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THE SURGERY OF CODIMENSION-ONE FOLIATIONS

GEN-ICHI OSHIKIRI

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1. Introduction. In [6], H. Rosenberg and W. Thurston posed the following problems:

1. If $(W^3, \mathscr{F}')$ is obtained from $(V^3, \mathscr{F})$ by performing surgery along a transversal simple closed curve, then is $(W^3, \mathscr{F}')$ foliated cobordant to $(V^3, \mathscr{F})$?

2. Are the Reeb foliations of S^3 null-cobordant?

Recently, F. Sergeraert [7] gave a positive answer to the second problem.

In this paper, we give a positive answer to the first problem using Sergeraert's result. In general, the existence of a codimension-one foliation on $D^2 \times S^{2n-1}$ enables us to perform foliated surgeries of type $(2, 2n)$ on $(M^{2n+1}, \mathscr{F})$ along closed transversals. We also give a relation between foliated cobordisms and the foliated surgery of this type. Furthermore, we see that the technic used in the proof of our theorem gives some information about foliated cobordism classes of the foliations obtained from spinnable structures.

We work mostly in the smooth category and all the foliations we consider will be smooth, of codimension one unless otherwise stated.

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2. Definitions and notations. Given a foliation of an oriented manifold M, we denote by $\mathscr{F}$ the set of all the leaves of the foliation, and by the pair $(M, \mathscr{F})$ we denote the oriented diffeomorphism class of the foliation of the manifold M defined by $\mathscr{F}$. Thus two foliations $(M_0, \mathscr{F}_0)$ and $(M_1, \mathscr{F}_1)$ are identified if and only if there exists an orientation preserving diffeomorphism $M_0 \to M_1$, which maps each leaf of $\mathscr{F}_0$ into a leaf of $\mathscr{F}_1$. By $-(M, \mathscr{F})$, we denote the diffeomorphism class of the same foliation as $(M, \mathscr{F})$ but with the opposite orientation of the underlying manifold. We denote by $\mathscr{F}|A$ the restriction of $\mathscr{F}$ to $A \subset M$.

DEFINITION. Two foliations of closed oriented n-manifolds $(M_0^n, \mathscr{F}_0)$

and $(M_1^n, \mathscr{F}_1)$ are said to be *foliated cobordant* or simply *cobordant* (we denote this by $(M_0^n, \mathscr{F}_0) \overset{f}{\sim} (M_1^n, \mathscr{F}_1))$, if there exists a foliation of a compact oriented $(n+1)$-manifold $(W^{n+1}, \mathscr{F})$ transverse to the boundary such that $\partial W = M_0 \cup (-M_1)$, $(M_0, \mathscr{F} | M_0) = (M_0, \mathscr{F}_0)$ and $(-M_1, \mathscr{F} | -M_1) = -(M_1, \mathscr{F}_1)$. We denote this by $\partial(W, \mathscr{F}) = (M_0, \mathscr{F}_0) - (M_1, \mathscr{F}_1)$.

We also consider a foliation of a manifold-with-corner M^n. Such a foliation is defined by a maximal set of charts of M^n modelled on a quadrant $\boldsymbol{Q}^n = \{(x_1, \cdots, x_{n-1}, x_n) \in \boldsymbol{R}^n; x_{n-1} \geqq 0, x_n \geqq 0\}$ with the standard foliation whose leaves are given by $\{x_{n-1} = \text{const.}\}$.

Given a foliation $(M, \mathscr{F})$ and a manifold N, we denote by $N \times (M, \mathscr{F})$ or by $(N \times M, N \times \mathscr{F})$ the foliation of $N \times M$ whose leaves are given by $\{N \times L; L \in \mathscr{F}\}$.

The set of cobordism classes of foliated manifolds of a fixed dimension and codimension forms an abelian group under disjoint union.

