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TRANSFORMATION OF CONDITIONAL WIENER INTEGRALS UNDER TRANSLATION AND THE CAMERON-MARTIN TRANSLATION THEOREM

J. YEH

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1. Introduction. Consider the Wiener measure space $(C[0, t], \mathfrak{W}^*, m_w)$ where $C[0, t]$ is the Wiener space consisting of all real valued continuous functions x on the interval $[0, t]$ in R^1 with $x(0) = 0$ for fixed $t \in (0, \infty)$; $\mathfrak{W}$ is the algebra of Borel cylinders in $C[0, t]$, i.e., the collection of all subsets W of $C[0, t]$ of the type

$$(1.1) \qquad W = \{x \in C[0, t]; [x(s_1), \cdots, x(s_n)] \in B\}$$

where n is an arbitrary positive integer, $0 = s_0 < s_1 < \cdots < s_n \leqq t$, and B is an arbitrary member of the σ-algebra $\mathfrak{B}^n$ of the Borel sets in the n-dimensional Euclidean space R^n; m_w is a probability measure on the algebra $\mathfrak{W}$ defined for W as in (1.1) by

$$(1.2) \quad m_w(W) = \left\{(2\pi)^n \prod_{j=1}^n (s_j - s_{j-1})\right\}^{-1/2} \int_B \exp\left\{-2^{-1} \sum_{j=1}^n (\xi_j - \xi_{j-1})^2 (s_j - s_{j-1})^{-1}\right\} m_L(d\xi)$$

where $\xi = (\xi_1, \cdots, \xi_n) \in R^n$, $\xi_0 = 0$ and m_L is the Lebesgue measure on $(R^n, \mathfrak{B}^n)$; $\mathfrak{W}^*$ is the σ-algebra of Carathéodory measurable subsets of $C[0, t]$ with respect to the outer measure derived from the measure m_w on the algebra. Needless to say $\mathfrak{W}^*$ contains the σ-algebra $\sigma(\mathfrak{W})$ generated by $\mathfrak{W}$ and the Wiener measure space is a complete measure space. The $\mathfrak{W}^*$-measurability and m_w-integrability of a functional on $C[0, t]$ will be referred to as the Wiener measurability and Wiener integrability.

By a conditional Wiener integral we mean specifically the conditional expectation $E^w(Y|X)$ of a real or complex valued Wiener integrable functional Y conditioned by the functional X on the Wiener space defined by

$$(1.3) \qquad X[x] = x(t) \qquad \text{for} \quad x \in C[0, t]$$

where the conditional expectation $E^w(Y|X)$ is not given as an equivalence class of random variables on the probability space $(C[0, t], \mathfrak{W}^*, m_w)$ but as an equivalence class of random variables on the probability space

$(R^1, \mathfrak{B}^1, P_X)$ where P_X is the probability distribution of X defined by

$$(1.4) \qquad P_X(B) = m_w(X^{-1}(B)) = (2\pi t)^{-1/2}\int_B \exp\{-(2t)^{-1}\xi^2\} m_L(d\xi) \quad \text{for } B \in \mathfrak{B}^1 .$$

We shall use the same notation $E^w(Y|X)$ to mean also the individual representatives of the equivalence class i.e., the versions of the conditional expectation.

In [6] and [7] we derived several Fourier inversion formulae for conditional Wiener integrals and applied one of these to give an alternate proof of the Kac-Feynman Formula. Evaluations of conditional Wiener integrals for certain types of functionals were also included there. In this paper we present an analogue of the Cameron-Martin Translation Theorem for the conditional Wiener integral. Our main result (see Theorem 2 in §3) is the following: Let Y be a Wiener integrable functional and let X be as in (1.3). Let $x_0 \in C[0, t]$ be absolutely continuous on $[0, t]$ with $\int_{[0,t]} [x_0'(s)]^2 m_L(ds) < \infty$. Then for arbitrary versions of the conditional Wiener integrals $E^w(Y|X)$ and $E^w[Y[\cdot + x_0]J|X]$ we have

