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UMBILICS OF CONFORMALLY FLAT SUBMANIFOLDS

YOSHIHISA KITAGAWA

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0. Introduction. For $n \geqq 4$, let M be an n-dimensional conformally flat submanifold of the $(n+p)$-dimensional Euclidean space E^{n+p}. Recently under the assumption that M has the positive sectional curvature and $p \leqq n-3$, Sekizawa [3] proved that M contains an open subset on which there exists an involutive distribution of dimension $\geqq n-p$ such that each leaf of this distribution is totally umbilic in M and in E^{n+p}. In this note we show that the result of Sekizawa remains true without the assumption that the sectional curvature is positive.

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1. Statement of results. For $n \geqq 4$, let M be an n-dimensional conformally flat submanifold of the $(n+p)$-dimensional Euclidean space E^{n+p}. We denote the induced Riemannian metric on M by $\langle\ ,\ \rangle$, the Riemannian connection by ∇, the Ricci tensor by Ric, the scalar curvature by S, and the second fundamental form by α. The symmetric tensor Ψ is defined by

$$\Psi(X, Y) = [\mathrm{Ric}(X, Y) - \langle X, Y\rangle S/2(n-1)]/(n-2)$$

for $X, Y \in T_xM$. We now recall the notion of umbilic subspace of T_xM introduced in [3]. A subspace V of T_xM is said to be umbilic if $\dim V \geqq 2$ and $\alpha(X, X) = \alpha(Y, Y)$ for all unit vectors X and Y in V. Then our first result is the following.

PROPOSITION 1. *For $n \geqq 4$, let M be an n-dimensional conformally flat submanifold of the $(n+p)$-dimensional Euclidean space E^{n+p}. If $p \leqq n-3$ and $\mathscr{U}_x$ is the set of all vectors $X \in T_xM$ such that $\|\alpha(X, X)\|^2 = 2\|X\|^2\Psi(X, X)$, then*

(a) *$\mathscr{U}_x$ is the largest umbilic subspace of T_xM, and $\dim \mathscr{U}_x \geqq n-p$.*

(b) *For each unit vector $X \in \mathscr{U}_x$, the subspace $\{Y \in T_xM: \alpha(Y, Z) = \langle Y, Z\rangle\alpha(X, X)$ for all $Z \in T_xM\}$ is equal to $\mathscr{U}_x$.*

Let $p \leqq n-3$. Then by Proposition 1 we can define a distribution $\mathscr{U}$ by $M \ni x \mapsto \mathscr{U}_x$. We call $\mathscr{U}$ the umbilic distribution. The umbilic distribution $\mathscr{U}$ is not smooth in general. However, we can prove the following.

PROPOSITION 2. *For $n \geqq 4$ and $p \leqq n-3$, let M be an n-dimensional conformally flat submanifold of the $(n+p)$-dimensional Euclidean space E^{n+p}. Then there exists an open subset of M on which the umbilic distribution $\mathscr{U}$ is smooth.*

Finally our main result is the following.

THEOREM. *For $n \geqq 4$ and $p \leqq n-3$, let M be an n-dimensional conformally flat submanifold of the $(n+p)$-dimensional Euclidean space E^{n+p}. If U is an open subset of M on which the umbilic distribution $\mathscr{U}$ is smooth, then $\mathscr{U}|U$ is involutive and each leaf L of $\mathscr{U}|U$ is totally umbilic in M and in E^{n+p}. In particular, L is a Riemannian manifold of constant curvature.*

REMARK 1. Let M^* be the union of all open subsets of M on which the umbilic distribution $\mathscr{U}$ is smooth. Using Proposition 2, we see that M^* is dense in M.

REMARK 2. Moore [2] states the above theorem without proof. Its complete proof seems not to have been published yet.

2. Proof of Proposition 1.

Since M is conformally flat, the Weyl conformal curvature tensor vanishes. Hence by the Gauss equation we have

$$\begin{aligned}(1)\qquad &\langle\alpha(X, Z), \alpha(Y, W)\rangle - \langle X, Z\rangle\Psi(Y, W) - \Psi(X, Z)\langle Y, W\rangle \\ &= \langle\alpha(Y, Z), \alpha(X, W)\rangle - \langle Y, Z\rangle\Psi(X, W) - \Psi(Y, Z)\langle X, W\rangle\end{aligned}$$

for all vectors X, Y, Z and W in T_xM. The formula (1) implies

$$(2)\qquad \langle\alpha(X, X), \alpha(Y, Y)\rangle = \Psi(X, X) + \Psi(Y, Y) + \|\alpha(X, Y)\|^2$$

for all orthonormal vectors X and Y in T_xM.