For a foliated manifold $(M^n, \mathscr{F})$ and an imbedding φ of S^1 into M^n, which is transverse to $\mathscr{F}$, with a trivial normal bundle, we define a foliated manifold $(M^n, \sigma_\varphi(\mathscr{F}))$, which is apparently concordant to $(M^n, \mathscr{F})$, as follows: let $T = T(\varphi(S^1))$ be a normal bundle of φ. Then there exists a foliation preserving diffeomorphism of $(T, \mathscr{F} | T)$ onto $(S^1 \times D^{n-1}, \{(x) \times D^{n-1}; x \in S^1\})$. Thus we can identify these foliations. Let f be a smooth function on $[0, 1]$ defined by $f(t) = 1$ on $[0, 1/4]$, $f(t) = -1$ on $[3/4, 1]$, $f(1/2) = 0$, and $df/dt < 0$ on $(1/4, 3/4)$. Parametrize D^{n-1} by (t, v) with $t \in [0, 1]$ and $v \in S^{n-2}$, identifying $(0, v)$ with $0 \in D^{n-1}$. Then $F: \boldsymbol{R} \times D^{n-1} \to \boldsymbol{R}$, defined by $F(z, (t, v)) = e^z f(t)$, is a submersion and this defines a foliation $\mathscr{F}'$ on $S^1 \times D^{n-1}$ which is a Reeb foliation on $S^1 \times D^{n-1}(1/2)$, where $D^{n-1}(1/2)$ is the closed disk with the radius $1/2$ and concentric with the closed unit disk, and which coincides with $\mathscr{F}$ near $S^1 \times \partial D^{n-1}$. Therefore we can define $\sigma_\varphi(\mathscr{F})$ on M^n to be $\mathscr{F}$ on $M - \operatorname{int} T$ and $\mathscr{F}'$ on $T \cong S^1 \times D^{n-1}$ (cf. J. Wood [12]). The foliation $(M^n, \sigma_\varphi(\mathscr{F}))$ constructed above is said to be *σ-modified along $\varphi(S^1)$ from $(M^n, \mathscr{F})$*.

Next we define $(s(M^{2n+1}), s_\alpha(\mathscr{F}))$ for $(M^{2n+1}, \mathscr{F})$ if $\dim M = 2n+1$. Here we also use an imbedding φ of S^1 into M^{2n+1} satisfying the above conditions. Remove the small tubular neighborhood $S^1 \times D^{2n}(1/2)$ of φ in $T(\varphi(S^1))$ from M^{2n+1} and add $D^2 \times S^{2n-1}$. This is the surgery of type $(2, 2n)$. The constructed manifold is denoted by $s(M^{2n+1})$. We define $s_\alpha(\mathscr{F})$ on $s(M^{2n+1})$ as follows: the manifold $D^2 \times S^{2n-1}$ has codimension-one foliations tangent to the boundary (I. Tamura [9]). We take one of these foliations and denote it by $\mathscr{F}_\alpha$. Define $s_\alpha(\mathscr{F})$ to be $\sigma_\varphi(\mathscr{F})$ on $M^{2n+1} - \operatorname{int}(S^1 \times D^{2n}(1/2))$ and $\mathscr{F}_\alpha$ on $D^2 \times S^{2n-1}$.

DEFINITION. We call this modification a *foliated surgery of type* (2, $2n$).

3. Statement of results.

THEOREM. *Let* $(M^{2n+1}, \mathscr{F})$ *be a foliated* $(2n+1)$*-manifold. Then,* $(s(M^{2n+1}), s_\alpha(\mathscr{F})) \overset{f}{\sim} (M^{2n+1}, \mathscr{F}) + (S^{2n+1}, \mathscr{F}'_\alpha)$, *where "+" means a disjoint union, and* $\mathscr{F}'_\alpha$ *on* S^{2n+1} *is obtained from* $\mathscr{F}_\alpha$ *on* $D^2 \times S^{2n-1}$ *and a Reeb foliation* $\mathscr{F}_R$ *on* $S^1 \times D^{2n}$ *by the canonical identification of* $\partial(D^2 \times S^{2n-1})$ *and* $\partial(S^1 \times D^{2n})$.

COROLLARY 1. *The answer to the Problem* 1 *in H. Rosenberg and W. Thurston* [6] *is affirmative.*

Indeed, in this case $(S^{2n+1}, \mathscr{F}'_\alpha) = (S^3, \mathscr{F}_R)$ and the latter is null-cobordant by the result of F. Sergeraert [7].