$$(1.5) \qquad \begin{aligned} &E^w(Y|X)(\xi) \\ &\quad = E^w[Y[\cdot + x_0]J|X](\xi - x_0(t)) \exp\{-(2t)^{-1}[x_0(t)]^2\} \exp\{t^{-1}\xi x_0(t)\} \end{aligned}$$

for a.e. ξ in $(R^1, \mathfrak{B}^1, m_L)$ where

$$(1.6) \qquad J(x) = \exp\left\{-2^{-1}\int_{[0,t]} [x_0'(s)]^2 m_L(ds)\right\} \exp\left\{-\int_0^t x_0'(s)dx(s)\right\}$$

for a.e. x in $(C[0, t], \mathfrak{W}^*, m_w)$.

The proof of this translation theorem is based on the Cameron-Martin Translation Theorem and a transformation theorem for conditional expectation in general which we present as Theorem 1 in §2. An example of evaluation of conditional Wiener integral by means of the translation theorem is given in §3.

2. Transformation of conditional expectations.

DEFINITION. Let $(\Omega, \mathfrak{B}, P)$ be a probability space and let X be a measurable transformation of $(\Omega, \mathfrak{B})$ into a measurable space $(S, \mathfrak{F})$. The conditional expectation of a real valued integrable random variable Z on $(\Omega, \mathfrak{B}, P)$ given X, written $E(Z|X)$, is defined to be any real valued $\mathfrak{F}$-measurable and P_X integrable function Ψ on S such that

$$(2.1) \qquad \int_{X^{-1}(F)} Z(\omega)P(d\omega) = \int_F \Psi(\xi)P_X(d\xi) \qquad \text{for} \quad F \in \mathfrak{F}$$

where P_X is the probability distribution of X defined by

$$P_X(F) = P(X^{-1}(F)) \quad \text{for} \quad F \in \mathfrak{F} \tag{2.2}$$

By the Radon-Nikodym Theorem, such a function Ψ always exists and is determined uniquely up to a null set in $(S, \mathfrak{F}, P_X)$. We write $E(Z|X)$ to mean either the class of all such functions Ψ or a particular member of the class. It should be clear from the context which one of the two is meant. Thus we have

$$\int_{X^{-1}(F)} Z(\omega)P(d\omega) = \int_F E(Z|X)(\xi)P_X(d\xi) \quad \text{for} \quad F \in \mathfrak{F} . \tag{2.3}$$

The conditional Wiener integral $E^w(Y|X)$ introduced in §1 is a particular case of the conditional expectation $E(Z|X)$ for which the probability space $(\Omega, \mathfrak{B}, P)$ is the Wiener measure space $(C[0, t], \mathfrak{W}^*, m_w)$ and X is given by (1.3).

LEMMA 1. *Given a real valued integrable random variable Z on a probability space $(\Omega, \mathfrak{B}, P)$ and a measurable transformation X of $(\Omega, \mathfrak{B})$ into a measurable space $(S, \mathfrak{F})$. Let T be a measurable transformation of $(\Omega, \mathfrak{B})$ into itself. If there exists a one-to-one transformation h of S onto itself such that both h and h^{-1} are measurable transformations of $(S, \mathfrak{F})$ into itself and if furthermore h satisfies the condition*

$$(X \circ T)(\omega) = (h \circ X)(\omega) \quad \textit{for a.e. } \omega \textit{ in } \quad (\Omega, \mathfrak{B}, P) \tag{2.4}$$

then for arbitrary versions of the conditional expectations $E(Z|X \circ T)$ and $E(Z|X)$ we have

$$E(Z|X \circ T)(\xi) = E(Z|X)(h^{-1}(\xi)) \quad \textit{for a.e. } \xi \textit{ in } (S, \mathfrak{F}, P_{X \circ T}) . \tag{2.5}$$