LEMMA 1. *If X and Y are unit vectors in T_xM such that $\alpha(X, X) = \alpha(Y, Y)$, then $\Psi(X, X) = \Psi(Y, Y)$.*

PROOF. Since $p \leqq n-3$ implies $\dim \operatorname{Ker} \alpha(X, \cdot) \geqq 3$, there exists a unit vector $Z \in \operatorname{Ker} \alpha(X, \cdot)$ orthogonal to X and Y. Using (2), we see that $\Psi(X, X) + \Psi(Z, Z) = \langle\alpha(X, X), \alpha(Z, Z)\rangle = \langle\alpha(Y, Y), \alpha(Z, Z)\rangle = \Psi(Y, Y) + \Psi(Z, Z) + \|\alpha(Y, Z)\|^2$. Hence $\Psi(X, X) \geqq \Psi(Y, Y)$. By the symmetry of X and Y, we get $\Psi(X, X) = \Psi(Y, Y)$. q.e.d.

LEMMA 2. *If V is an umbilic subspace of T_xM, then $V \subset \mathscr{U}_x$.*

PROOF. For each unit vector $X \in V$, there exists a unit vector $Y \in V$ orthogonal to X. Since V is umbilic, we have $\alpha(X, X) = \alpha(Y, Y)$ and $\alpha(X, Y) = 0$. Lemma 1 implies $\Psi(X, X) = \Psi(Y, Y)$. Hence by (2) we see that $\|\alpha(X, X)\|^2 = \langle \alpha(X, X), \alpha(Y, Y) \rangle = \Psi(X, X) + \Psi(Y, Y) = 2\Psi(X, X)$. q.e.d.

Let $T_xM^\perp$ be the normal space to M at x, and define a Lorentzian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $T_xM^\perp \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ by

$$\langle\!\langle (\xi_1, s_1, t_1), (\xi_2, s_2, t_2) \rangle\!\rangle = \langle \xi_1, \xi_2 \rangle + s_1t_2 + t_1s_2 \tag{3}$$

for $(\xi_i, s_i, t_i) \in T_xM^\perp \oplus \boldsymbol{R} \oplus \boldsymbol{R}$. Now we define a symmetric bilinear map $\beta\colon T_xM \times T_xM \to T_xM^\perp \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ by

$$\beta(X, Y) = (\alpha(X, Y), \langle X, Y \rangle, -\Psi(X, Y)) . \tag{4}$$

The formula (1) implies that β is flat with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$ in the sense of [2, p. 91]. Furthermore, $p \leqq n - 3$ implies $\dim T_xM > \dim (T_xM^\perp \oplus \boldsymbol{R} \oplus \boldsymbol{R})$, and (4) implies $\beta(X, X) \neq 0$ for all nonzero $X \in T_xM$. Hence by [2, Proposition 2] there exists a nonzero null vector $e \in T_xM^\perp \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ and a nonzero symmetric bilinear map $f\colon T_xM \times T_xM \to \boldsymbol{R}$ such that $\dim N(\beta - fe) \geqq n - p \geqq 3$, where $N(\beta - fe) = \{X \in T_xM\colon (\beta - fe)(X, Y) = 0$ for all $Y \in T_xM\}$.

Let $e = (\xi, s, t)$. Since e is a null vector, we have $\|\xi\|^2 + 2st = 0$. For all $X \in N(\beta - fe)$ and $Y \in T_xM$, we see that $\alpha(X, Y) = f(X, Y)\xi$, $\langle X, Y \rangle = f(X, Y)s$ and $-\Psi(X, Y) = f(X, Y)t$. Hence we have the following:

$$\alpha(X, Y) = \langle X, Y \rangle \xi/s \tag{5}$$

$$\Psi(X, Y) = -\langle X, Y \rangle t/s \tag{6}$$

$$\|\alpha(X, Y)\|^2 = 2\langle X, Y \rangle \Psi(X, Y) \tag{7}$$

for $X \in N(\beta - fe)$ and $Y \in T_xM$.

