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ON THE TRANSFORMATION OF SOME CLASSES OF MARTINGALES BY A CHANGE OF LAW

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1. Introduction. Let M be a continuous local martingale with $M_0 = 0$, and let us denote by $\langle M\rangle$ the continuous increasing process such that $M^2 - \langle M\rangle$ is also a local martingale. Then the solution Z of the stochastic integral equation:

$$Z_t = 1 + \int_0^t Z_s dM_s$$

is given by the formula $Z_t = \exp(M_t - \langle M\rangle_t/2)$, so that it is a positive local martingale with $Z_0 = 1$. However, it is not always a martingale. The problem of finding sufficient conditions for the process Z to be a martingale, which is proposed by I. V. Girsanov, is important in certain questions concerning the absolute continuity of probability measures of diffusion processes. In Section 3, we shall give a new sufficient condition for the problem of Girsanov. Namely, it will be proved that if M is a BMO-martingale, then Z is an L^p-bounded martingale for some $p > 1$. The theory of H^p and BMO martingales was developed in [3] and [4], and it is well-known nowadays that $(H^1)^* \cong \mathrm{BMO}$, that is, the dual space of H^1 is isomorphic to BMO. In Section 4, Z is assumed to be a uniformly integrable martingale. Then we can define a change of the underlying probability measure dP by the formula $d\hat{P} = Z_\infty dP$. If $\mathscr{H}$ is a class of continuous local martingales, with respect to $d\hat{P}$ we denote by $\hat{\mathscr{H}}$ the class corresponding to $\mathscr{H}$. Our interest here lies in investigating the relations between $\mathscr{H}$ and $\hat{\mathscr{H}}$. In the section we shall prove that if M is a BMO-martingale, then $\mathrm{BMO} \cong \mathrm{BMO}^\wedge$ and $H^1 \cong \hat{H}^1$. In addition, it is shown that $H^2 \cong \hat{H}^2$ holds in general. In Section 5 we shall give a generalization of the classical inequalities of J. L. Doob.

Let us denote by C a positive constant and by C_x a positive constant depending only on the indicated parameter x. Both letters are not necessarily the same in each occurrence.

2. Preliminaries.

1) Definitions and notations. Let (Ω, F, P) be a complete probability space, and let $(F_t)_{(0 \leqq t < \infty)}$ be a non-decreasing right continuous family of sub-σ-fields of F with $F = \bigvee_{t \geqq 0} F_t$ such that F_0 contains all null sets. Throughout the paper we shall deal only with continuous local martingales. The reader is assumed to be familiar with the martingale theory as given in [3] and [10]. See Getoor and Sharpe [4] for the theory of conformal martingales.

For any process $X = (X_t, F_t)$, we denote by X^* the quantity $\sup_t |X_t|$. If T is a stopping time, X^T is the process $(X_{t \wedge T})$ stopped at T. Let $\mathscr{L}$ be the class of all continuous local martingales X over (F_t) with $X_0 = 0$. For X and Y in $\mathscr{L}$, we define $\langle X, Y \rangle = (\langle X+Y \rangle - \langle X \rangle - \langle Y \rangle)/2$. Then, as is well-known, $XY - \langle X, Y \rangle$ belongs to $\mathscr{L}$. For $X \in \mathscr{L}$ and a locally bounded previsible process H, $H \circ X$ is the unique element of $\mathscr{L}$ such that for all $Y \in \mathscr{L}$, $\langle H \circ X, Y \rangle_t = \int_0^t H_s d\langle X, Y \rangle_s$. The process $H \circ X$ is called the stochastic integral of H relative to X. We also write $(H \circ X)_t = \int_0^t H_s dX_s$.

DEFINITION 1. For any $X \in \mathscr{L}$ and $0 < p < \infty$, let

$$\|X\|_{H^p} = (E[\langle X \rangle_\infty^{p/2}])^{1/p} .$$

We denote by H^p the class of all $X \in \mathscr{L}$ such that $\|X\|_{H^p} < \infty$. If $1 \leqq p < \infty$, H^p is a real Banach space with norm $\| \quad \|_{H^p}$.

Recall now the inequality of B. Davis:

$$(1/4\sqrt{2})E[X^*] \leqq E[\langle X \rangle_\infty^{1/2}] \leqq 2E[X^*] , \qquad X \in \mathscr{L} .$$

For the proof, see [4]. This implies that if $X \in H^1$, X is uniformly integrable. This inequality of Davis is of fundamental importance in the martingale theory.

DEFINITION 2. For any $X \in \mathscr{L}$, let

$$\|X\|_{\mathrm{BMO}} = \sup_t \|(E[\langle X \rangle_\infty - \langle X \rangle_t | F_t])^{1/2}\|_\infty .$$

Let BMO consist of those $X \in \mathscr{L}$ which satisfy $\|X\|_{\mathrm{BMO}} < \infty$. The energy inequalities (see [10]) give

$$E[\langle X \rangle_\infty^n] \leqq n! \, \|X\|_{\mathrm{BMO}}^{2n} , \qquad n = 1, 2, \cdots .$$

Therefore, $\mathrm{BMO} \subset H^p$ for every p. The space BMO, which can be identified with the dual space of H^1, is complete with norm $\| \quad \|_{\mathrm{BMO}}$. The following is an example of BMO-martingales.

EXAMPLE 1. Let $B = (B_t, F_t, P_x)_{x \in R}$ be a one dimensional Brownian motion and let $T_a = \inf(t; |B_t| = a)$, $(a > 0)$. It is easy to see that T_a is a stopping time. Then the BMO-norm of the martingale B^{T_a} with respect to the measure P_0 is equal to a. In fact, if $|x| < a$, $E_x[T_a] = a^2 - x^2$ because $|B_{T_a}| = a$ and $E_x[B_{T_a}^2 - T_a] = x^2$. Now let θ_t be the shift operators of the process $B = (B_t)$. Then $T_a - t = T_a \circ \theta_t$ on $(t < T_a)$ by the definition of T_a. It is also clear that $\langle B^{T_a} \rangle_t = t \wedge T_a$, P_0-a.s., so that using the Markovian character, we have

$$\begin{aligned} E_0[T_a - t \wedge T_a | F_t] &= E_0[T_a \circ \theta_t | F_t] I_{(t<T_a)} \\ &= E_{B_t}[T_a] I_{(t<T_a)} = (a^2 - B_t^2) I_{(t<T_a)} . \end{aligned}$$

Therefore we have $||B^{T_a}||_{\mathrm{BMO}} = a$.

Now for $M \in \mathscr{L}$, let us consider the process Z defined by the formula

$$Z_t = e^{M_t - \langle M \rangle_t / 2} , \qquad t \geqq 0 .$$

It is a positive supermartingale such that $Z - 1 \in \mathscr{L}$. As $Z_0 = 1$, $E[Z_t] \leqq 1$ for every t. Thus Z is a martingale if and only if $E[Z_t] = 1$ for every t. Let $Z_\infty = \lim Z_t$. The existence of this limit is guaranteed by the martingale convergence theorem due to Doob. Fatou's lemma shows that it is finite with probability 1. Similarly, for each real number a, the process $Z^{(a)}$ defined by $Z_t^{(a)} = \exp(aM_t - a^2 \langle M \rangle_t / 2)$ is also a positive local martingale. As $Z_t Z_t^{(-1)} = \exp(-\langle M \rangle_t)$, $Z_\infty = 0$ implies $\langle M \rangle_\infty = \infty$. Conversely, if $\langle M \rangle_\infty = \infty$, then $Z_\infty = 0$, for $Z_t = (Z_t^{(1/2)})^2 \exp(-\langle M \rangle_t / 4)$. We now remark that $Z^{(-1)}$ is not necessarily a martingale even if Z is bounded. Here is an example.

EXAMPLE 2. Let $B = (B_t, F_t)$ be a one dimensional Brownian motion starting at 0, defined on a probability space (Ω, F, P). We set $T = \inf(t; B_t \geqq 1)$, which is a stopping time such that $0 < T < \infty$. Now let $g: [0, 1[\to [0, \infty[$ be an increasing homeomorphism, and set

$$\tau_t = \begin{cases} g(t) \wedge T & \text{if } 0 \leqq t < 1 \\ T & \text{if } 1 \leqq t < \infty . \end{cases}$$

Then these τ_t are stopping times with $\tau_0 = 0$ and $\tau_1 = T$ such that for a.e. $\omega \in \Omega$ the sample functions $\tau.(\omega)$ are non-decreasing and continuous. Thus, the process M defined by $M_t = B_{\tau_t}$ is a continuous local martingale over (F_{τ_t}). As $\tau_t \leqq T$, we have $M_t \leqq 1$, so that Z_t is bounded by e. On the other hand, as $M_1 = B_T = 1$, we have $E[Z_1^{(-1)}] \leqq E[\exp(-M_1)] < 1$. This implies that $Z^{(-1)}$ is not a martingale.

