamma \xi^h$$

$$+ \Big(\sum_h B^h_\alpha \nabla^\alpha \xi^h \Big)^2$$

$$+ \sum_i B^{\beta i} \nabla^\alpha \xi^i \sum_h (B^h_\beta \nabla_\alpha \xi^h + B^h_\alpha \nabla_\beta \xi^h)$$

$$- m \sum_i N^i \nabla_\beta \xi^i \sum_h N^h \nabla^\beta \xi^h - \sum_h \nabla_\alpha \xi^h \nabla^\alpha \xi^h \Big] .$$

If we put

$$\underset{X}{f} = \sum_h \underset{X}{C}{}^h \xi^h \qquad X = m+2, \cdots, n , \tag{4.2}$$

where $\underset{X}{C}$ are as defined in §3, we get

$$
\begin{aligned}
&k^{\beta}{}_{\gamma}k^{\gamma\alpha}\sum_{h}\nabla_{\beta}\nabla_{\sigma}\xi^{h}\nabla_{\alpha}\nabla^{\sigma}\xi^{h}\\
&\quad = k^{\beta}{}_{\gamma}k^{\gamma\alpha}\sum_{h,i}\nabla_{\beta}\nabla_{\sigma}\xi^{i}\nabla_{\alpha}\nabla^{\sigma}\xi^{h}\Big(N^{i}N^{h} + \sum_{X}\underset{X}{C}{}^{i}\underset{X}{C}{}^{h} + B^{\rho i}B^{h}_{\rho}\Big)\\
&\quad = k^{\beta}{}_{\gamma}k^{\gamma\alpha}\sum_{i}N^{i}\nabla_{\beta}\nabla_{\sigma}\xi^{i}\sum_{h}N^{h}\nabla_{\alpha}\nabla^{\sigma}\xi^{h}\\
&\qquad + k^{\beta}{}_{\gamma}k^{\gamma\alpha}\sum_{X}\nabla_{\beta}\nabla_{\sigma}\underset{X}{f}\nabla_{\alpha}\nabla^{\sigma}\underset{X}{f}\\
&\qquad + k^{\beta}{}_{\gamma}k^{\gamma\alpha}\sum_{h,i}B^{i}_{\rho}B^{\rho h}\nabla_{\beta}\nabla_{\sigma}\xi^{i}\nabla_{\alpha}\nabla^{\sigma}\xi^{h}\,,
\end{aligned}
$$

hence B is reduced to

$$
B \cong \det(h_{\mu}{}^{\lambda})k^{\beta}{}_{\gamma}k^{\gamma\alpha}\sum_{X}\nabla_{\beta}\nabla_{\sigma}\underset{X}{f}\nabla_{\alpha}\nabla^{\sigma}\underset{X}{f} + B_{1} + B_{2} + B_{3}\,, \tag{4.3}
$$

where

$$
\begin{aligned}
B_{1} = \det(h_{\mu}{}^{\lambda})\Big[&\Big(k^{\beta\alpha}\sum_{h}N^{h}\nabla_{\beta}\nabla_{\alpha}\xi^{h}\Big)^{2}\\
&- k^{\beta\sigma}k^{\alpha\rho}\sum_{i}N^{i}\nabla_{\beta}\nabla_{\alpha}\xi^{i}\sum_{h}N^{h}\nabla_{\sigma}\nabla_{\rho}\xi^{h}\Big]\,,
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
B_{2} = \det(h_{\mu}{}^{\lambda})\Big[&-2k^{\beta\alpha}\sum_{i}N^{i}\nabla_{\beta}\nabla_{\alpha}\xi^{i}\sum_{h}B^{h}_{\sigma}\nabla^{\sigma}\xi^{h}\\
&- 2k^{\beta\alpha}\sum_{i}B^{i}_{\gamma}\nabla_{\beta}\nabla_{\alpha}\xi^{i}\sum_{h}N^{h}\nabla^{\gamma}\xi^{h}\Big]\,,
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
B_{3} = \det(h_{\mu}{}^{\lambda})\Big[&\Big(\sum_{h}B^{h}_{\alpha}\nabla^{\alpha}\xi^{h}\Big)^{2}\\
&+ \sum_{i}B^{\beta i}\nabla^{\alpha}\xi^{i}\sum_{h}\big(B^{h}_{\beta}\nabla_{\alpha}\xi^{h} + B^{h}_{\alpha}\nabla_{\beta}\xi^{h}\big)\\
&- m\sum_{i}N^{i}\nabla_{\beta}\xi^{i}\sum_{h}N^{h}\nabla^{\beta}\xi^{h} - \sum_{h}\nabla_{\alpha}\xi^{h}\nabla^{\alpha}\xi^{h}\Big]\,.
\end{aligned} \tag{4.6}
$$