As is explained in [5], every foliation $(S^n, \mathscr{F})$ is not null-cobordant if $n \equiv 1 \pmod{4}$, hence $(s(M^{4n+1}), s_\alpha(\mathscr{F}))$ is not cobordant to $(M^{4n+1}, \mathscr{F})$. But T. Mizutani [5] constructed foliations on S^{4n+1} twice which are null-cobordant. On the other hand, I. Tamura and T. Mizutani [10] constructed foliations on S^n for $n \equiv 3 \pmod{4}$, which is null-cobordant. Thus we have

COROLLARY 2. *We can choose* $\mathscr{F}_\alpha$ *on* $D^2 \times S^{2n-1}$ *so that* $(s(M^{2n+1}), s_\alpha(\mathscr{F})) \overset{f}{\sim} (M^{2n+1}, \mathscr{F})$ *for n odd, and* $2\{(s(M^{2n+1}), s_\alpha(\mathscr{F})) - (M^{2n+1}, \mathscr{F})\} \overset{f}{\sim} 0$ *for n even.*

Next we consider the foliations obtained from specially spinnable structures (I. Tamura [9]). Recall that an n-manifold M^n is called spinnable if there exists an $(n-2)$-dimensional submanifold X of M^n satisfying the following conditions:

(1) The normal bundle of X is trivial.

(2) Let $X \times D^2$ be a tubular neighborhood of X. Then $C = M^n - (X \times \operatorname{int} D^2)$ is the total space of a differentiable fiber bundle ξ over a circle.

(3) Let $p: C \to S^1$ be the projection of ξ. Then the following diagram commutes:

$$\begin{array}{ccc} X \times S^1 & \xrightarrow{\iota} & C \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & = & S^1, \end{array}$$

where ι denotes the inclusion map and p' denotes the natural projection onto the second factor.

The submanifold X is called an axis and a fiber F of ξ is called a generator. Obviously $\partial F = X$ holds if $\partial M = \emptyset$. The fiber bundle $\xi = \{C, p, S^1, F\}$ is called a spinning bundle, and the pair (X, ξ) is called a spinnable structure on M^n. If $X = S^{n-2}$, then (X, ξ) is called a specially spinnable structure on M^n.

Suppose that M^n is a spinnable manifold with (X, ξ) as above, and that $X \times D^2$ has a foliation tangent to the boundary. Obviously C, the total space of ξ, has a foliation tangent to the boundary $\partial C = X \times S^1$, which is induced from the fibration over S^1 outside a neighborhood of ∂C. Then M^n has a foliation $\mathscr{F}$. $(M^n, \mathscr{F})$ is said to be the foliation obtained from a spinnable structure (X, ξ). If $X = S^{n-2}$, then M^n is obtained from a fibration over S^1 by a surgery on a cross-section ([9], Theorem 6). Thus we have

COROLLARY 3. *Let $(M^{2n+1}, \mathscr{F})$ be a foliation obtained from a specially spinnable structure (S^{2n-1}, ξ). Put $\mathscr{F}_\alpha = \mathscr{F} | S^{2n-1} \times D^2$, then*

$$(M^{2n+1}, \mathscr{F}) \overset{f}{\sim} (a \text{ bundle foliation}) + (S^{2n+1}, \mathscr{F}'_\alpha)\,,$$

where the bundle foliation is obtained from $C \cup D^{2n} \times S^1$ by the canonical identification of $\partial C = S^{2n-1} \times S^1$ with $\partial D^{2n} \times S^1$, which naturally gives a fibration over S^1.

In particular if $\dim M = 3$, all the bundle foliations are null-cobordant by Thurston's results [11], and the axis X is a disjoint union of S^1's. Thus we have

COROLLARY 4. *If $(M^3, \mathscr{F})$ is obtained from a spinnable structure $(\bigcup_{i=1}^k S^1_i, \xi)$, then $(M^3, \mathscr{F}) \overset{f}{\sim} \sum_{i=1}^k (S^3_i, \mathscr{F}'_i)$, where $\mathscr{F}_i | S^1_i \times D^2 = \mathscr{F} | S^1_i \times D^2$.*

Similar results have also been obtained by K. Fukui [2].