In what follows, given $M \in \mathscr{L}$, Z denotes the process $(\exp(M_t - \langle M \rangle_t / 2))$, unless otherwise stated.

DEFINITION 3. Let $1 < p < \infty$. We say that Z satisfies the (A_p) condition if

$$\sup_t \|E[(Z_t/Z_\infty)^{1/(p-1)} | F_t]\|_\infty < \infty .$$

If Z satisfies (A_p), then $Z_\infty > 0$ a.s., so that $\langle M\rangle_\infty < \infty$ a.s.. If $1 < p < r$, (A_p) implies (A_r) by Hölder's inequality. For simplicity, let us say that (A_∞) holds, if Z satisfies (A_p) for some $p > 1$. By Lemma 5, if Z satisfies (A_∞), then the process $Z^{(a)}$, defined as before, also satisfies the condition. The (A_p) condition has already appeared many times in the literature in connection with several different questions (for example, see [12]).

2) **Preliminary lemmas.** Here we collect several lemmas which are of use in subsequent sections. The following inequality is called Fefferman's inequality.

LEMMA 1. *If* $X \in H^1$ *and* $Y \in \mathrm{BMO}$, *then*

$$E\left[\int_0^\infty |d\langle X, Y\rangle_s|\right] \leqq \sqrt{2}\|X\|_{H^1}\|Y\|_{\mathrm{BMO}} .$$

PROOF. It is proved in [4], but for the reader's convenience we shall recall briefly the proof.

By using the usual stopping argument, we may assume X in H^2. Then we have

$$E\left[\int_0^\infty |d\langle X, Y\rangle_s|\right]^2 \leqq E\left[\int_0^\infty \langle X\rangle_s^{-1/2} d\langle X\rangle_s\right] E\left[\int_0^\infty \langle X\rangle_s^{1/2} d\langle Y\rangle_s\right] .$$

The first term on the right hand side is smaller that $2\|X\|_{H^1}$. On the other hand, by integration by parts, the second term is

$$E\left[\langle X\rangle_\infty^{1/2}\langle Y\rangle_\infty - \int_0^\infty \langle Y\rangle_s d\langle X\rangle_s^{1/2}\right] = E\left[\int_0^\infty (\langle Y\rangle_\infty - \langle Y\rangle_s) d\langle X\rangle_s^{1/2}\right]$$
$$= E\left[\int_0^\infty E[\langle Y\rangle_\infty - \langle Y\rangle_s | F_s] d\langle X\rangle_s^{1/2}\right] ,$$

which is dominated by $\|Y\|_{\mathrm{BMO}}^2\|X\|_{H^1}$. Thus the lemma is proved.

Fefferman's inequality implies that $\mathrm{BMO} \subset (H^1)^*$. The following result is also proved in [4].

LEMMA 2. *Let* $X \in \mathscr{L}$. *Then we have*

$$\|X\|_{H^1} \leqq \sup\{E[\langle X, Y\rangle_\infty]; Y \in \mathrm{BMO}, \|Y\|_{\mathrm{BMO}} \leqq 1\} .$$

PROOF. Let (T_n) be a non-decreasing sequence of stopping times with $\lim_n T_n = \infty$ a.s., such that $X^{T_n} \in H^1$ for each n. In addition, it is easy to see that $\langle X^{T_n}, Y\rangle = \langle X, Y^{T_n}\rangle$, $\|Y^{T_n}\|_{\mathrm{BMO}} \leqq \|Y\|_{\mathrm{BMO}}$ and $\lim_n \|X^{T_n}\|_{H^1} = \|X\|_{H^1}$. Therefore we may assume that $X \in H^1$. Let now

ε be an arbitrary positive real number, and define $Y_t = \int_0^t D_{s-}dX_s$, where $D_t = E[(\varepsilon + \langle X\rangle_\infty)^{-1/2}|F_t]$. Then, by an elementary calculation, we get $||Y||_{\mathrm{BMO}} \leqq 1$. Furthermore, $\langle X\rangle$ being continuous, we have

$$E[\langle X, Y\rangle_\infty] = E\left[\int_0^\infty D_{s-}d\langle X\rangle_s\right] = E\left[\int_0^\infty D_s d\langle X\rangle_s\right]$$
$$= E[(\varepsilon + \langle X\rangle_\infty)^{-1/2}\langle X\rangle_\infty] ,$$

which increases to $||X||_{H^1}$ as $\varepsilon \to 0$. This completes the proof.

P. A. Meyer proved in [11] the following inequality.

LEMMA 3. *Let* $X \in \mathscr{L}$. *Then*

$$||X||_{\mathrm{BMO}} \leqq \sup\{E[\langle X, X\rangle_\infty]; Y \in H^1, ||Y||_{H^1} \leqq 1\} .$$

PROOF. We prove it, following the idea of Meyer. Let us denote by d its right hand side, and T be any stopping time. It is sufficient to show that

$$E[\langle X\rangle_\infty - \langle X\rangle_T; A] \leqq d^2P(A) \qquad \text{for} \quad A \in F_T .$$

For simplicity, set $U = \langle X\rangle_\infty - \langle X\rangle_T$. The stopping argument enables us to assume that $X \in \mathrm{BMO}$, and so $E[UI_A] < \infty$. The process H given by $H_t = I_{A\cap(T<t)}$ is a previsible process such that $H^2 = H$. Then we have $\langle H\circ X, X\rangle_\infty = \langle H\circ X\rangle_\infty = UI_A$, so that

$$E[UI_A] \leqq d||H\circ X||_{H^1} = dE[I_A\sqrt{UI_A}] .$$

By Schwarz' inequality the right hand side is smaller than

$$dP(A)^{1/2}E[UI_A]^{1/2} .$$

Consequently we get $E[UI_A] \leqq d^2P(A)$.

The next inequality, which was established by A. M. Garsia for discrete martingales in [3], plays an important role in our investigation.

LEMMA 4. *If* $||X||_{\mathrm{BMO}} < 1$, *then*

$$E[e^{\langle X\rangle_\infty - \langle X\rangle_t}|F_t] \leqq (1 - ||X||^2_{\mathrm{BMO}})^{-1} .$$

PROOF. For simplicity, let us denote by d the right hand side of this inequality. It suffices to show that for every $A \in F_t$

$$E[e^{\langle X\rangle_\infty - \langle X\rangle_t}; A] \leqq dP(A) .$$

We may assume that $P(A) > 0$. To show this, let us set $dP' = (I_A/P(A))dP$ and $F'_s = F_{t+s}$. Then it is not difficult to see that for $X \in \mathrm{BMO}$ the process X' defined by $X'_s = X_{t+s} - X_t$ is also a BMO-martingale over (F'_t) with respect to dP' and that $\langle X'\rangle_s = \langle X\rangle_{t+s} - \langle X\rangle_t$. Therefore we have

$$E[e^{\langle X\rangle_\infty - \langle X\rangle_t}; A] = E'[e^{\langle X'\rangle_\infty}]P(A) ,$$

where $E'[\ \ ]$ denotes the expectation over Ω with respect to dP'. An elementary calculation shows that the BMO-norm of X' is smaller than $\|X\|_{\mathrm{BMO}}$. Then, by the energy inequalities, we have

$$E'[e^{\langle X'\rangle_\infty}] = \sum_{n=0}^{\infty}\frac{1}{n!}E'[\langle X'\rangle_\infty^n] \leqq \sum_{n=0}^{\infty}\|X'\|_{\mathrm{BMO}}^{2n} \leqq \sum_{n=0}^{\infty}\|X\|_{\mathrm{BMO}}^{2n} = d ,$$

completing the proof.

This estimate is the best possible, as the following example shows.