Now in order to prove

$$
B_{1} + B_{2} + B_{3} \cong -\det(h_{\mu}{}^{\lambda})\sum_{X}\nabla_{\alpha}\underset{X}{f}\nabla^{\alpha}\underset{X}{f} \tag{4.7}
$$

we first prove

$$
\begin{aligned}
B_{1} \cong \det(h_{\mu}{}^{\lambda})\Big[&k^{\beta\gamma}\sum_{h,i}(B^{\alpha i}N^{h} + B^{\alpha h}N^{i})(\nabla_{\alpha}\xi^{i}\nabla_{\beta}\nabla_{\gamma}\xi^{h} - \nabla_{\beta}\xi^{i}\nabla_{\alpha}\nabla_{\gamma}\xi^{h})\\
&+ (m-1)\sum_{i}N^{i}\nabla_{\beta}\xi^{i}\sum_{h}N^{h}\nabla^{\beta}\xi^{h}\Big]\,.
\end{aligned} \tag{4.8}
$$

The next calculation resembles that carried out in §3, but is a little more complicated: We get

$$-\det(h_\mu{}^\lambda)k^{\beta\sigma}k^{\alpha\rho}\sum_i N^i\nabla_\beta\nabla_\alpha\xi^i\sum_h N^h\nabla_\sigma\nabla_\rho\xi^h$$
$$\cong\sum_{h,i}\nabla_\alpha\xi^i[\det(h_\mu{}^\lambda)k^{\beta\sigma}k^{\alpha\rho}N^iN^h\nabla_\beta\nabla_\sigma\nabla_\rho\xi^h$$
$$+\nabla_\beta(\det(h_\mu{}^\lambda)k^{\beta\sigma}k^{\alpha\rho}N^iN^h)\nabla_\sigma\nabla_\rho\xi^h]$$

and into the second member we substitute

$$\nabla_\beta\nabla_\sigma\nabla_\rho\xi^h=\nabla_\beta\nabla_\rho\nabla_\sigma\xi^h$$
$$=\nabla_\rho\nabla_\beta\nabla_\sigma\xi^h-(h_{\rho\sigma}h_\beta{}^\gamma-h_{\beta\sigma}h_\rho{}^\gamma)\nabla_\gamma\xi^h$$

which is the result of the Ricci identity and the Gauss equation. Again we get

$$\sum_{h,i}\nabla_\alpha\xi^i\det(h_\mu{}^\lambda)k^{\beta\sigma}k^{\alpha\rho}N^iN^h\nabla_\rho\nabla_\beta\nabla_\sigma\xi^h$$
$$\cong-\sum_{h,i}\det(h_\mu{}^\lambda)k^{\beta\sigma}k^{\alpha\rho}N^iN^h\nabla_\rho\nabla_\alpha\xi^i\nabla_\beta\nabla_\sigma\xi^h$$
$$-\sum_{h,i}\nabla_\rho(\det(h_\mu{}^\lambda)k^{\beta\sigma}k^{\alpha\rho}N^iN^h)\nabla_\alpha\xi^i\nabla_\beta\nabla_\sigma\xi^h\ ,$$

hence

$$B_1\cong\sum_{h,i}\nabla_\beta[\det(h_\mu{}^\lambda)(k^{\beta\sigma}k^{\alpha\rho}-k^{\rho\sigma}k^{\alpha\beta})N^iN^h]\nabla_\alpha\xi^i\nabla_\sigma\nabla_\rho\xi^h$$
$$+(m-1)\det(h_\mu{}^\lambda)\sum_i N^i\nabla_\alpha\xi^i\sum_h N^h\nabla^\alpha\xi^h\ .$$