LEMMA 3. *$\alpha(X, X) = \xi/s$ for all unit vectors $X \in \mathscr{U}_x$.*

PROOF. For each unit vector $X \in \mathscr{U}_x$, there exists a unit vector $Y \in N(\beta - fe)$ orthogonal to X. By (5) and (7) we have $\alpha(Y, Y) = \xi/s$ and $\|\alpha(Y, Y)\|^2 = 2\Psi(Y, Y)$. Using (2) and $\|\alpha(X, X)\|^2 = 2\Psi(X, X)$, we see that $\|\alpha(X, X) - \xi/s\|^2 = \|\alpha(X, X)\|^2 + \|\alpha(Y, Y)\|^2 - 2\langle \alpha(X, X), \alpha(Y, Y) \rangle = 2\Psi(X, X) + 2\Psi(Y, Y) - 2\langle \alpha(X, X), \alpha(Y, Y) \rangle = -2\|\alpha(X, Y)\|^2$. Hence by (7) we get $\|\alpha(X, X) - \xi/s\|^2 = 0$. q.e.d.

LEMMA 4. *If $X \in \mathscr{U}_x$, then $\alpha(X, Y) = 0$ for all $Y \in T_xM$ orthogonal to X.*

PROOF. We may assume that X is a unit vector in $\mathscr{U}_x$, and Y is a unit vector orthogonal to X. Since $\dim N(\beta - fe) \geqq 3$, there exists a unit vector $Z \in N(\beta - fe)$ orthogonal to X and Y. Then by (5) and Lemma 3 we have $\alpha(X, X) = \xi/s = \alpha(Z, Z)$. Hence by Lemma 1 we have $\Psi(X, X) = \Psi(Z, Z)$. Using (2), we see that $\|\alpha(X, Y)\|^2 = \langle \alpha(X, X), \alpha(Y, Y)\rangle - \Psi(X, X) - \Psi(Y, Y) = \langle \alpha(Z, Z), \alpha(Y, Y)\rangle - \Psi(Z, Z) - \Psi(Y, Y) = \|\alpha(Z, Y)\|^2$. Hence by (5) we get $\|\alpha(X, Y)\|^2 = 0$. q.e.d.

Let N be a subspace of T_xM defined by $N = \{X \in T_xM: \alpha(X, Y) = \langle X, Y\rangle \xi/s$ for all $Y \in T_xM\}$. Since (5) implies $N \supset N(\beta - fe)$, we see that $\dim N \geqq n - p \geqq 3$ and N is an umbilic subspace of T_xM. Thus by Lemma 2 we have $N \subset \mathscr{U}_x$. Lemmas 3 and 4 imply $\mathscr{U}_x \subset N$ and we get $\mathscr{U}_x = N$. Hence Lemma 2 implies (a), and Lemma 3 implies (b). This completes the proof of Proposition 1.

3. Proof of Proposition 2.

For $n \geqq 4$ and $p \leqq n - 3$, let M be an n-dimensional conformally flat submanifold of the $(n + p)$-dimensional Euclidean space E^{n+p}. Then by Proposition 1 we can define a normal vector $\eta(x)$ at $x \in M$ by $\eta(x) = \alpha(X, X)$, where X is a unit vector in $\mathscr{U}_x$. We call η the normal curvature vector field.

LEMMA 5. *There exists an open subset of M on which the normal curvature vector field η is smooth.*

PROOF. Let $TM^{\perp}$ be the normal bundle over M. We consider the Whitney sum $TM^{\perp} \oplus \boldsymbol{R}_M \oplus \boldsymbol{R}_M$, where $\boldsymbol{R}_M$ is the trivial real line bundle over M. For each fiber $T_xM^{\perp} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$, the Lorentzian metric $\langle\!\langle\ ,\ \rangle\!\rangle$ and the symmetric bilinear map $\beta: T_xM \times T_xM \to T_xM^{\perp} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ were defined by (3) and (4). We introduce a function λ on TM by $\lambda(X) = \operatorname{rank} \beta(X, \cdot)$ for $X \in TM$. Let $V_0 \in TM$ be a maximum point of λ and let $x_0 = \pi(V_0)$, $\lambda_0 = \lambda(V_0)$, where π is the canonical projection $\pi: TM \to M$. Choose a smooth tangent vector field V on M such that $V(x_0) = V_0$. Since the function $\lambda(V)$ defined on M is lower semi-continuous, there exists a neighborhood U of x_0 such that $\lambda(V) = \lambda_0$ on U.