EXAMPLE 3. Firstly, let G^0 be the class of all topological Borel sets in $R_+ = [0, \infty[$, and S be the identity mapping of R_+ onto R_+. We define a probability measure $d\mu$ on R_+ such that $\mu(S>t)=e^{-t}$. Let G be the completion of G^0 with respect to $d\mu$, and similarly G_t the completion of the Borel field generated by $S \wedge t$. It is clear that S is a stopping time over (G_t). We now construct in the usual way a probability system $(\Omega, F, P; (F_t))$ by taking the product of the system $(R_+, G, d\mu; (G_t))$ with another system $(\Omega', F', P'; (F'_t))$ which carries a one dimensional Brownian motion $B = (B_t)$ starting at 0. Then S is also a stopping time over (F_t) so that $X = B^S$ is a continuous martingale. As $\langle X\rangle_t = S \wedge t$, we get

$$E[\langle X\rangle_\infty - \langle X\rangle_t | F_t] = e^t \int_t^\infty (x-t)e^{-x}dxI_{(t<S)} = I_{(t<S)} ,$$

from which $\|X\|_{\mathrm{BMO}} = 1$. Let now $0 < \varepsilon < 1$. Then by Lemma 4

$$E[e^{(1-\varepsilon)\langle X\rangle_\infty}] \leqq (1-(1-\varepsilon)\|X\|_{\mathrm{BMO}}^2)^{-1} = \varepsilon^{-1} .$$

But the left hand side is

$$\int_{R_+} e^{(1-\varepsilon)S}d\mu = \int_0^\infty e^{-\varepsilon x}dx = \varepsilon^{-1} .$$

Thus the inequality given in Lemma 4 cannot be improved.

We finish this section with the following result obtained by Kazamaki [6]. Quite recently, the extension to right continuous local martingales was given by C. Doléans-Dade and P. A. Meyer [1] and by Kazamaki [8].

LEMMA 5. *Let $M \in \mathscr{L}$. Then M is a* BMO-*martingale if and only if Z satisties (A_∞).*

PROOF. Suppose firstly that $\|M\|_{\mathrm{BMO}} < \infty$, and choose $p > 1$ such that $\|M\|_{\mathrm{BMO}}^2 < 2(\sqrt{p}-1)^2$. Now we are going to show that Z satisfies

(A_p). Indeed, set $p_0 = \sqrt{p} + 1$. The exponent conjugate q_0 is $(\sqrt{p} + 1)/\sqrt{p}$, so that $1/q_0(\sqrt{p} - 1)^2 - p_0/(p-1)^2 = 1/(p-1)$. By Hölder's inequality

$$\begin{aligned} E[(Z_t/Z_\infty)^{1/(p-1)} \mid F_t] &= E[\exp(-(M_\infty - M_t)/(p-1) - p_0(\langle M\rangle_\infty - \langle M\rangle_t)/2(p-1)^2) \\ &\qquad \times \exp((\langle M\rangle_\infty - \langle M\rangle_t)/2q_0(\sqrt{p}-1)^2) \mid F_t] \\ &\leqq E[\exp(-p_0(M_\infty - M_t)/(p-1) - p_0^2(\langle M\rangle_\infty - \langle M\rangle_t)/2(p-1)^2) \mid F_t]^{1/p_0} \\ &\qquad \times E[\exp((\langle M\rangle_\infty - \langle M\rangle_t)/2(\sqrt{p}-1)^2) \mid F_t]^{1/q_0} . \end{aligned}$$

By the supermartingale inequality, the first term on the right hand side is smaller than 1. In addition, according to Lemma 4, the second term is dominated by $(1 - \|M\|_{\mathrm{BMO}}^2/2(\sqrt{p}-1)^2)^{-1}$.

Conversely, let us assume that Z satisfies the (A_{p-1}) condition for some $p > 2$. Let (T_n) be a non-decreasing sequence of stopping times with $\lim_n T_n = \infty$ such that each process M^{T_n} is a uniformly integrable martingale. We now claim that each Z^{T_n} satisfies (A_p). To see this, we apply Hölder's inequality with exponents $(p-1)/(p-2)$ and $p-1$:

$$\begin{aligned} E[(Z_{t\wedge T_n}/Z_{T_n})^{1/(p-1)} \mid F_{t\wedge T_n}] &= E[(Z_{t\wedge T_n}/Z_\infty)^{1/(p-1)}(Z_\infty/Z_{T_n})^{1/(p-1)} \mid F_{t\wedge T_n}] \\ &\leqq E[(Z_{t\wedge T_n}/Z_\infty)^{1/(p-2)} \mid F_{t\wedge T_n}]^{(p-2)/(p-1)} \\ &\qquad \times E[Z_\infty/Z_{T_n} \mid F_{t\wedge T_n}]^{1/(p-1)} . \end{aligned}$$

The first term on the right hand side is dominated by some constant C_p because Z satisfies (A_{p-1}). In addition, as Z is a positive supermartingale, the second term is smaller than 1. Consequently, for every n, Z^{T_n} satisfies the (A_p) condition. Then by Jensen's inequality

$$\begin{aligned} &E[(Z_{t\wedge T_n}/Z_{T_n})^{1/(p-1)} \mid F_{t\wedge T_n}] \\ &\quad \geqq \exp(E[-M_{T_n} + M_{t\wedge T_n} + (\langle M\rangle_{T_n} - \langle M\rangle_{t\wedge T_n})/2 \mid F_{t\wedge T_n}]/(p-1)) \\ &\quad = \exp(E[\langle M\rangle_{T_n} - \langle M\rangle_{t\wedge T_n} \mid F_{t\wedge T_n}]/2(p-1)) , \end{aligned}$$

from which $\|M^{T_n}\|_{\mathrm{BMO}}^2 \leqq 2(p-1)\log C_p$ for every n. Letting $n \to \infty$, we get $M \in \mathrm{BMO}$. Thus the lemma is completely established.

By this lemma it is immediate to see that even if Z is bounded, it does not always satisfy (A_∞). See Example 2.

3. On the problem of Girsanov. If $M \in \mathscr{L}$, when can one assert that $Z_t = \exp(M_t - \langle M\rangle_t/2)$ is a martingale? In 1960 this problem was posed by I. V. Girsanov. A. A. Novikov [13] gave an answer to the effect that if $\exp(\langle M\rangle_t/2) \in L^1$ for every t, then the process Z is a martingale. Recently, by making a partial modification of Novikov's proof, Kazamaki [7] showed that if $(\exp(M_t/2))$ is a submartingale, then Z is a martingale. Note that Kazamaki's condition is weaker than Novikov's, because $E[\exp(M_t/2)] \leqq E[\exp(\langle M\rangle_t/2)]^{1/2}$ by Schwarz' inequality. Furthermore, there exists a BMO-martingale M, which does not satisfy Novikov's condition, although $\exp(M_t/2)$ is a submartingale, as the following example shows.

EXAMPLE 4. Let S, $B = (B_t, F_t)$ and (Ω, F, P) be as in Example 3. Then $X_t = \sqrt{2}B_{S\wedge t}$ is a BMO-martingale over (F_t). By the result of Novikov

$$\int_{\Omega'} \exp(B_u/\sqrt{2} - u/4)dP' = 1$$

for every $u \geqq 0$, and so by Fubini's theorem we have

$$E[\exp(X_\infty/2)] = E[\exp(B_S/\sqrt{2})] = \int_0^\infty \exp(u/4)d\mu \int_{\Omega'} \exp(B_u/\sqrt{2} - u/4)dP'$$
$$= \int_0^\infty \exp(-3u/4)du < \infty .$$

Let now (τ_t) be a continuous change of time such that $\tau_0 = 0$ and $\tau_1 = S$, and consider the martingale $M_t = X_{\tau_t}$. It is a BMO-martingale over (F_{τ_t}), and the process $\exp(M_t/2)$ is a submartingale. But, $\exp(\langle M\rangle_1/2)$ is not integrable because $\langle M\rangle_1 = 2S$.

We now give a new sufficient condition for the problem of Girsanov as follows.