As we have

$$\nabla_\beta[\det(h_\mu{}^\lambda)(k^{\beta\sigma}k^{\alpha\rho}-k^{\rho\sigma}k^{\alpha\beta})N^iN^h]$$
$$=\det(h_\mu{}^\lambda)[(k^{\rho\omega}k^{\sigma\nu}k^{\alpha\beta}-k^{\beta\sigma}k^{\alpha\omega}k^{\rho\nu})\nabla_\beta h_{\omega\nu}N^iN^h$$
$$-(k^{\beta\sigma}k^{\alpha\rho}-k^{\rho\sigma}k^{\alpha\beta})h_\beta{}^\gamma(B^i_\gamma N^h+B^h_\gamma N^i)]$$

in view of (3.2), (3.3), we get (4.8).

Now we can reduce B_1+B_2 to a formula where only the first derivatives of ξ^h are contained: We get from (4.5) and (4.8)

$$B_1+B_2\cong\det(h_\mu{}^\lambda)\Big[-\sum_{h,i}k^{\gamma\beta}(B^{\alpha i}N^h+B^{\alpha h}N^i)\nabla_\gamma(\nabla_\alpha\xi^i\nabla_\beta\xi^h)$$
$$+(m-1)\sum_i N^i\nabla_\beta\xi^i\sum_h N^h\nabla^\beta\xi^h\Big]$$
$$\cong\sum_{h,i}\nabla_\gamma\Big[\det(h_\mu{}^\lambda)k^{\gamma\beta}(B^{\alpha i}N^h+B^{\alpha h}N^i)\nabla_\alpha\xi^i\nabla_\beta\xi^h$$
$$+(m-1)\det(h_\mu{}^\lambda)\sum_i N^i\nabla_\beta\xi^i\sum_h N^h\nabla^\beta\xi^h$$
$$=\det(h_\mu{}^\lambda)\Big[(m+1)\sum_i N^i\nabla_\beta\xi^i\sum_h N^h\nabla^\beta\xi^h$$
$$-\Big(\sum_i B^i_\alpha\nabla^\alpha\xi^i\Big)^2$$
$$-\sum_{h,i}B^i_\beta\nabla^\beta\xi^hB^h_\alpha\nabla^\alpha\xi^i\Big]\ .$$

Substituting this result and (4.6) into $B_1 + B_2 + B_3$ we get

$$\begin{aligned} &B_1 + B_2 + B_3 \\ &\cong \det(h_\mu{}^\lambda)\Big[\sum_i N^i \nabla_\beta \xi^i \sum_h N^h \nabla^\beta \xi^h \\ &\quad + \sum_{h,i} B^{\alpha i} B^h_\alpha \nabla_\beta \xi^i \nabla^\beta \xi^h - \sum_h \nabla_\alpha \xi^h \nabla^\alpha \xi^h\Big] \end{aligned}$$

and this is equivalent to (4.7).

From (4.3) and (4.7) we get the following lemma.

LEMMA 4.1. *Let M_0 be a C^∞ submanifold of E^n lying in a linear subspace E^{m+1} as a closed hypersurface with positive second fundamental form. It we consider an infinitesimal deformation of such a submanifold M_0 in E^n, we get for the second derivative of γ defined in §1*

$$\left(\frac{d^2\gamma}{dt^2}\right)_0 = \int_{M_0} \det(h_\mu{}^\lambda)\Big[k^\beta{}_\gamma k^{\gamma\alpha} \sum_X \nabla_\beta \nabla_\sigma \underset{X}{f} \nabla_\alpha \nabla^\sigma \underset{X}{f} - \sum_X \nabla_\alpha \underset{X}{f} \nabla^\alpha \underset{X}{f}\Big] v \ ,$$

where v is the volume form of (M_0, g), $h_{\mu\lambda}$ is the second fundamental form of M_0 as a hypersurface of E^{m+1}, $k^{\mu\lambda}$ is given by $k^{\mu\alpha} h_{\lambda\alpha} = \delta^\mu_\lambda$,

$$\underset{X}{f} = \sum_h \underset{X}{C^h} \xi^h \qquad X = m+2, \cdots, n \ ,$$