PROOF. Since $X \circ T$ is a measurable transformation of $(\Omega, \mathfrak{B})$ into $(S, \mathfrak{F})$, the conditional expectation $E(Z|X \circ T)$ exists and for an arbitrary version of it we have by (2.3)

$$\int_F E(Z|X \circ T)(\xi)P_{X \circ T}(d\xi) = \int_{(X \circ T)^{-1}(F)} Z(\omega)P(d\omega) \quad \text{for} \quad F \in \mathfrak{F} . \tag{2.6}$$

By (2.4) there exists a null set Ω_0 in $(\Omega, \mathfrak{B}, P)$ such that $X \circ T = h \circ X$ on Ω_0^c. Now

$$(X \circ T)^{-1}(F) = \{(X \circ T)^{-1}(F) \cap \Omega_0^c\} \cup \{(X \circ T)^{-1}(F) \cap \Omega_0\}$$

and similarly

$$(h \circ X)^{-1}(F) = \{(h \circ X)^{-1}(F) \cap \Omega_0^c\} \cup \{(h \circ X)^{-1}(F) \cap \Omega_0\} .$$

The second set in the union on the right side of each of the above two equalities is a null set in $(\Omega, \mathfrak{B}, P)$. On the other hand since $X \circ T = h \circ X$ on Ω_0^c the first set in the union on the right side of the first equality is equal to that of the second equality. Thus for the symmetric difference of the two sets on the left side of the two equalities we have

$$P[\{(X \circ T)^{-1}(F)\} \triangle \{(h \circ X)^{-1}(F)\}] = 0 .$$

From this we have

$$\begin{aligned} (2.7) \qquad \int_{(X\circ T)^{-1}(F)} Z(\omega)P(d\omega) &= \int_{(h\circ X)^{-1}(F)} Z(\omega)P(d\omega) \\ &= \int_{X^{-1}(h^{-1}(F))} Z(\omega)P(d\omega) = \int_{h^{-1}(F)} E(Z|X)(\xi)P_X(d\xi) \quad \text{for} \quad F \in \mathfrak{F} \end{aligned}$$

where the last equality holds for an arbitrary version of $E(Z|X)$ by (2.3) since $h^{-1}(F) \in \mathfrak{F}$ for our $F \in \mathfrak{F}$.

Next consider the measurable transformation h of the probability space $(S, \mathfrak{F}, P_X)$ into the measurable space $(S, \mathfrak{F})$. In terms of the probability measure P_h on $(S, \mathfrak{F})$ induced by h we have

$$\begin{aligned} \int_{h^{-1}(F)} E(Z|X)(\xi)P_X(d\xi) &= \int_{h^{-1}(F)} E(Z|X)[h^{-1}(h(\xi))]P_X(d\xi) \\ &= \int_F E(Z|X)(h^{-1}(\eta))P_h(d\eta) \quad \text{for} \quad F \in \mathfrak{F} . \end{aligned}$$

But

$$\begin{aligned} P_h(F) &= P_X(h^{-1}(F)) = P[X^{-1}(h^{-1}(F))] \\ &= P[(h \circ X)^{-1}(F)] = P_{h\circ X}(F) \quad \text{for} \quad F \in \mathfrak{F} \end{aligned}$$

i.e., the probability distributions on $(S, \mathfrak{F})$ of the measurable transformations h and $h \circ X$ on the probability spaces $(S, \mathfrak{F}, P_X)$ and $(\Omega, \mathfrak{B}, P)$ respectively are identical. Therefore

$$(2.8) \qquad \int_{h^{-1}(F)} E(Z|X)(\xi)P_X(d\xi) = \int_F E(Z|X)(h^{-1}(\xi))P_{h\circ X}(d\xi) \quad \text{for} \quad F \in \mathfrak{F} .$$

From (2.6). (2.7), and (2.8) we have for arbitrary versions of $E(Z|X \circ T)$ and $E(Z|X)$

$$(2.9) \qquad \int_F E(Z|X \circ T)(\xi)P_{X\circ T}(d\xi) = \int_F E(Z|X)(h^{-1}(\xi))P_{h\circ X}(d\xi) \quad \text{for} \quad F \in \mathfrak{F} .$$

Now (2.4) implies

$$(2.10) \qquad P_{h\circ X} = P_{X\circ T} \quad \text{on} \quad (S, \mathfrak{F}) .$$

From (2.9) and (2.10) we have (2.5).