LEMMA 6. *If M is a* BMO-*martingale, then Z is a uniformly integrable martingale.*

PROOF. We may assume that $0 < ||M||_{\mathrm{BMO}} < \infty$. Firstly we show that if $||M||_{\mathrm{BMO}} < \sqrt{2}$, then Z is uniformly integrable. Let c be a positive number. Then applying Schwarz' inequality we have $E[\exp(cM_t)] \leqq E[\exp(2c^2\langle M\rangle_t)]^{1/2}$. Now let $0 < \delta < 1/\sqrt{2}||M||_{\mathrm{BMO}} - 1/2$ and $c = 1/2 + \delta$. As $||\sqrt{2}cM||_{\mathrm{BMO}} < 1$, it follows from Lemma 4 that

$$E[\exp((1/2 + \delta)M_t)] \leqq E[\exp(2c^2\langle M\rangle_t)]^{1/2} \leqq (1 - 2c^2||M||^2_{\mathrm{BMO}})^{-1/2} .$$

Namely, $\sup_t E[\exp((1/2 + \delta)M_t)] < \infty$. Set now $p = 1 + 4\delta > 1$. So the exponent conjugate to p is $q = (1 + 4\delta)/4\delta$. Then by Hölder's inequality we get

$$E[Z_t^r] = E[\exp(\sqrt{r/p}M_t - r\langle M\rangle_t/2)\exp((r - \sqrt{r/p})M_t)]$$
$$\leqq E[\exp(\sqrt{pr}M_t - pr\langle M\rangle_t/2)]^{1/p}E[\exp((r - \sqrt{r/p})qM_t)]^{1/q} , \quad r > 0 .$$

The first term on right hand side is bounded by 1, because the process

$\exp(\sqrt{pr}M_t - pr\langle M\rangle_t/2)$ is nothing but the positive local martingale $Z^{(\sqrt{pr})}$. If $r = (1+2\delta)^2/(1+4\delta) > 1$, by a simple calculation we have $(r - \sqrt{r/p})q = 1/2 + \delta$, so that $\sup_t E[Z_t^r] < \infty$. Therefore Z is a uniformly integrable martingale if $\|M\|_{\mathrm{BMO}} < \sqrt{2}$. Now we are going to deal with the general case. Let us choose a number a such that $0 < a < \mathrm{Min}(1, 2/\|M\|_{\mathrm{BMO}}^2)$. Then, as $\|aM\|_{\mathrm{BMO}} < \sqrt{2}$, the process $Z^{(a)}$ is a uniformly integrable martingale. Therefore, for any stopping time T

$$
\begin{aligned}
1 &= E[Z_\infty^{(a)}/Z_T^{(a)} \mid F_T] \\
&= E[\exp(a(M_\infty - M_T) - a(\langle M\rangle_\infty - \langle M\rangle_T)/2 \exp(a(1-a)(\langle M\rangle_\infty - \langle M\rangle_T)/2) \mid F_T] .
\end{aligned}
$$

Applying Hölder's inequality with exponents $1/a$ and $1/(1-a)$ to the right hand side we can obtain:

$$1 \leqq E[Z_\infty/Z_T \mid F_T] E[\exp(a(\langle M\rangle_\infty - \langle M\rangle_T)/2) \mid F_T]^{(1-a)/a} .$$

By Lemma 4 the second term on the right hand side is smaller than

$$(1 - a\|M\|_{\mathrm{BMO}}^2/2)^{-(1-a)/a} = \{(1 - a\|M\|_{\mathrm{BMO}}^2/2)^{-2/a\|M\|_{\mathrm{BMO}}^2}\}^{(1-a)\|M\|_{\mathrm{BMO}}^2/2} ,$$

which converges to $\exp(\|M\|_{\mathrm{BMO}}^2/2)$ as $a \to 0$. Consequently, we have

$$Z_T \leqq E[Z_\infty \mid F_T] \exp(\|M\|_{\mathrm{BMO}}^2/2) .$$

This implies that Z is a uniformly integrable martingale.

Our aim in this section is to prove the following:

THEOREM 1. *If M is a* BMO*-martingale, then the "reverse Hölder inequality"*

$$E[Z_\infty^{1+\varepsilon} \mid F_t] \leqq C_\varepsilon Z_t^{1+\varepsilon}$$

holds for every t, with positive constants C_ε and ε.

REMARK. Quite recently, C. Doléans-Dade and P. A. Meyer [2] proved, assuming the uniform integrability of the process Z, that the reverse Hölder inequality holds if Z satisfies (A_∞). In [2] they make a systematic study of the subject about the (A_p) condition from a more general point of view.

PROOF. Our proof is an adaptation of the proof given in [2]. Now let $M \in \mathrm{BMO}$. Then, by Lemmas 5 and 6, Z is a uniformly integrable martingale which satisfies (A_p) for some $p > 1$. We denote by $d\hat{P}$ the weighted probability measure $Z_\infty dP$ and by $\hat{E}[\ ]$ the expectation over Ω with respect to $d\hat{P}$. Clearly, if $A \in F_t$, $\hat{P}(A) = \int_A Z_t dP$ so that for every $\hat{P}$-integrable random variable V we have

$$\hat{E}[V \mid F_t] = E[Z_\infty V \mid F_t]/Z_t \quad \text{a.s., under } dP \text{ and } d\hat{P} .$$

We shall use this formula many times in the sequel. Let K be a constant $\geqq 1$ depending only on p such that

$$Z_t E[Z_\infty^{-1/(p-1)} | F_t]^{p-1} \leqq K ,$$

which follows from the definition of (A_p). Now we set $a = 1/2^p K$ and $b_\varepsilon = 2\varepsilon/(1+\varepsilon)a^{1+\varepsilon}$ and let us choose $\varepsilon > 0$ such that $b_\varepsilon < 1$. Then we claim that $E[Z_\infty^{1+\varepsilon} | F_t] \leqq C_\varepsilon Z_t^{1+\varepsilon}$ where $C_\varepsilon = (3 - b_\varepsilon)/(1 - b_\varepsilon)$.

Firstly, we show that the basic inequality

$$E[Z_\infty; Z_\infty > \lambda] \leqq 2\lambda P(Z_\infty > a\lambda)$$

is valid for every $\lambda > 0$. Indeed, let $T = \inf(t; Z_t > \lambda)$, which is a stopping time with $Z_T \leqq \lambda$ a.s.. In addition, $Z_T = \lambda$ on $(T < \infty)$ because Z is continuous. Let us consider the martingale X defined by $X_t = P(Z_\infty \leqq aZ_T | F_t)$. As $X_T = Z_T \hat{E}[X_\infty / Z_\infty | F_T]$, we apply Hölder's inequality with exponents p and $q = p/(p-1)$ to the right hand side:

$$\begin{aligned} X_T^p &\leqq Z_T^p \hat{E}[Z_\infty^{-q} | F_T]^{p-1} \hat{E}[X_\infty^p | F_T] \\ &= Z_T E[Z_\infty^{-1/(p-1)} | F_T]^{p-1} \hat{E}[X_\infty^p | F_T] \leqq K E[Z_\infty X_\infty^p | F_T]/Z_T . \end{aligned}$$

But $Z_\infty X_\infty^p \leqq aZ_T$ by the definition of X. Thus $X_T \leqq (aK)^{1/p} = 1/2$ and so $P(Z_\infty > a\lambda) \geqq P(T < \infty)/2$ because $1/2 \leqq 1 - X_T = P(Z_\infty > aZ_T | F_T)$ and $(T < \infty) \in F_T$. Consequently we get

$$\begin{aligned} E[Z_\infty; Z_\infty > \lambda] &\leqq E[Z_\infty; T < \infty] = E[Z_T; T < \infty] = \lambda P(T < \infty) \\ &\leqq 2\lambda P(Z_\infty > a\lambda) . \end{aligned}$$

Now let $U_n = \mathrm{Min}(Z_\infty, n)$ for $n \geqq 1$. It is clear that $U_n \to Z_\infty$ as $n \to \infty$. It is also immediate to see that for each n the inequality

$$E[U_n; U_n > \lambda] \leqq 2\lambda P(U_n > a\lambda)$$

is valid. Then, multiplying both sides of this inequality by $\varepsilon\lambda^{\varepsilon-1}$ and integrating on the interval $[1, \infty[$, we find that

$$\int_{\{U_n > 1\}} (U_n^{1+\varepsilon} - U_n) dP \leqq b_\varepsilon \int_{\{U_n > a\}} U_n^{1+\varepsilon} dP \leqq b_\varepsilon \int_{\{U_n > 1\}} U_n^{1+\varepsilon} dP + b_\varepsilon .$$

As $E[U_n] \leqq E[Z_\infty] \leqq 1$ and $E[U_n^{1+\varepsilon}] < \infty$, we have

$$(1 - b_\varepsilon) \int_{\{U_n > 1\}} U_n^{1+\varepsilon} dP \leqq b_\varepsilon + 1 \leqq 2 .$$

That is, $E[U_n^{1+\varepsilon}] \leqq 1 + 2/(1 - b_\varepsilon) = C_\varepsilon$. From Fatou's lemma it follows that $E[Z_\infty^{1+\varepsilon}] \leqq C_\varepsilon$.