4. **r_t-surgery.** First we define two smooth functions f_0 and f_1 on $(0, 1)$ as follows: $f_t(s) = (-1)^t f_t(1-s)$ for $s \in (0, 1)$, $t = 0$ and 1, $f_t(s) = 0$ for $s \in (1/4, 3/4)$, $df_t/ds < 0$ on $(0, 1/4)$, and $f_t(s) = \exp(1/s)$ on $(0, \varepsilon)$ for a sufficiently small $\varepsilon > 0$.

Then, for each $t \in \{0, 1\}$, we get a foliation $\mathscr{F}'_t$ on $[0, 1] \times \boldsymbol{R}$ whose leaves are given by $\{0, 1\} \times \boldsymbol{R}$, the graph of f_t and its translations along $\boldsymbol{R}$. Clearly this foliation is invariant under the action of $\boldsymbol{Z}$ on $[0, 1] \times \boldsymbol{R}$ by $n \cdot (s, u) = (s, u + n)$ for $n \in \boldsymbol{Z}$, $s \in [0, 1]$ and $u \in \boldsymbol{R}$. Hence we have a foliation $\mathscr{F}_t$, for each t, on $[0, 1] \times S^1$, regarding S^1 as $\boldsymbol{R}/\boldsymbol{Z}$.

Recall that the oriented circle of a Reeb foliation on $S^1 \times D^n$ is the circle $S^1 \times \{0\} \subset S^1 \times D^n$ with the specified orientation so that the orientation of the circle coincides with the direction to which the leaves wind the boundary $S^1 \times \partial D^n$ (T. Mizutani [4]).

For a foliated manifold $(M^{n+1}, \mathscr{F})$ and each $t \in \{0, 1\}$, we define a foliated manifold $(r(M^{n+1}), r_t(\mathscr{F}))$ on the assumption that there exists an imbedding φ of $S^1 \times S^{n-1} \times [0, 1]$ into M^{n+1} such that $\mathscr{F} | \mathrm{im}\, \varphi = S^{n-1} \times (S^1 \times [0, 1], \mathscr{F}_t)$. The manifold $r(M^{n+1})$ is obtained from M^{n+1} by the round surgery of index n using the above φ (D. Asimov [1]), i.e.,

$$r(M^{n+1}) = (M^{n+1} - \mathrm{int}\, \varphi(S^1 \times S^{n-1} \times [0, 1])) \cup S^1 \times D^n \times S^0$$

with the canonical identification of $\partial(\varphi(S^1 \times S^{n-1} \times [0, 1]))$ with $\partial(S^1 \times D^n \times S^0)$. On $S^1 \times D^n \times S^0$ we consider two foliations according as $t = 0$ or 1:

$$(S^1 \times D^n \times S^0, \mathscr{F}_R \cup \mathscr{F}_R)\ ,$$

whose oriented circles coincide if $t = 0$ and are opposite if $t = 1$. And on $M^{n+1} - \mathrm{int}\,(\mathrm{im}\, \varphi)$ we consider the restriction of $\mathscr{F}$. Then we get a foliated manifold $(r(M^{n+1}), r_t(\mathscr{F}))$ for each $t \in \{0, 1\}$.

DEFINITION. We call this modification an *r_t-surgery.*

PROPOSITION. $(M^{n+1}, \mathscr{F}) \overset{f}{\sim} (r(M^{n+1}), r_t(\mathscr{F}))$, *for each* t.