For each point x in U, $V(x)$ is a regular element of β in the sense of [2, p. 92]. As in the proof of [2, Proposition 2], we see that the restriction of $\langle\!\langle\ ,\ \rangle\!\rangle$ to $\beta(V(x), T_xM)$ is degenerate. Thus we have $\dim \mathscr{L}_x \geqq 1$, where $\mathscr{L}_x = \{e \in \beta(V(x), T_xM): \langle\!\langle e, \tilde{e}\rangle\!\rangle = 0$ for all $\tilde{e} \in \beta(V(x), T_xM)\}$. Since $\langle\!\langle\ ,\ \rangle\!\rangle$ is Lorentzian, we have $\dim \mathscr{L}_x \leqq 1$. Hence $\dim \mathscr{L}_x = 1$ and we see that $\mathscr{L} = \bigcup_{x \in U} \mathscr{L}_x$ is a smooth subbundle of $TM^{\perp} \oplus \boldsymbol{R}_M \oplus \boldsymbol{R}_M | U$.

It is not difficult to show by linear algebra that there exists an open subset $U_0 \subset U$ on which there exists a local frame $(e_1, \cdots, e_{p+2})$ of $TM^{\perp} \oplus \boldsymbol{R}_M \oplus \boldsymbol{R}_M$ such that

$$e_1 \in \mathscr{L}, \qquad \langle\!\langle e_i, e_j \rangle\!\rangle = \begin{cases} 1 - \delta_{ij} & \text{for} \quad 1 \leqq i,\ j \leqq 2\ . \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

For each point x in U_0, there exist symmetric bilinear functions f^i: $T_xM \times T_xM \to \boldsymbol{R}$ such that $\beta = \sum_{i=1}^{p+2} f^i e_i$. As in the proof of [2, Proposition 2], we have

$$(8) \qquad \dim N_x(\beta - f^1 e_1) \geqq n - p \geqq 3\ ,$$

where $N_x(\beta - f^1e_1) = \{X \in T_xM\colon (\beta - f^1e_1)(X, Y) = 0 \ \text{ for all } \ Y \in T_xM\}$. We write $e_1 = (\xi, s, t)$, where ξ is a smooth normal vector field on U_0 and s and t are smooth functions on U_0. Then we have

$$(9) \qquad \alpha(X, Y) = \langle X, Y\rangle \xi(x)/s(x)$$

for $X \in N_x(\beta - f^1e_1)$ and $Y \in T_xM$. The formulas (8) and (9) imply that $N_x(\beta - f^1e_1)$ is an umbilic subspace of T_xM. Hence by Proposition 1 and (9) we have $\eta(x) = \xi(x)/s(x)$. Thus the normal curvature vector field η is smooth on U_0. q.e.d.

Let $L(TM; TM^\perp)$ be a vector bundle over M with fiber $L(T_xM; T_xM^\perp)$, where $L(T_xM; T_xM^\perp)$ is the space of linear maps $T_xM \to T_xM^\perp$. By Lemma 5 there exists an open subset U of M on which the normal curvature vector field η is smooth. For each point x in U, we define a linear map $\phi_x\colon T_xM \to L(T_xM; T_xM^\perp)$ by $[\phi_x(X)](Y) = \alpha(X, Y) - \langle X, Y\rangle\eta(x)$. Then we obtain a smooth bundle map $\phi\colon TM|U \to L(TM; TM^\perp)|U$. By Proposition 1 we have $\mathscr{U}_x = \operatorname{Ker}\phi_x$. Hence there exists an open subset $U_0 \subset U$ such that $U_0 \ni x \mapsto \mathscr{U}_x$ is smooth. This completes the proof of Proposition 2.