Secondly, let S be a stopping time, and let A be an arbitrary element of F_S such that $P(A) > 0$. As in the proof of Lemma 4, we set $dP' =$

$I_A dP/P(A)$ and $F'_t = F_{S+t}$. $E'[\ \ ]$ denotes the expectation over Ω with respect to dP'. Consider now the process Z' defined by $Z'_t = Z_{S+t}/Z_S$. Clearly $0 < Z'_t$ and $E'[Z'_\infty]=1$. Furthermore, it is a uniformly integrable martingale over (F'_t) relative to dP' such that for the same constant K as before

$$Z'_t E'[(Z'_\infty)^{-1/(p-1)} \,|\, F'_t]^{p-1} \leqq K\,, \qquad P'\text{-a.s.}\,.$$

Therefore, by the same argument as above we obtain $E'[(Z'_\infty)^{1+\varepsilon}] \leqq C_\varepsilon$, that is, $E[(Z_\infty/Z_S)^{1+\varepsilon}; A] \leqq C_\varepsilon P(A)$. This is valid for any $A \in F_S$, so that we have the desired inequality. Hence the theorem is established.

In the proof of Proposition 3, we shall show that, if Z is a uniformly integrable martingale satisfying the reverse Hölder inequality, then M is a BMO-martingale.

COROLLARY. *Let a be a real number. If M is a* BMO-*martingale, then $Z^{(a)}$ is an L^p-bounded martingale for some $p > 1$.*

PROOF. If M is a BMO-martingale, so is aM. Then the conclusion follows immediately from Theorem 1.

Let $M \in \mathscr{L}$. Obviously, if it is bounded from above, then the process $\exp(M_t/2)$ is a submartingale. But there exists a continuous martingale M, bounded from above, which is not a BMO-martingale. See Example 2. We now remark that, even if M is a BMO-martingale, $\exp(M_t/2)$ is not necessarily a submartingale. We end this section with such examples.

EXAMPLE 5. Let S, $B = (B_t, F_t)$, (Ω, F, P) be as in Example 3, and let (τ_t) be a continuous change of time such that $\tau_0 = 0$ and $\tau_1 = S$. Then $2\sqrt{2}B_{S\wedge t}$ is a BMO-martingale over (F_t), and so $M_t = 2\sqrt{2}B_{S\wedge\tau_t}$ is a BMO-martingale over (F_{τ_t}). But it follows from Fubini's theorem that $\exp(M_1/2) = \exp(\sqrt{2}B_S)$ is not integrable. Namely, $\exp(M_t/2)$ is not a submartingale.

EXAMPLE 6. Let $B = (B_t, F_t)$ be a complex Brownian motion starting at 0 and let $T = \inf(t; |B_t| = 1)$. Then $\log(1 - B^T)$ is a conformal martingale on $[0, T[$, because $\log(1 - z)$ is analytic in the unit disc $|z| < 1$. Its imaginary part is bounded, so that by the main theorem of R. K. Getoor and M. J. Sharpe [4] the real part $\log|1-B^T|$ is a BMO-martingale. Now let $X = -\log|1 - B^T|$. As is well-known, B_T is uniformly distributed on the unit circle $|z| = 1$. Therefore we get

$$\begin{aligned} E[\exp(X_\infty/2)] &= E[\exp(-\log|1 - B_T|)] \\ &= (2\pi)^{-1}\int_0^{2\pi}\{2(1 - \cos\theta)\}^{-1/2} d\theta = \infty\,. \end{aligned}$$

Let us define a change of time (τ_t) with $\tau_0 = 0$ and $\tau_1 = T$ as in Example 2. Then $M_t = X_{\tau_t}$ is a desired BMO-martingale.

4. Transformation of the spaces BMO and H^1 by a change of law. Let $M \in \mathscr{L}$ and consider the process $Z_t = \exp(M_t - \langle M \rangle_t/2)$ as usual. In this section, Z is assumed to be a uniformly integrable martingale with $Z_\infty > 0$. $d\hat{P}$ denotes always the weighted probability measure $Z_\infty dP$. It is obvious that the measures dP and $d\hat{P}$ are mutually absolutely continuous. We shall consider the process W defined by $W_t = 1/Z_t$. It is a uniformly integrable martingale with respect to $d\hat{P}$, for $\hat{E}[W_\infty | F_t] = E[Z_\infty W_\infty | F_t]/Z_t = W_t$. Clearly, $0 < W_t$, $W_0 = 1$ and $W_\infty d\hat{P} = dP$. If $\mathscr{H}$ is a subclass of $\mathscr{L}$, $\hat{\mathscr{H}}$ denotes the class of continuous local martingales relative to $d\hat{P}$, which corresponds to $\mathscr{H}$. So $\hat{\mathscr{L}}$ is the class of all $\hat{P}$-continuous local martingales X' over (F_t) with $X'_0 = 0$. Our interest here lies in investigating the relations between $\mathscr{H}$ and $\hat{\mathscr{H}}$. The following lemma plays a very important role in our discussion.

LEMMA 7. *For any $X \in \mathscr{L}$, $\hat{X} = X - \langle X, M \rangle$ belongs to $\hat{\mathscr{L}}$ and $\langle \hat{X} \rangle = \langle X \rangle$ under either probability measure. Furthermore, the mapping $i: X \to \hat{X}$ is linear and bijective.*

PROOF. To see $\hat{X} \in \hat{\mathscr{L}}$, it is enough to check that $Z\hat{X} \in \mathscr{L}$. $\hat{X}$ is a semi-martingale with respect to dP, and $\langle X, M \rangle_t = \int_0^t Z_s^{-1} d\langle X, Z \rangle_s$ because $M_t = \int_0^t Z_s^{-1} dZ_s$. Then, by Ito's formula we have

$$
\begin{aligned}
Z_t\hat{X}_t &= Z_0\hat{X}_0 + \int_0^t \hat{X}_s dZ_s + \int_0^t Z_s d\hat{X}_s + \langle Z, X \rangle_t \\
&= \int_0^t \hat{X}_s dZ_s + \int_0^t Z_s dX_s ,
\end{aligned}
$$

which belongs to $\mathscr{L}$. Similarly, we can check the equality $\langle \hat{X} \rangle = \langle X \rangle$. From these facts follows the linearity and the injectivity of the mapping i. So it remains to show the surjectivity. As $\hat{M} = M - \langle M \rangle$ and $\langle \hat{M} \rangle = \langle M \rangle$, we have

$$
W_t = \exp(-\hat{M}_t - \langle \hat{M} \rangle_t/2) ,
$$

so that for any $X' \in \hat{\mathscr{L}}$, $X = X' + \langle X, \hat{M} \rangle$ belongs to $\mathscr{L}$. On the other hand, $\hat{X} = X - \langle X, M \rangle$ is in $\hat{\mathscr{L}}$. Therefore $X' - \hat{X} = \langle X, M \rangle - \langle X', \hat{M} \rangle$ is also a $\hat{P}$-continuous local martingale with finite variation on each finite interval. This implies that $X' = \hat{X}$. Thus the lemma is proved.

J. H. Van Schuppen and E. Wong [14] tried to extend this transformation to right continuous local martingales, and the generalization was completely established by E. Lenglart [9]. Note that "the stochastic integral $H\circ\hat{X}$ relative to $d\hat{P}$" coincides with "the stochastic integral of H with respect to the semi-martingale $\hat{X}$ relative to dP".