LEMMA 2. *Let $(\Omega, \mathfrak{B}, \mu)$ be a finite measure space and let T be a* *measurable transformation of* $(\Omega, \mathfrak{B})$ *into itself. If there exists a real valued* $\mathfrak{B}$*-measurable function* J *on* Ω *such that*

$$\mu(A) = \int_{T^{-1}(A)} J(\omega)\mu(d\omega) \quad \textit{for} \quad A \in \mathfrak{B} \tag{2.11}$$

then for every real valued $\mathfrak{B}$*-measurable function* Z *on* Ω *we have*

$$\int_{\Omega} Z(\omega)\mu(d\omega) = \int_{\Omega} (Z \circ T)(\omega)J(\omega)\mu(d\omega) \tag{2.12}$$

in the sense that the existence of one side implies that of the other and the equality of the two.

PROOF. Consider first the case where Z is a simple function on $(\Omega, \mathfrak{B}, \mu)$, i.e.,

$$Z = \sum_{j=1}^{n} c_j \chi_{A_j} \text{ where } c_j \in R^1 \text{ and } A_j \in \mathfrak{B} \text{ for } j = 1, 2, \cdots, n .$$

In this case we have

$$\begin{aligned}
\int_{\Omega} Z(\omega)\mu(d\omega) &= \sum_{j=1}^{n} c_j \mu(A_j) = \sum_{j=1}^{n} c_j \int_{T^{-1}(A_j)} J(\omega)\mu(d\omega) \\
&= \int_{\Omega} \sum_{j=1}^{n} c_j \chi_{T^{-1}(A_j)}(\omega) J(\omega)\mu(d\omega) = \int_{\Omega} \sum_{j=1}^{n} c_j \chi_{A_j}(T(\omega)) J(\omega)\mu(d\omega) \\
&= \int_{\Omega} (Z \circ T)(\omega) J(\omega)\mu(d\omega)
\end{aligned}$$

so that (2.12) holds in this case.

When Z is a nonnegative valued $\mathfrak{B}$-measurable function on Ω, there exists a monotone increasing sequence of nonnegative simple functions $\{Z_n, n = 1, 2, \cdots\}$ on $(\Omega, \mathfrak{B}, \mu)$ such that $Z_n \uparrow Z$ on Ω. By the above result and by the Monotone Convergence Theorem, (2.12) holds for this case too. Finally for a real valued $\mathfrak{B}$-measurable function Z on Ω we apply the above result to the positive and negative parts of Z.

THEOREM 1. *Given a real valued integrable random variable* Z *on a probability space* $(\Omega, \mathfrak{B}, P)$ *and a measurable transformation* X *of* $(\Omega, \mathfrak{B})$ *into a measurable space* $(S, \mathfrak{F})$. *Let* T *be a measurable transformation of* $(\Omega, \mathfrak{B})$ *into itself which satisfies the following two conditions:*

1° *There exists a real valued* $\mathfrak{B}$*-measurable function* J *on* Ω *such that*

$$P(A) = \int_{T^{-1}(A)} J(\omega)P(d\omega) \quad \textit{for} \quad A \in \mathfrak{B} .$$

2° *There exists a one-to-one transformation* h *of* S *onto itself such that both* h *and* h^{-1} *are measurable transformations of* $(S, \mathfrak{F})$ *into itself* *and furthermore*

$$(X \circ T)(\omega) = (h \circ X)(\omega) \quad \textit{for a.e. } \omega \textit{ in } \quad (\Omega, \mathfrak{B}, P)\,.$$