4. Proof of Theorem.

Let $\mathscr{U}$ be the umbilic distribution and let η be the normal curvature vector field. For each point x in M, by Proposition 1 we have

$$(10) \qquad \mathscr{U}_x = \{X \in T_xM\colon \alpha(X, Y) = \langle X, Y\rangle\eta(x) \quad \text{for all } \ Y \in T_xM\}\ .$$

We define a distribution $\mathscr{U}^\perp$ by $\mathscr{U}^\perp\colon M \ni x \mapsto \mathscr{U}_x^\perp$, where $\mathscr{U}_x^\perp$ is the orthogonal complement of $\mathscr{U}_x$ in T_xM. Let U be an open subset of M on which $\mathscr{U}$ is smooth. Then η and $\mathscr{U}^\perp$ are also smooth on U.

Let X and Y be smooth sections in $\mathscr{U}|U$ and let Z be a smooth section in $\mathscr{U}^\perp|U$. Then we have the following:

$$\begin{aligned}(\tilde{\nabla}_X\alpha)(Y, Z) &= \langle \nabla_X Y, Z\rangle\eta - \alpha(\nabla_X Y, Z)\ ,\\ (\tilde{\nabla}_Y\alpha)(X, Z) &= \langle \nabla_Y X, Z\rangle\eta - \alpha(\nabla_Y X, Z)\ ,\\ (\tilde{\nabla}_Z\alpha)(X, Y) &= \langle X, Y\rangle D_Z\eta\ .\end{aligned}$$

We refer the reader to [1, Chapter 7] for the definitions of $\tilde{\nabla}$ and D. Since the Codazzi equation implies $(\tilde{\nabla}_X\alpha)(Y, Z) = (\tilde{\nabla}_Y\alpha)(X, Z) = (\tilde{\nabla}_Z\alpha)(X, Y)$, we have the following:

$$\alpha([X, Y], Z) = \langle [X, Y], Z\rangle \eta \, , \tag{11}$$

$$\langle \nabla_X Y, Z\rangle \eta - \alpha(\nabla_X Y, Z) = \langle X, Y\rangle D_Z \eta \, . \tag{12}$$

By (10) and (11) we see that $[X, Y]$ belongs to $\mathscr{U}|U$. Hence $\mathscr{U}|U$ is involutive.

Let L be a leaf of $\mathscr{U}|U$ and let x be a point in L. We denote by γ the second fundamental form with respect to the immersion $L \subset M$. For all smooth sections X and Y in $\mathscr{U}|U$, we see that $\gamma(X(x), Y(x))$ is the $\mathscr{U}_x^\perp$-component of $(\nabla_X Y)(x)$. Hence by (12) we have

$$\langle \gamma(X_x, Y_x), Z_x\rangle \eta(x) - \alpha(\gamma(X_x, Y_x), Z_x) = \langle X_x, Y_x\rangle D_{Z_x}\eta$$

for $X_x, Y_x \in \mathscr{U}_x$ and $Z_x \in \mathscr{U}_x^\perp$. If X_x and Y_x are unit vectors in $\mathscr{U}_x$, the above formula implies

$$\begin{aligned}\langle \gamma(X_x, X_x), Z_x\rangle \eta(x) &- \alpha(\gamma(X_x, X_x), Z_x) \\ &= \langle \gamma(Y_x, Y_x), Z_x\rangle \eta(x) - \alpha(\gamma(Y_x, Y_x), Z_x)\end{aligned}$$

for $Z_x \in \mathscr{U}_x^\perp$. Hence by (10) we have $\gamma(X_x, X_x) - \gamma(Y_x, Y_x) \in \mathscr{U}_x$. Since $\gamma(X_x, X_x) - \gamma(Y_x, Y_x) \in \mathscr{U}_x^\perp$, we get $\gamma(X_x, X_x) = \gamma(Y_x, Y_x)$. Hence L is totally umbilic in M.

We denote by δ the second fundamental form with respect to the immersion $L \subset E^{n+p}$. Then we have $\delta = \alpha + \gamma$ on $\mathscr{U}_x$. For all unit vectors X_x and Y_x in $\mathscr{U}_x$, we see that $\delta(X_x, X_x) = \alpha(X_x, X_x) + \gamma(X_x, X_x) = \alpha(Y_x, Y_x) + \gamma(Y_x, Y_x) = \delta(Y_x, Y_x)$. Hence L is totally umbilic in E^{n+p}. This completes the proof of Theorem.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980 JAPAN