PROPOSITION 1. *If $Z^*\in L^1$, then for any $X\in\mathscr{L}$*

$$\|\hat{X}\|_{\hat{H}^2}\leqq(2E[Z^*])^{1/2}\|X\|_{\mathrm{BMO}}\ .$$

PROOF. Let $X\in\mathrm{BMO}$ and choose a non-decreasing sequence (T_n) of stopping times with $\lim_n T_n=\infty$ such that $\hat{X}^{T_n}\in\hat{H}^2$ for every $n\geqq 1$. Then for each n we have

$$\hat{E}[\langle\hat{X}\rangle_{T_n}]=E[Z_{T_n}\langle X\rangle_{T_n}]=E\left[\int_0^{T_n}Z_s d\langle X\rangle_s\right]=E[\langle Z\circ X,X\rangle_{T_n}]$$
$$\leqq\sqrt{2}E\left[\left(\int_0^{T_n}Z_s^2 d\langle X\rangle_s\right)^{1/2}\right]\|X\|_{\mathrm{BMO}}\ ,$$

which follows from Lemma 1. The expectation on the right hand side is smaller than

$$E\left[(Z^*)^{1/2}\left(\int_0^{T_n}Z_s d\langle X\rangle_s\right)^{1/2}\right]\leqq E[Z^*]^{1/2}E\left[\int_0^{T_n}Z_s d\langle X\rangle_s\right]^{1/2}$$
$$=E[Z^*]^{1/2}\hat{E}[\langle\hat{X}\rangle_{T_n}]^{1/2}\ .$$

Therefore, as $\hat{E}[\langle\hat{X}\rangle_{T_n}]<\infty$, we have $\hat{E}[\langle\hat{X}\rangle_{T_n}]^{1/2}\leqq\sqrt{2}E[Z^*]^{1/2}\|X\|_{\mathrm{BMO}}$, for $n\geqq 1$. Letting $n\to\infty$ and using Fatou's lemma, we are done.

Proposition 1 shows that if $Z^*\in L^1$, then the mapping $i\colon\mathrm{BMO}\to\hat{H}^2$ is continuous.

PROPOSITION 2. *$Z^*\in L^1$ if and only if $\hat{M}\in\hat{H}^2$.*

PROOF. We define $\log^+ x$, as usual, as 0 if $x<1$ and $\log x$ if $x\geqq 1$. We begin with the proof of the "if" part. From the definition of $d\hat{P}$ it follows that

$$E[Z_\infty\log^+ Z_\infty]=\hat{E}[\log^+ Z_\infty]=\hat{E}[M_\infty-\langle M\rangle_\infty/2;\ Z_\infty\geqq 1]\ .$$

By Lemma 7 the right hand side is

$$\hat{E}[\hat{M}_\infty+\langle\hat{M}\rangle_\infty/2;\ Z_\infty\geqq 1]\leqq\hat{E}[\langle\hat{M}\rangle_\infty]^{1/2}+\hat{E}[\langle\hat{M}\rangle_\infty]/2\ .$$

Therefore, if $\hat{M}\in\hat{H}^2$, we have $Z^*\in L^1$ by the classical inequality of Doob.

To see the "only if" part, we need the inequality:

$$E[Z_\infty\log Z_\infty]\leqq 4\sqrt{2\pi}(E[Z^*]+1)\ ,$$

which follows from a result given by S. Watanabe [15]. Following his idea, we show this inequality. Firstly, let us choose Y in $\mathscr{L}$ in such a way that $U_t=Z_t+iY_t$ is a conformal martingale; that is, $\langle Z\rangle=\langle Y\rangle$

and $\langle Z, Y\rangle = 0$. Then $V_t = U_t \log U_t$ is also a conformal martingale, for $f(z) = z \log z$ is analytic in $D = \{z; \operatorname{Re} z > 0\}$. Therefore, $\operatorname{Re} V_t = Z_t \log|U_t| - Y_t \arg U_t$ is a continuous local martingale. By using the stopping argument we may assume that $Z_t \log|U_t|$ and Y_t are in H^2. Then $E[Z_\infty \log|U_\infty|] = E[Y_\infty \arg U_\infty]$. In addition, $U_\infty \in D$, hence $|\arg U_\infty| \leqq \pi/2$. We now apply Davis' inequality:

$$\begin{aligned} E[Z_\infty \log Z_\infty] &\leqq E[Z_\infty \log|U_\infty|] \leqq (\pi/2)E[|Y_\infty|] \\ &\leqq 2\sqrt{2\pi}E[\langle Z\rangle_\infty^{1/2}] \leqq 4\sqrt{2\pi}E[(Z-1)^*] \leqq 4\sqrt{2\pi}(E[Z^*]+1)\,. \end{aligned}$$

Therefore, if $Z^* \in L^1$, then $E[Z_\infty \log Z_\infty] < \infty$.

Now we are going to show that $\hat{M} \in \hat{H}^2$. The stopping argument enables us to assume that $\hat{M}$ is $\hat{P}$-uniformly integrable. Then, as $\hat{E}[\hat{M}_\infty] = 0$, we have

$$\hat{E}[\langle\hat{M}\rangle_\infty] = 2\hat{E}[\hat{M}_\infty + \langle\hat{M}\rangle_\infty/2] = 2E[Z_\infty(M_\infty - \langle M\rangle_\infty/2)] = 2E[Z_\infty \log Z_\infty]\,,$$

and we are done.

Now let $\mathscr{N} = \bigcap_{p>0} H^p$. As is well-known, if $1 < p < \infty$, H^p coincides with the class of all L^p-bounded continuous martingales.

PROPOSITION 3. *Assume that* $M \in \mathrm{BMO}$. *Then* $X \in \mathscr{N}$ *if and only if* $\hat{X} \in \hat{\mathscr{N}}$.

PROOF. By the corollary to Theorem 1, Z is an L^{p_0}-bounded martingale for some $p_0 > 1$. It follows from Hölder's inequality that for each X

$$\hat{E}[\langle\hat{X}\rangle_\infty^p] = E[Z_\infty\langle X\rangle_\infty^p] \leqq \|Z_\infty\|_{p_0}\|\langle X\rangle^p\|_{q_0}\,,$$

where $1/p_0 + 1/q_0 = 1$. This implies that if $X \in \mathscr{N}$, then $\hat{X} \in \hat{\mathscr{N}}$.

To see the converse, it is enough to show that $\hat{M} \in \mathrm{BMO}$. As $M \in \mathrm{BMO}$, according to Theorem 1, it satisfies the reverse Hölder inequality, that is, $E[Z_\infty^{1+\varepsilon}|F_t] \leqq C_\varepsilon Z_t^{1+\varepsilon}$ for some $\varepsilon > 0$. This can be rewritten as follows:

$$\hat{E}[(W_t/W_\infty)^\varepsilon|F_t] \leqq C_\varepsilon\,.$$

Namely, W satisfies the (A_p) condition relative to $d\hat{P}$ for each $p > 1$ with $1/(p-1) < \varepsilon$. Consequently, using again Lemma 5, we obtain the fact that $\hat{M} \in \mathrm{BMO}^\wedge$. This completes the proof.

It should be noted that Proposition 3 does not hold without the condition "$M \in \mathrm{BMO}$". In the following we give such an example.

EXAMPLE 7. Consider a one dimensional Brownian motion $B = (B_t, F_t)$ starting at 0 and defined on a probability space $(\Omega, F, d\mu)$. Let $T = \inf(t; B_t \geqq 1)$. Then the process B^T stopped at T is a continuous martingale, which is not uniformly integrable with respect to $d\mu$. Clearly, the process Y given by $Y_t = \exp(B_{t\wedge T} - (t \wedge T)/2)$ is a bounded martingale. So $dP = Y_\infty d\mu$ is a probability measure on Ω. Now let $M = -B^T + \langle B^T \rangle$ and $Z_t = \exp(M_t - \langle M \rangle_t/2)$. The process Z is a P-uniformly integrable martingale with $Z_t = 1/Y_t$, and the weighted probability measure $d\hat{P} = Z_\infty dP$ equals $d\mu$. By Lemma 7, M is a P-local martingale with $\langle M \rangle = \langle B^T \rangle$. Let us consider the P-local martingale $X = M/\sqrt{2}$. Then from the fact $B_T = 1$ follows

$$E[\exp(\langle X \rangle_\infty)] = \int_\Omega \exp(\langle M \rangle_\infty/2) \exp(B_T - \langle B \rangle_T/2) d\mu$$
$$= \int_\Omega \exp(B_T) d\mu = e \ .$$

That is, $X \in \mathscr{N}$. However, $\hat{X} = \hat{M}/\sqrt{2} = -B^T/\sqrt{2}$ is not uniformly integrable with respect to $d\mu$. It follows from Proposition 3 that M is not a BMO-martingale.