Then for every measurable transformation g of $(S, \mathfrak{F})$ into $(R^1, \mathfrak{B}^1)$ we have

$$\int_S g(\xi) E(Z\,|\,X)(\xi) P_X(d\xi) = \int_S g(\xi) E[(Z \circ T)J\,|\,X](h^{-1}(\xi)) P_{X \circ T}(d\xi) \tag{2.13}$$

in the sense that the existence of one side implies that of the other and the equality of the two. If in addition to 1° and 2°, T satisfies the condition

3° $P_{X \circ T} \ll P_X$ *on* $(S, \mathfrak{F})$

then for arbitrary versions of $E(Z|X)$ and $E[(Z \circ T)J|X]$ we have

$$E(Z\,|\,X)(\xi) = E[(Z \circ T)J\,|\,X](h^{-1}(\xi))(dP_{X \circ T}/dP_X)(\xi) \quad \textit{for a.e. } \xi \textit{ in } \quad (S, \mathfrak{F}, P_X)\,. \tag{2.14}$$

PROOF. Assume 1° and 2°. Let g be a measurable transformation of $(S, \mathfrak{F})$ into $(R^1, \mathfrak{B}^1)$. According to Proposition 3 on p. 635 of [6] we have

$$E[(g \circ X)Z] = \int_S g(\xi) E(Z\,|\,X)(\xi) P_X(d\xi) \tag{2.15}$$

in the sense that the existence of one side implies that of the other and the equality of the two. In the rest of the proof whenever an equality holds in the above sense we shall say for brevity that the equality holds in the restricted sense. Now by Lemma 2,

$$E[(g \circ X)Z] = E[(g \circ X \circ T)(Z \circ T)J] \tag{2.16}$$

in the restricted sense. Applying (2.15) to the right side of (2.16) where $X \circ T$ and $(Z \circ T)\,J$ correspond respectively to X and Z on the left side of (2.15), we obtain

$$E[(g \circ X)Z] = \int_S g(\xi) E[(Z \circ T)J\,|\,X \circ T](\xi) P_{X \circ T}(d\xi) \tag{2.17}$$

in the restricted sense. According to Lemma 1,

$$E[(Z \circ T)J\,|\,X \circ T](\xi) = E[(Z \circ T)J\,|\,X](h^{-1}(\xi)) \quad \text{for a.e. } \xi \text{ in } \quad (S, \mathfrak{F}, P_{X \circ T}) \tag{2.18}$$

for arbitrary versions of the two conditional expectations involved. Using (2.18) in (2.17) we have

$$E[(g \circ X)Z] = \int_S g(\xi)E[(Z \circ T)J \mid X](h^{-1}(\xi))P_{X \circ T}(d\xi) \tag{2.19}$$

in the restricted sense. Combining (2.15) and (2.19) we have (2.13).

To prove (2.14) note that under the assumption of 3°, the equality (2.13) becomes

$$\int_S g(\xi)E(Z \mid X)(\xi)P_X(d\xi)$$
$$= \int_S g(\xi)E[(Z \circ T)J \mid X](h^{-1}(\xi))(dP_{X \circ T}/dP_X)(\xi)P_X(d\xi)$$

in the restricted sense. By letting $g = \chi_F$ for arbitrary $F \in \mathfrak{F}$ we have

$$\int_F E(Z \mid X)(\xi)P_X(d\xi)$$
$$= \int_F E[(Z \circ T)J \mid X](h^{-1}(\xi))(dP_{X \circ T}/dP_X)(\xi)P_X(d\xi) \quad \text{for} \quad F \in \mathfrak{F} .$$

From this we have (2.14).