PROOF. We define a subset L_a in $\boldsymbol{R}^n \times \boldsymbol{R}$ by $L_a = \{(x, y) \in \boldsymbol{R}^n \times \boldsymbol{R}$: $|x| \leqq 1, |y| \leqq 1, (|x| - 1)^2 + (|y| - 1)^2 \geqq a^2\}$, and put $K_a = \{(x, y) \in L_a$: $(|x| - 1)^2 + (|y| - 1)^2 = a^2\}$, $K_+^a = K_a \cap \{y > 0\}$, and $K_-^a = K_a \cap \{y < 0\}$. In particular, we put $L = L_{1/2}$, $K_+ = K_+^{1/2}$, and $K_- = K_-^{1/2}$. Clearly $L \times S^1$ is a manifold-with-corner. On this manifold we define a foliation with leaves $K_+ \times S^1$, $K_- \times S^1$, which coincides with $\mathscr{F}_R$ on $D^n \times \{\pm 1\} \times S^1$, and with $S^{n-1} \times ([-1/2, 1/2] \times S^1, \mathscr{F}_t)$ on $S^{n-1} \times [-1/2, 1/2] \times S^1$, where we identify $[-1/2, 1/2] \times S^1$ with $[0, 1] \times S^1$. First, we consider a smooth function g on $(1/8, 1/2)$ such that $g(s) = 0$ on $(1/8, 1/4)$, $dg/ds > 0$ on $(1/4, 1/2)$, and $g(s) = \exp(2/(1 - 2s))$ on $(-\varepsilon + 1/2, 1/2)$ for a sufficiently small $\varepsilon > 0$. Next we define two functions g_0 and g_1 on $(1/8, 1/2) \times \{\pm 1\}$ to be

$$g_0 | (1/8, 1/2) \times \{+1\} = g_0 | (1/8, 1/2) \times \{-1\} = g\ , \quad \text{and}$$
$$g_1 | (1/8, 1/2) \times \{+1\} = -g_1 | (1/8, 1/2) \times \{-1\} = g\ .$$

Note that $L' - L_{7/8}$, where $L' = L - K_+ - K_-$, is canonically diffeomorphic to $S^{n-1} \times (1/8, 1/2) \times [0, 1] \times \{\pm 1\}$. Thus if we lift g_0 and g_1 onto $S^{n-1} \times (1/8, 1/2) \times [0, 1] \times \{\pm 1\}$, then we get two smooth functions f_0' and f_1' on $L' - L_{7/8}$. Furthermore, these functions are constant on $L_{3/4} - L_{7/8}$. Hence we can extend f_0' and f_1' to functions f_0 and f_1 on L' by defining $f_t = 0$ on $L_{3/4}$. Thus we get two smooth functions f_0 and f_1 on L'. The same construction as $\mathscr{F}_t$ on $S^1 \times [0, 1]$ gives the desired foliation on $L \times S^1$

for each $t \in \{0, 1\}$. We denote this by $\mathscr{F}_t^*$. We can obtain a foliated cobordism between $(M^{n+1}, \mathscr{F})$ and $(r(M^{n+1}), r_t(\mathscr{F}))$ as follows: the cobordism W between M and $r(M)$ is obtained from

$$(M^{n+1} - \operatorname{int} \varphi(S^1 \times S^{n-1} \times [0, 1])) \times [0, 1] \cup L \times S^1$$

by the canonical identification of $\varphi(S^1 \times S^{n-1} \times \{0, 1\}) \times [0, 1]$ with $(K_+ \cup K_-) \times S^1$. Define a foliation on W to be $(\mathscr{F} | M - \operatorname{int} \varphi(S^1 \times S^{n-1} \times [0, 1])) \times [0, 1]$ on $(M - \operatorname{int} \varphi(S^1 \times S^{n-1} \times [0, 1])) \times [0, 1]$ and $\mathscr{F}_t^*$ on $L \times S^1$. Obviously this is the desired foliated cobordism. This completes the proof.