PROPOSITION 4. *ϕ: $X \to Z^{-1/2} \circ \hat{X}$ is an isometric isomorphism of H^2 onto $\hat{H}^2$.*

PROOF. Let $X \in H^2$. Lemma 7 says that $\hat{X}$ is in $\hat{\mathscr{L}}$. Let $T_n \uparrow \infty$ be stopping times such that $\hat{X}^{T_n} \in \hat{H}^2$ for every n. Since $W_t = 1/Z_t$ is a uniformly integrable martingale with respect to $d\hat{P}$, we have

$$\hat{E}[\langle Z^{-1/2} \circ \hat{X} \rangle_{T_n}] = \hat{E}\left[\int_0^{T_n} W_s d\langle \hat{X} \rangle_s\right] = \hat{E}[W_{T_n} \langle \hat{X} \rangle_{T_n}] = E[\langle X \rangle_{T_n}] \ , \quad \text{for} \quad n \geqq 1 \ .$$

Letting $n \to \infty$ and using the monotone convergence theorem, we obtain $\hat{E}[\langle Z^{-1/2} \circ \hat{X} \rangle_\infty] = E[\langle X \rangle_\infty] < \infty$, so that $Z^{-1/2} \circ \hat{X} \in \hat{H}^2$. This implies that the mapping ϕ: $H^2 \to \hat{H}^2$ given by $\phi(X) = Z^{-1/2} \circ \hat{X}$ is well-defined. Clearly it is linear and injective. From the above calculation it follows that $\|\phi(X)\|_{\hat{H}^2} = \|X\|_{H^2}$. Thus, it remains to prove the surgectivity. To see this, let $X' \in \hat{H}^2$. By Lemma 7, $\hat{U} = X'$ and $\langle U \rangle = \langle X' \rangle$ for some $U \in \mathscr{L}$. We now set $X = Z^{1/2} \circ U$ and choose stopping times $T_n \uparrow \infty$ such that $U^{T_n} \in H^2$ for every n. Then we have

$$E[\langle X \rangle_{T_n}] = E\left[\int_0^{T_n} Z_s d\langle U \rangle_s\right] = E[Z_{T_n} \langle U \rangle_{T_n}]$$
$$= \hat{E}[\langle X' \rangle_{T_n}] \leqq \hat{E}[\langle X' \rangle_\infty] \ .$$

From Fatou's lemma it follows that $X \in H^2$. Moreover, we have

$$\phi(X) = Z^{-1/2} \circ (Z^{1/2} \circ \hat{U}) = \hat{U} = X' \ .$$

Consequently, the mapping ϕ is surjective.

Let $1 \leqq p < \infty$. In particular, if Z_∞ is bounded, then $i: X \to \hat{X}$ is a continuous linear mapping of H^p into $\hat{H}^p$. Therefore, it is evident that if $0 < c \leqq Z_\infty \leqq C$, then the mapping i is an isomorphism of H^p onto $\hat{H}^p$.

THEOREM 2. *If* $M \in \mathrm{BMO}$, *then* $i: X \to \hat{X}$ *is an isomorphism of* BMO *onto* BMO^.

PROOF. Let $M \in \mathrm{BMO}$. By Lemma 5, Z satisfies (A_p) for some $p > 1$. We now need the following inequality due to Kazamaki [8]:

$$||X||_{\mathrm{BMO}} \leqq C_p ||\hat{X}||_{\mathrm{BMO}^\wedge}\,, \qquad \text{for } X \in \mathscr{L}.$$

To show this, let us assume that $0 < ||\hat{X}||_{\mathrm{BMO}^\wedge} < \infty$, and set $a = (2p||\hat{X}||^2_{\mathrm{BMO}^\wedge})^{-1}$. As $||\sqrt{ap}\,\hat{X}||^2_{\mathrm{BMO}^\wedge} = 1/2$, Lemma 4 yields

$$\hat{E}[\exp(ap(\langle\hat{X}\rangle_\infty - \langle\hat{X}\rangle_t))\,|\,F_t] \leqq 2\,.$$

By using a simple inequality $x \leqq e^{ax}/a$ and Hölder's inequality, we have

$$\begin{aligned} E[\langle X\rangle_\infty - \langle X\rangle_t\,|\,F_t] &\leqq E[(Z_t/Z_\infty)^{1/p}(Z_\infty/Z_t)^{1/p}\exp(a(\langle X\rangle_\infty - \langle X\rangle_t))\,|\,F_t]/a \\ &\leqq E[(Z_t/Z_\infty)^{1/(p-1)}\,|\,F_t]^{1/q} \\ &\quad\times E[(Z_\infty/Z_t)\exp(ap(\langle X\rangle_\infty - \langle X\rangle_t))\,|\,F_t]^{1/p}/a\,, \end{aligned}$$

with $1/p + 1/q = 1$. Clearly, $1/a = 2p||\hat{X}||^2_{\mathrm{BMO}^\wedge}$. Since Z satisfies (A_p), the first expectation on the right hand side is smaller than some constant K_p. The second one can be written as $\hat{E}[\exp(ap(\langle\hat{X}\rangle_\infty - \langle\hat{X}\rangle_t))\,|\,F_t]$, which is bounded by 2. Thus, $||X||^2_{\mathrm{BMO}} \leqq C_p||\hat{X}||^2_{\mathrm{BMO}^\wedge}$.

As mentioned in the proof of Proposition 3, if $M \in \mathrm{BMO}$, then $\hat{M} \in$ BMO^. Therefore we get $c||X||_{\mathrm{BMO}} \leqq ||\hat{X}||_{\mathrm{BMO}^\wedge} \leqq C||X||_{\mathrm{BMO}}$ for $X \in \mathscr{L}$. Here, the positive constants c and C do not depend on X. Then, combining this inequality with Lemma 7, we see that the spaces BMO and BMO^ are isomorphic via the mapping i.

We remark that, without the condition "$M \in \mathrm{BMO}$", the conclusion of Theorem 2 no longer follows. In the next theorem, let $1 \leqq p \leqq \infty$ and $H^\infty = \mathrm{BMO}$. We denote by q the exponent conjugate to p; namely, $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$.

THEOREM 3. $j: \hat{X} \to X$ *is a continuous mapping of* $\hat{H}^p$ *into* H^p *if and only if* $\psi: X \to Z^{-1} \circ \hat{X}$ *is a continuous mapping of* H^q *into* $\hat{H}^q$.

PROOF. We deal only with the case $p = \infty$; the proof for the other cases is similar. Firstly, let us assume that the mapping j is continuous, that is, $||Y||_{\mathrm{BMO}} \leqq ||j||\,||\hat{Y}||_{\mathrm{BMO}^\wedge}$ for every $\hat{Y} \in$ BMO^. Let $X \in H^1$ and $\hat{Y} \in$ BMO^. Since $W_t = 1/Z_t$ is a uniformly integrable martingale with respect to $d\hat{P}$, we have

$$\hat{E}[\langle Z^{-1}\circ\hat{X},\ \hat{Y}\rangle_\infty] = \hat{E}\left[\int_0^\infty W_s d\langle\hat{X},\ \hat{Y}\rangle_s\right] = \hat{E}[W_\infty\langle X,\ Y\rangle_\infty] = E[\langle X,\ Y\rangle_\infty]\ .$$

By Lemma 1 this is smaller than $\sqrt{2}||X||_{H^1}||Y||_{\mathrm{BMO}}$. Therefore, from Lemma 2 follows the inequality

$$||Z^{-1}\circ\hat{X}||_{\hat{H}^1} \leqq \sqrt{2}||j||\ ||X||_{H^1}$$

for every $X\in H^1$.

Conversely, suppose that ψ: $X\to Z^{-1}\circ\hat{X}$ is a continuous mapping of H^1 into $\hat{H}^1$. Let $X\in H^1$ and $\hat{Y}\in \mathrm{BMO}^\wedge$. By using the stopping argument we may assume that $Y\in\mathrm{BMO}$. Then, by the same calculation as above, we have

$$\begin{aligned} E[\langle X,\ Y\rangle_\infty] &= \hat{E}[\langle Z^{-1}\circ\hat{X},\ \hat{Y}\rangle_\infty] \leqq \sqrt{2}||Z^{-1}\circ\hat{X}||_{\hat{H}^1}||\hat{Y}||_{\mathrm{BMO}^\wedge} \\ &\leqq \sqrt{2}||\psi||\ ||X||_{H^1}||\hat{Y}||_{\mathrm{BMO}^\wedge}\ . \end{aligned}$$

In addition, by Lemma 3,

$$\begin{aligned} ||Y||_{\mathrm{BMO}} &\leqq \sup\{E[\langle Y,\ X\rangle_\infty];\ X\in H^1,\ ||X||_{H^1}\leqq 1\} \\ &\leqq \sqrt{2}||\psi||\ ||\hat{Y}||_{\mathrm{BMO}^\wedge}\ . \end{aligned}$$

Thus our claim is established.