3. Translation of conditional Wiener integrals. Consider the Wiener measure space $(C[0, t], \mathfrak{W}^*, m_w)$. With a fixed element x_0 in $C[0, t]$ consider the transformation T of $C[0, t]$ into itself defined by

$$T[x] = x + x_0 \quad \text{for} \quad x \in C[0, t] . \tag{3.1}$$

According to the Cameron-Martin Translation Theorem [2], if x_0' exists on $[0, t]$ and $x_0' \in \text{B.V.}\, [0, t]$ then

$$T(\Gamma),\ T^{-1}(\Gamma) \in \mathfrak{W}^* \quad \text{for} \quad \Gamma \in \mathfrak{W}^* \tag{3.2}$$

and furthermore

$$m_w(\Gamma) = \int_{T^{-1}(\Gamma)} J(x)m_w(dx) \tag{3.3}$$

where

$$J(x) = \exp\left\{-2^{-1}\int_{[0,t]} [x_0'(s)]^2 m_L(ds)\right\} \exp\left\{-\int_0^t x_0'(s)dx(s)\right\} \quad \text{for} \quad x \in C[0, t] \tag{3.4}$$

the integral in the second exponential factor above being a Riemann-Stieltjes integral. G. Sunouchi [5] and I. E. Segal [4] showed that (3.2) and (3.3) still hold when x_0 is absolutely continuous on $[0, t]$ and $\int_{[0,t]} [x_0'(s)]^2 m_L(ds) < \infty$ provided that the integral in the second exponential factor in (3.4) is a Paley-Wiener-Zygmund-Riemann-Stieltjes integral (see [3]). The P.W.Z.R.S integral exists for a.e. x in $(C[0, t], \mathfrak{W}^*, m_w)$. On the exceptional null set of the Wiener measure space J may be defined arbitrarily since our measure space is a complete measure space. It is also known that when x_0' exists on $[0, t]$ and $x_0' \in$ B.V. $[0, t]$, the P.W.Z.R.S. integral is equal to the R.S. integral for a.e. x in the Wiener measure space.

THEOREM 2. *Let Y be a real valued integrable functional on the Wiener measure space $(C[0, t], \mathfrak{W}^*, m_w)$ and let $X[x] = x(t)$ for $x \in C[0, t]$. Let x_0 be an element in $C[0, t]$ which is absolutely continuous on $[0, t]$ with $\int_{[0,t]} [x_0'(s)]^2 m_L(ds) < \infty$. Then for arbitrary versions of the conditional Wiener integrals $E^w(Y|X)$ and $E^w[Y[\cdot + x_0]J|X]$ we have*

$$(3.5) \qquad \begin{aligned} &E^w(Y|X)(\xi) \\ &\quad = E^w[Y[\cdot + x_0]J|X](\xi - x_0(t)) \exp\{-(2t)^{-1}[x_0(t)]^2\} \exp\{t^{-1}\xi x_0(t)\} \\ &\quad\quad \text{for a.e. } \xi \text{ in } (R^1, \mathfrak{B}^1, m_L) \end{aligned}$$

where

$$(3.6) \qquad J(x) = \exp\left\{-2^{-1}\int_{[0,t]} [x_0'(s)]^2 m_L(ds)\right\} \exp\left\{-\int_0^t x_0'(s)dx(s)\right\} \quad \text{for a.e. } x \text{ in } (C[0, t], \mathfrak{W}^*, m_w) .$$

PROOF. Let T be the transformation of $C[0, t]$ into itself defined by

$$T[x] = x + x_0 \quad \text{for} \quad x \in C[0, t] .$$

Then

$$(X \circ T)[x] = X[x + x_0] = x(t) + x_0(t) = (h \circ X)[x] \quad \text{for} \quad x \in C[0, t]$$

where h is the one-to-one transformation of R^1 onto R^1 defined by

$$h(\xi) = \xi + x_0(t) \quad \text{for} \quad \xi \in R^1$$

for which we have

$$h^{-1}(\xi) = \xi - x_0(t) \quad \text{for} \quad \xi \in R^1 .$$

Clearly both h and h^{-1} are measurable transformations of $(R^1, \mathfrak{B}^1)$ into itself. Thus our T satisfies the condition $2°$ in Theorem 1 where $(S, \mathfrak{F})$ is now $(R^1, \mathfrak{B}^1)$. The condition $1°$ in Theorem 1 is also satisfied by our T according to the Cameron-Martin Translation Theorem.