5. Proof of Theorem. Before the proof we define two foliations on $S^1 \times [0, 1]$. Let h be a smooth function on $(0, 1]$ such that $h(s) = 0$ on $[1/2, 1]$, $dh/ds < 0$ on $(0, 1/2)$ and $h(s) = \exp(1/s)$ on $(0, \varepsilon)$ for a sufficiently small $\varepsilon > 0$. The same construction as $\mathscr{F}_t$ on $S^1 \times [0, 1]$ in §4 gives a foliation on $S^1 \times [0, 1]$. We denote this by $\mathscr{G}$, which contains $S^1 \times \{0\}$ as a leaf and is transverse to $S^1 \times \{1\}$. If we regard the inclusion $S^1 \times \{1/2\} \to S^1 \times [0, 1]$ as an imbedding φ of S^1 into the foliated manifold $(S^1 \times [0, 1], \mathscr{G})$, then we have a new foliation $\sigma_\varphi(\mathscr{G})$ on $S^1 \times [0, 1]$ (see §2). We denote it by $\mathscr{H}$. We can assume that $\mathscr{H}_1 = \mathscr{H} | [0, 1/3] \times S^1$ and $\mathscr{H}_2 = \mathscr{H} | [1/3, 2/3] \times S^1$ are equivalent to $\mathscr{F}_0$ on $[0, 1] \times S^1$.

Now we prove our theorem.

Proof of Theorem. As $(M^{2n+1}, \mathscr{F})$ is concordant to $(M^{2n+1}, \sigma_\varphi(\mathscr{F}))$, we have only to show that

$$(s(M^{2n+1}), s_\alpha(\mathscr{F})) \overset{f}{\sim} (M^{2n+1}, \sigma_\varphi(\mathscr{F})) + (S^{2n+1}, \mathscr{F}'_\alpha) .$$

We may assume that on $T(\varphi(S^1)) - \operatorname{int}(\varphi(S^1) \times D^{2n}(1/2))$ $(\subset s(M^{2n+1}))$, which is diffeomorphic to $S^1 \times [0, 1] \times S^{2n-1}$, the foliation $s_\alpha(\mathscr{F})$ is equivalent to $(S^1 \times [0, 1], \mathscr{G}) \times S^{2n-1}$, where the S^1-factor is the same as ∂D^2 and $D^2 \times S^{2n-1}$ is attached along $S^1 \times \{0\} \times S^{2n-1}$. As is easily seen, $(S^1 \times [0, 1], \mathscr{G}) \times S^{2n-1}$ is concordant to $(S^1 \times [0, 1], \mathscr{H}) \times S^{2n-1} \cong (S^1 \times [0, 1/3], \mathscr{H}_1) \times S^{2n-1} \cup (S^1 \times [1/3, 2/3], \mathscr{H}_2) \times S^{2n-1} \cup (S^1 \times [2/3, 1], \mathscr{H} | S^1 \times [2/3, 1]) \times S^{2n-1}$. As $(S^1 \times [0, 1/3], \mathscr{H}_1) \times S^{2n-1}$ and $(S^1 \times [1/3, 2/3], \mathscr{H}_2) \times S^{2n-1}$ are diffeomorphic to $(S^1 \times [0, 1], \mathscr{F}_0) \times S^{2n-1}$, we can perform r_0-surgeries on them. By Proposition in §4 we get

$$\begin{aligned}(s(M^{2n+1}), s_\alpha(\mathscr{F})) \overset{f}{\sim} (M^{2n+1}, \sigma_\varphi(\mathscr{F})) &+ (S^{2n} \times S^1, \mathscr{F}') \\ &+ (S^{2n+1}, \mathscr{F}'_\alpha) ,\end{aligned}$$

where $\mathscr{F}'$ on $S^{2n} \times S^1$ is obtained from two Reeb foliations on $D^{2n} \times S^1$ by the canonical identification along $\partial D^{2n} \times S^1$. Obviously $(S^{2n} \times S^1, \mathscr{F}')$

is null-cobordant. This completes the proof.

6. Concluding remarks. H. Lawson [3] defined $\bigstar$-operations in $\mathscr{F}\Omega_1$, and one of these operations gives a different proof of our theorem, that is, $(s(M^{2n+1}), s_\alpha(\mathscr{F})) = (M^{2n+1}, \mathscr{F}) \bigstar (S^{2n+1}, \mathscr{F}'_\alpha)$ for a suitable choice of closed transversals in M^{2n+1} and S^{2n+1}. None of our technic is needed.