The mapping j defined above is nothing else but the inverse of the mapping i. Combining Theorems 2 and 3, we get:

COROLLARY. *If $M\in\mathrm{BMO}$, then the spaces H^1 and $\hat{H}^1$ are isomorphic via the mapping ψ.*

We remark that it is impossible to remove the condition "$M\in\mathrm{BMO}$". In other words, $Z^{-1}\circ\hat{X}\notin\hat{H}^1$ for some $X\in H^1$. Here is an example.

EXAMPLE 8. Let $S, B=(B_t, F_t)$ and (Ω, F, P) be as in Example 3, except that we use here the distribution $d\mu = I_{[1,\infty]}(u)u^{-2}du$ of S instead. Let $M = B^S$. Then it is immediate to see that $Z_t = \exp(M_t - \langle M\rangle_t/2)$ is a uniformly integrable martingale. As $E[\langle M\rangle_\infty^{1/2}] = \int_1^\infty u^{-3/2}du = 2$, we have $M\in H^1$. But it does not belong to H^2, for $E[\langle M\rangle_\infty] = \int_1^\infty u^{-1}du = \infty$. By Proposition 3, $M\notin H^2$ if and only if W^* is not integrable with respect to $d\hat{P}$. In addition, $W = 1 - W\circ\hat{M}$, and so $W\circ\hat{M} = Z^{-1}\circ\hat{M}\notin H^1$.

Finally, we point out the fact that i: $X\to\hat{X}$ is not always a continuous mapping of H^2 onto $\hat{H}^2$, even if M is a BMO-martingale. Indeed, if the mapping i were continuous, then by Theorem 3 $\hat{H}^2\ni\hat{X}\to Z\circ X\in H^2$ must be continuous. This would imply that if $X\in H^2$, then $Z\circ X\in H^2$. However, for the BMO-martingale $M = B^S$ considered in Example 3, $Z\circ M\notin H^2$.

5. A generalization of Doob's inequalities. In this section, let us assume that $M \in \mathrm{BMO}$. Then by Theorem 1 the process Z satisfies the reverse Hölder inequality: $E[Z_\infty^{1+\varepsilon} | F_t] \leqq C_\varepsilon Z_t^{1+\varepsilon}$ for some $\varepsilon > 0$. By combining this result with Lemma 7, we can give a generalization of the classical inequalities due to J. L. Doob. The inequality (1) given in the following theorem was essentially proved by M. Izumisawa and N. Kazamaki [5].

THEOREM 4. (1) *Let $p > 1 + 1/\varepsilon$. Then the inequality*

$$E\left[\sup_t |X_t - \langle X, M\rangle_t|^p\right] \leqq C_{p,\varepsilon} \sup_t E[|X_t - \langle X, M\rangle_t|^p]$$

is valid for all $X \in \mathscr{L}$.

(2) *In particular, if $Z_\infty/Z_t \leqq C$, then there exists a constant $c > 0$ such that the inequality*

$$cE\left[\sup_t |X_t - \langle X, M\rangle_t|\right]$$
$$\leqq e/(e-1) + (e/(e-1)) \sup_t E[|X_t - \langle X, M\rangle_t| \log^+ |X_t - \langle X, M\rangle_t|]$$

is valid for all $X \in \mathscr{L}$.

PROOF. We begin with the proof of (1). Let $X \in \mathscr{L}$ and $0 < \delta < p - (1 + 1/\varepsilon)$. Then $1 < p_0 = (p-\delta)/(p-\delta-1) < 1+\varepsilon$ and $q_0 = p_0/(p_0 - 1) = p - \delta > 1$. It follows from the assumption that $E[Z_\infty^{p_0} | F_t] \leqq C_{p,\varepsilon} Z_t^{p_0}$. Lemma 7 says that $\hat{X} = X - \langle X, M\rangle \in \hat{\mathscr{L}}$. By using the stopping argument we may assume that $\hat{X} \in \hat{H}^p$. Then $\hat{X}_t = \hat{E}[\hat{X}_\infty | F_t] = E[Z_\infty \hat{X}_\infty / Z_t | F_t]$, and so by Hölder's inequality with exponents p_0 and q_0 we obtain:

$$|\hat{X}_t|^{p-\delta} \leqq E[(Z_\infty/Z_t)^{p_0} | F_t]^{p-\delta-1} E[|\hat{X}_\infty|^{p-\delta} | F_t]$$
$$\leqq C_{p,\varepsilon} E[|\hat{X}_\infty|^{p-\delta} | F_t] .$$

We now apply the classical theorem of Doob to the martingale $E[|\hat{X}_\infty|^{p-\delta} | F_t]$ to obtain

$$E\left[\sup_t |\hat{X}_t|^p\right] \leqq C_{p,\varepsilon} E\left[\sup_t E[|\hat{X}_\infty|^{p-\delta} | F_t]^{p/(p-\delta)}\right]$$
$$\leqq C_{p,\varepsilon} E[|\hat{X}_\infty|^p] .$$

Finally, we show (2). For simplicity, we may assume that $\hat{X}$ is a uniformly integrable martingale relative to $d\hat{P}$. Then from the assumption it follows that $|\hat{X}_t| = |\hat{E}[\hat{X}_\infty | F_t]| \leqq E[Z_\infty |\hat{X}_\infty| / Z_t | F_t] \leqq CE[|\hat{X}_\infty| \,|F_t]$, and so by applying the theorem of Doob to the martingale $E[|\hat{X}_\infty| \,| F_t]$, we obtain (2).

If $Z_\infty/Z_t \leqq C$, then the inequality (1) is valid for any $p > 1$ and M

belongs to the class BMO. The classical inequalities of Doob correspond to the case $M=0$.

REFERENCES

[1] C. DOLÉANS-DADE AND P. A. MEYER, Une caractérisation de BMO, Séminaire de Probabilités XI, Université de Strasbourg, Lecture Notes in Math., 581 (1977), Springer-Verlag, 383–389.

[2] C. DOLÉANS-DADE AND P. A. MEYER, Inégalités de normes avec poids, Séminaire de Probabilités XIII, Université de Strasbourg, Lecture Notes in Math., Berlin-Heidelberg-New York, Springer-Verlag, (to appear).

[3] A. M. GARSIA, Martingale inequalities, Seminar Notes on Recent Progress, Benjamin, (1973).

[4] R. K. GETOOR AND M. J. SHARPE, Conformal martingales, Inventiones Math., 16 (1972), 271–308.

[5] M. IZUMISAWA AND N. KAZAMAKI, Weighted norm inequalities for martingales, Tôhoku Math. J., 29 (1977), 115–124.

[6] N. KAZAMAKI, A characterization of BMO-martingales, Séminaire de Probabilités X, Université de Strasbourg, Lecture Notes in Math., 511 (1976), Springer-Verlag, 536–538.

[7] N. KAZAMAKI, On a problem of Girsanov, Tôhoku Math. J., 29 (1977), 597–600.

[8] N. KAZAMAKI, On transforming the class of BMO-martingales by a change of law, Tôhoku Math. J., 31 (1979), 117–125.

[9] E. LENGLART, Transformation des martingales locales par changement absolument continu de probabilités, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 37 (1977), 65–70.

[10] P. A. MEYER, Probabilités et Potentiel, (1966), Hermann, Paris.

[11] P. A. MEYER, Complément sur la dualité entre H^1 et BMO, Séminaire de Probabilités IX, Université de Strasbourg, Lecture Notes in Math., 465 (1975), Springer-Verlag, 237–238.

[12] B. MUCKENHOUPT, Weighted norm inequalities for the Hardy maximal functions, Trans. Amer. Math. Soc., 165 (1972), 207–226.

[13] A. A. NOVIKOV, On an identity for stochastic integrals, Theory Prob. Appli., 17 (1972), 717–720.

[14] J. H. VAN SCHUPPEN AND E. WONG, Transformation of local martingales under a change of law, Ann. of Prob., 2 (1974), 879–888.

[15] S. WATANABE, A remark on the integrabilty of $\sup_t X_t$ for martingales, Proc. of the second Japan-USSR symposium on Probability Theory, no. 2 (1972), 147–155.

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