To verify condition $3°$ of Theorem 1 for our T, recall that the random variables $X[x] = x(t)$ and $(X \circ T)[x] = x(t) + x_0(t)$ for $x \in C[0, t]$ are normally distributed and in fact

$$P_X = N(0, t) \quad \text{and} \quad P_{X \circ T} = N(x_0(t), t) \quad \text{on} \quad (R^1, \mathfrak{B}^1) .$$

Thus $P_{X\circ T} \ll P_X$ on $(R^1, \mathfrak{B}^1)$ and as a version of the Radon-Nikodym derivative of $P_{X\circ T}$ with respect to P_X we have

$$\begin{aligned}(dP_{X\circ T}/dP_X)(\xi) &= (dP_{X\circ T}/dm_L)(\xi)[(dP_X/dm_L)(\xi)]^{-1} \\ &= \exp\{-(2t)^{-1}[\xi - x_0(t)]^2\}\exp\{(2t)^{-1}\xi^2\} \\ &= \exp\{-(2t)^{-1}[x_0(t)]^2\}\exp\{t^{-1}\xi x_0(t)\} \quad \text{for} \quad \xi \in R^1 .\end{aligned}$$

With this, (3.5) follows from (2.14).

COROLLARY. *Consider the Wiener measure space $(C[0, t], \mathfrak{W}^*, m_w)$. Let $X[x] = x(t)$ for $x \in C[0, t]$ and let x_0 be an element in $C[0, t]$ which is absolutely continuous on $[0, t]$ with $\int_{[0,t]} [x_0'(s)]^2 m_L(ds) < \infty$. Then one version of the conditional Wiener integral $E_x^w\left[\exp\left\{-\int_0^t x_0'(s)dx(s)\right\}\middle| X\right]$ which is an equivalence class of real valued random variables on $(R^1, \mathfrak{B}^1, P_X)$ where $P_X = N(0, t)$ is given by*

$$(3.7)\quad \begin{aligned}&E_x^w\left[\exp\left\{-\int_0^t x_0'(s)dx(s)\right\}\middle| X\right](\xi) \\ &\quad = \exp\left\{2^{-1}\int_{[0,t]}[x_0'(s)]^2 m_L(ds)\right\}\exp\{-(2t)^{-1}[x_0(t)]^2\}\exp\{-t^{-1}\xi x_0(t)\} \\ &\quad \textit{for} \quad \xi \in R^1 .\end{aligned}$$

PROOF. Let Y be the functional which is identically equal to 1 on $C[0, t]$. Then one version of the conditional Wiener integral $E^w(Y|X)$ is given by

$$(3.8)\qquad E^w(Y|X)(\xi) = 1 \quad \text{for} \quad \xi \in R^1 .$$

Let J be as given by (3.6). Since $Y[x + x_0] = 1$ for every $x \in C[0, t]$ we have

$$(3.9)\quad \begin{aligned}&E^w[Y[\cdot + x_0]J|X](\xi) \\ &\quad = \exp\left\{-2^{-1}\int_{[0,t]}[x_0'(s)]^2 m_L(ds)\right\}E_x^w\left[\exp\left\{-\int_0^t x_0'(s)dx(s)\right\}\middle| X\right](\xi)\end{aligned}$$

for a.e. ξ in $(R^1, \mathfrak{B}^1, P_X)$ for arbitrary versions of the two conditional Wiener integrals involved. Since P_X and m_L are equivalent on $(R^1, \mathfrak{B}^1)$, (3.9) holds for a.e. ξ in $(R^1, \mathfrak{B}^1, m_L)$. Since a translate of a null set in $(R^1, \mathfrak{B}^1, m_L)$ is again a null set in $(R^1, \mathfrak{B}^1, m_L)$, if we replace ξ in both sides of (3.9) by $\eta - x_0(t)$ then the equality holds for a.e. η in $(R^1, \mathfrak{B}^1, m_L)$. Using this equality and (3.8) in (3.5) we have