However, our technic gives the proof of the following statement, which gives some information about foliated cobordism classes of the foliations obtained from spinnable structures:

STATEMENT. If $(M, \mathscr{F})$ is obtained from a spinnable structure $(X, \xi = \{C, p, S^1, F\})$, then

$$(M, \mathscr{F}) \overset{f}{\sim} (\text{a bundle foliation}) + (M', \mathscr{F}') ,$$

where the total space of the bundle foliation is obtained from $C \cup F' \times S^1$, F' being any compact oriented manifold with $\partial F' = X = \partial F$, by the identification of $\partial C = X \times S^1$ and $\partial F' \times S^1 = X \times S^1$ with the canonical fibration over S^1, and where $(M', \mathscr{F}')$ is obtained from a spinnable structure $(X, \xi' = \{F' \times S^1, p_2, S^1, F'\})$ with the same foliation on $X \times D^2$ as $\mathscr{F} | X \times D^2$ in M.

As an application of the above statement we show that the foliations on S^{2n+1} constructed by I. Tamura [8] are cobordant to bundle foliations.

Spinnable structures on S^{2n+1} constructed in [8] are $(V_{2n,2}, \xi)$ on S^{4n-1}, and $(S^{2n} \times S^{2n-1}, \xi')$ on S^{4n+1}. $V_{2n,2}$ is the Stiefel manifold $SO(2n)/SO(2n-2)$. We also treat the case where a spinnable structure on S^{4n-1} is $(S^{2n-1} \times S^{2n-2}, \xi'')$. Then S^{2n+1} has a spinnable structure $(S^\alpha \times S^\beta, \xi')$, where α (resp. β) is the odd (resp. even) integer of $\{n-1, n\}$, respectively. Foliations on $V_{2n,2} \times D^2$ and $S^\alpha \times S^\beta \times D^2$ are pullbacks of foliations on $S^{2n-1} \times D^2$ and $S^\alpha \times D^2$ by the fibrations $V_{2n,2} \times D^2 \to S^{2n-1} \times D^2$ and $S^\alpha \times S^\beta \times D^2 \to S^\alpha \times D^2$, respectively.

For $(S^{2n+1}, \mathscr{F})$ obtained from $(S^\alpha \times S^\beta, \xi')$, the above statement gives

$$(S^{2n+1}, \mathscr{F}) \overset{f}{\sim} S^\beta \times (S^\alpha, \mathscr{F}') + (\text{a bundle foliation})$$

if we choose F' to be $S^\beta \times D^{\alpha+1}$. Obviously $S^\beta \times (S^\alpha, \mathscr{F}')$ is null-cobordant.

For $(S^{4n-1}, \mathscr{F})$ obtained from $(V_{2n,2}, \xi)$, the above statement gives

$$(S^{4n-1}, \mathscr{F}) \overset{f}{\sim} (V_{2n,2} \times D^2 \cup N \times S^1, \mathscr{F}') + (\text{a bundle foliation})$$

if we choose F' to be N, a tubular neighborhood of the diagonal $\{(x, x) \in S^{2n-1} \times S^{2n-1} \colon x \in S^{2n-1}\}$ in $S^{2n-1} \times S^{2n-1}$. Note that $\partial N = V_{2n,2}$. By straightening the angle we get $\partial(N \times D^2) = V_{2n,2} \times D^2 \cup N \times S^1$. Considering the fibration $N \times D^2 \to S^{2n-1} \times D^2$, we can easily construct a foliation $\mathscr{F}^*$ on $N \times D^2$ with

$$\partial(N \times D^2, \mathscr{F}^*) = (V_{2n,2} \times D^2 \cup N \times S^1, \mathscr{F}') .$$

Thus many foliations constructed by Tamura and Mizutani, in [4] and [9] for example, are cobordant to bundle foliations.

On the other hand, using Thurston's result [11, Theorem 2], we can classify bundle foliations completely in $\mathscr{F}\Omega_1$. Hence many foliations constructed by Tamura and Mizutani can be classified. In particular, combining the above with this, we have different proofs of the Theorem in [10] and Theorem 2 in [5].

K. Fukui also treated these problems by constructing Γ_1-structures and using Thurston's results.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980 JAPAN