$\underset{X}{C}$ being orthonormal normal vectors of E^{m+1} which are constant vectors and ξ^h is the vector field of deformation.

5. Index of a Gauss-critical submanifold M_0. We first prove the following theorem.

THEOREM 5.1. *If (M, g) is a C^∞ closed hypersurface of E^{m+1} with the positive second fundamental form $h_{\mu\lambda}$, then the following inequality holds for any C^∞ function f:*

$$\int_M \det(h_\mu{}^\lambda)[k^\beta{}_\gamma k^{\gamma\alpha} \nabla_\beta \nabla_\sigma f \nabla_\alpha \nabla^\sigma f - \nabla_\alpha f \nabla^\alpha f] v \geqq 0 \ , \tag{5.1}$$

where v is the volume form of (M, g) and $k^{\mu\lambda}$ satisfies $k^{\mu\alpha} h_{\lambda\alpha} = \delta^\mu_\lambda$.

To prove Theorem 5.1 we need some lemmas.

LEMMA 5.2. *$k^{\beta\alpha} k^{\sigma\rho} a_{\beta\sigma} a_{\alpha\rho}$ is non-negative for any tensor $a_{\mu\lambda}$.*

PROOF. As $h_{\mu\lambda}$ is positive, $k^{\mu\lambda}$ is also positive and, if we fix any point, we can put $k^{\beta\alpha} = k^\beta \delta^{\beta\alpha}$ where $k^\beta > 0$ for $\beta = 1, \cdots, m$. Hence we get $k^{\beta\alpha} k^{\sigma\rho} a_{\beta\sigma} a_{\alpha\rho} = \sum_{\beta,\sigma} k^\beta k^\sigma (a_{\beta\sigma})^2 \geqq 0$.

LEMMA 5.3.

$$\int_M \det(h_\mu{}^\lambda)[mk^{\beta\alpha}k^{\sigma\rho}\nabla_\beta\nabla_\sigma f\nabla_\alpha\nabla_\rho f - (k^{\beta\alpha}\nabla_\beta\nabla_\alpha f)^2]v \geqq 0 .$$

PROOF. We have in view of Lemma 5.2

$$k^{\beta\alpha}k^{\sigma\rho}\Big(\nabla_\beta\nabla_\sigma f - \frac{1}{m}h_{\beta\sigma}k^{\nu\mu}\nabla_\nu\nabla_\mu f\Big) \times \Big(\nabla_\alpha\nabla_\rho f - \frac{1}{m}h_{\alpha\rho}k^{\lambda\kappa}\nabla_\lambda\nabla_\kappa f\Big) \geqq 0 ,$$

namely,

$$mk^{\beta\alpha}k^{\sigma\rho}\nabla_\beta\nabla_\sigma f\nabla_\alpha\nabla_\rho f - (k^{\beta\alpha}\nabla_\beta\nabla_\alpha f)^2 \geqq 0 .$$

Then this lemma is immediately obtained.