$$(3.10)\quad \begin{aligned}1 &= E_x^w\left[\exp\left\{-\int_0^t x_0'(s)dx(s)\right\}\middle| X\right](\eta - x_0(t))\exp\left\{-2^{-1}\int_{[0,t]}[x_0'(s)]^2 m_L(ds)\right\} \\ &\quad \times \exp\{-(2t)^{-1}[x_0(t)]^2\}\exp\{t^{-1}\eta x_0(t)\} \quad \text{for a.e. } \eta \text{ in } (R^1, \mathfrak{B}^1, m_L)\end{aligned}$$

for an arbitrary version of the conditional Wiener integral involved. Writing $\eta = \xi + x_0(t)$ and recalling that a translate of a null set in $(R^1, \mathfrak{B}^1, m_L)$ is again a null set in $(R^1, \mathfrak{B}^1, m_L)$, we have from (3.10)

$$E_x^w\left[\exp\left\{-\int_0^t x_0'(s)dx(s)\right\}\Big| X\right](\xi)$$
$$= \exp\left\{2^{-1}\int_{[0,t]}[x_0'(s)]^2 m_L(ds)\right\}\exp\{(2t)^{-1}[x_0(t)]^2\}\exp\{-t^{-1}x_0(t)[\xi + x_0(t)]\}$$
$$= \exp\left\{2^{-1}\int_{[0,t]}[x_0'(s)]^2 m_L(ds)\right\}\exp\{-(2t)^{-1}[x_0(t)]^2\}\exp\{-t^{-1}\xi x_0(t)\}$$

for a.e. ξ in $(R^1, \mathfrak{B}^1, m_L)$ for an arbitrary version of the conditional Wiener integral. Therefore there exists a version as given by (3.7) by the equivalence of P_X and m_L on $(R^1, \mathfrak{B}^1)$.

The following is an example of application of Theorem 2 in evaluating a conditional Wiener integral.

EXAMPLE. Let

$$(3.11)\qquad X[x] = x(t) \quad \text{and} \quad Y[x] = \exp\left\{\lambda\int_0^t p(s)dx(s)\right\} \quad \text{for} \quad x \in C[0, t]$$

where $\lambda \in R^1$ and p is a real valued function which is continuous and of bounded variation on $[0, t]$.

Define an element x_0 in $C[0, t]$ by

$$x_0(s) = -\lambda\int_0^s p(r)dr \quad \text{for} \quad s \in [0, t]\,.$$

Then x_0' exists on $[0, t]$ and

$$x_0' = -\lambda p \in \mathrm{B.V.}\,[0, t]\,.$$

With x_0 our Y can be written as

$$Y[x] = \exp\left\{-\int_0^t x_0'(s)dx(s)\right\} \quad \text{for} \quad x \in C[0, t]\,.$$

Thus, by (3.7) a version of $E^w(Y|X)$ is given by

$$(3.12)\qquad E_x^w(Y|X)(\xi)$$
$$= \exp\left\{2^{-1}\lambda^2\int_0^t[p(s)]^2 ds\right\}\exp\left\{-(2t)^{-1}\lambda^2\left[\int_0^t p(s)ds\right]^2\right\}\exp\left\{t^{-1}\lambda\xi\left[\int_0^t p(s)ds\right]\right\}$$
$$\text{for} \quad \xi \in R^1\,.$$

Therefore for every $B \in \mathfrak{B}^1$

$$\int_{\{x(t)\in B\}}\exp\left\{\lambda\int_0^t p(s)dx(s)\right\}m_w(dx)$$

$$= \int_B E_x^w(X|Y)(\xi)P_X(d\xi)$$

$$= \exp\left\{2^{-1}\lambda^2\int_0^t[p(s)]^2ds\right\}(2\pi t)^{-1/2}\int_B \exp\left\{-(2t)^{-1}\left[\xi - \lambda\int_0^t p(s)ds\right]^2\right\}m_L(d\xi)$$

by (3.12) and by the fact that $P_X = N(0, t)$ on $(R^1, \mathfrak{B}^1)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
IRVINE, CALIFORNIA