LEMMA 5.4.

$$\int_M [\det(h_\mu{}^\lambda)k^{\beta\alpha}\nabla_\beta\nabla_\alpha f k^{\sigma\rho}\nabla_\sigma\nabla_\rho f]v = \int_M \det(h_\mu{}^\lambda)[k^{\beta\alpha}k^{\sigma\rho}\nabla_\sigma\nabla_\beta f\nabla_\rho\nabla_\alpha f + (m-1)\nabla_\alpha f\nabla^\alpha f]v .$$

PROOF. Applying Green's theorem we get

$$\begin{aligned}
&\int_M [\det(h_\mu{}^\lambda)k^{\beta\alpha}\nabla_\beta\nabla_\alpha f k^{\sigma\rho}\nabla_\sigma\nabla_\rho f]v \\
&= -\int_M [\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho}\nabla_\alpha f\nabla_\beta\nabla_\rho\nabla_\sigma f]v \\
&\quad - \int_M [\nabla_\beta(\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho})\nabla_\alpha f\nabla_\sigma\nabla_\rho f]v \\
&= -\int_M \det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho}\nabla_\alpha f(\nabla_\rho\nabla_\beta\nabla_\sigma f - K_{\beta\rho\sigma}{}^\gamma\nabla_\gamma f)v \\
&\quad - \int_M [\nabla_\beta(\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho})\nabla_\alpha f\nabla_\sigma\nabla_\rho f]v \\
&= \int_M [\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho}\nabla_\rho\nabla_\alpha f\nabla_\beta\nabla_\sigma f]v \\
&\quad + \int_M [\nabla_\rho(\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho})\nabla_\alpha f\nabla_\beta\nabla_\sigma f]v \\
&\quad - \int_M [\nabla_\beta(\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho})\nabla_\alpha f\nabla_\rho\nabla_\sigma f]v \\
&\quad + (m-1)\int_M [\det(h_\mu{}^\lambda)\nabla_\alpha f\nabla^\alpha f]v .
\end{aligned}$$

As we have

$$\begin{aligned}
&\nabla_\rho(\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho})\nabla_\beta\nabla_\sigma f - \nabla_\beta(\det(h_\mu{}^\lambda)k^{\beta\alpha}k^{\sigma\rho})\nabla_\rho\nabla_\sigma f \\
&\quad = \nabla_\rho(\det(h_\mu{}^\lambda)(k^{\beta\alpha}k^{\sigma\rho} - k^{\rho\alpha}k^{\sigma\beta}))\nabla_\beta\nabla_\sigma f = 0 ,
\end{aligned}$$

we get Lemma 5.4.

As we assume $m > 1$, hence $m - 1 > 0$, we get from Lemma 5.3 and Lemma 5.4

$$\int_M \det (h_\mu{}^\lambda)(k^{\beta\alpha}k^{\sigma\rho}\nabla_\sigma\nabla_\beta f \nabla_\rho\nabla_\alpha f - \nabla_\alpha f \nabla^\alpha f)v \geqq 0 \,. \tag{5.2}$$

LEMMA 5.5.

$$\int_M \det (h_\mu{}^\lambda)(k^\beta{}_\gamma k^{\gamma\alpha}\nabla_\beta\nabla_\sigma f \nabla_\alpha\nabla^\sigma f - k^{\beta\alpha}k^{\sigma\rho}\nabla_\sigma\nabla_\beta f \nabla_\rho\nabla_\alpha f)v \geqq 0 \,.$$

PROOF. If $S^{\mu\lambda} = g^{\mu\beta}g^{\lambda\alpha}S_{\beta\alpha}$ we have

$$(S_{\beta\alpha} - S_{\alpha\beta})(S^{\beta\alpha} - S^{\alpha\beta}) \geqq 0 \,,$$

hence

$$S_{\beta\alpha}S^{\beta\alpha} - S_{\beta\alpha}S^{\alpha\beta} \geqq 0 \,.$$

Lemma 5.5 is proved if we put $S_{\gamma\sigma} = k^\beta{}_\gamma \nabla_\beta\nabla_\sigma f$.

From Lemma 5.5 and inequality (5.2) we get (5.1), hence Theorem 5.1 is proved.

The following Main Theorem is a direct consequence of Theorem 5.1 together with Lemma 4.1.

THEOREM 5.6. *If an m-dimensional C^∞ submanifold M of E^n lies in a linear subspace E^{m+1} as a closed hypersurface with the positive second fundamental form, then the index of M as a critical point of the integral* $\mathrm{Vol}^*(\Gamma(M))$ *is zero.*

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