DYNAMIC BEHAVIOR FROM BIFURCATION EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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Abstract. Necessary and sufficient conditions for existence of small periodic solutions of some evolution equations can be obtained by the Liapunov-Schmidt method. In a neighborhood of zero, this gives a function (the bifurcation function) to each zero of which corresponds a periodic solution of the original equations. If this function is scalar, we show that its sign between the zeros gives the complete description of the stability properties of the periodic solutions.

1. **Introduction.** For the determination of solutions of an equation near a given solution, the method of Liapunov-Schmidt is very effective and has been applied to boundary value problems for ordinary, partial and functional differential equations, the problem of Hopf bifurcation for such equations, as well as many other problems. For several problems, this method reduces the discussion to the zeros of a function, called the *bifurcation function*, from a neighborhood of zero in one finite dimensional space to another finite dimensional space. The zeros of this function correspond to solutions of the original problem near the given solution, and conversely. Thus, the bifurcation function is a precise quantitative measure of the number of solutions of the original problem.

If the original problem corresponds to an evolutionary equation, one also must determine the stability properties of these solutions. It is the purpose of this paper to show that the bifurcation function may also carry the qualitative and quantitative dynamic behavior of the original problem. More precisely, consider the problem of the existence of 2π-periodic solutions of 2π-periodic ordinary, parabolic or retarded functional differential equations for which the linear part of the unperturbed equation has one zero eigenvalue and the remaining ones with

negative real parts. In this case, the bifurcation function is a map G from R to R and we prove that the stability of a periodic solution corresponding to a zero of G is the same as the stability of the zero of G as a solution of $\dot{a} = G(a)$. This result also has immediate application to the problem of Hopf bifurcation since an appropriate change of variables using polar coordinates reduces the discussion to the above situation.

For differential equations containing only one parameter, the classical procedure for determining the existence and stability properties of periodic solutions is the method of averaging which consists in successively transforming the nonautonomous evolutionary equation to one which is almost autonomous. Under generic hypotheses on the autonomous part, one obtains the number and stability of the solutions. If the equation contains several parameters, the method of averaging can be applied by appropriately scaling each parameter in terms of a common scalar parameter. When the generic hypotheses are not satisfied; that is, at points in the original parameter space where bifurcations could possibly occur, special arguments are needed, if they can be done at all, to show that bifurcations actually do occur uniformly with respect to parameters. The bifurcation function truncated to a certain order coincides with the averaged equations on the center manifold truncated to the same order. This shows that the quantitative information on stability is also contained in the bifurcation function. Thus, averaging is not needed for this problem. Since the bifurcation function is obtained simply by equating coefficients in a Fourier series, it is much easier to apply than averaging.

In addition to showing that the bifurcation function gives as much information as averaging for this particular problem, we can show that the structure of the flow of the nonautonomous equation is completely determined by the scalar ordinary differential equation $\dot{x} = G(x)$ where G is the bifurcation function. This qualitative result could never be obtained from averaging.

If the center manifold has dimension 2 and G is the bifurcation function from R^2 to R^2, one cannot hope that the flow for the original system is completely determined by the equation $\dot{x} = G(x)$. However, it is interesting to attempt to discover which properties of the original equation are determined by this autonomous equation. We do not know the answer, but some information has already been obtained by Langford [8].

Let us now describe the problem in some detail. Suppose B is an

$n \times n$ matrix with zero as a simple eigenvalue and all other eigenvalues with negative real parts; suppose $F: R \times R^n \to R^n$ has continuous derivatives up through order $k \geq 2$, $F(t, 0) = 0$, $\partial F(t, 0)/\partial z = 0$, $h: R \times R^n \to R^n$ is continuous, $h(t, z)$ has continuous derivatives up through order k with respect to z and let

$$|h|_k = \sup_{(t,z)} \left\{ \left| \frac{\partial^j h}{\partial z^j} (t, z) \right|, \quad 0 \leq j \leq k \right\}.$$

Also, suppose $h(t + 2\pi, z) = h(t, z)$.

For a given function F, the problem is to determine the existence, number and stability of the 2π-periodic solutions of the equation

$$(1.1) \qquad \dot{z} = Bz + F(t, z) + h(t, z)$$

near $z = 0$ for every h with $|h|_k$ small. In the applications, one generally does not consider all perturbations h but only those which depend on a finite number of parameters in a specified manner. The results below apply equally as well to this case.

For the existence of 2π-periodic solutions, the method of Liapunov-Schmidt is very convenient. If

$$B = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \quad x \in R, \quad y \in R^{n-1}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix},$$

$$F(t, z) = \begin{pmatrix} v(t, x) \\ H(t, y) \end{pmatrix}, \quad h(t, z) = \begin{pmatrix} f(t, x, y) \\ g(t, x, y) \end{pmatrix}$$

this method consists only of the following. Fix a constant a, substitute a Fourier series for z as

$$z = \begin{pmatrix} a \\ y_0 \end{pmatrix} + \sum_{n \neq 0} b_n e^{int}$$

and determine the coefficients $y_0(a, h)$, $b_n(a, h)$, $n = \pm 1, \pm 2, \cdots$ so that all of the Fourier coefficients of the 2π-periodic function

$$\dot{z}(t) - Bz(t) - F(t, z(t)) - h(t, z(t))$$

are zero except the constant term in the first component. The function

$$(1.2) \qquad z(t, a, h) = \begin{pmatrix} a \\ y_0(a, h) \end{pmatrix} + \sum_{n \neq 0} b_n(a, h) e^{int}$$

will then be a solution of (1.1) provided that a satisfies the bifurcation equation

(1.3) $0 = G(a, h) \overset{\text{def}}{=} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \{v(t, x(t, a, h)) + f(t, z(t, a, h))\}dt$.

All 2π-periodic solutions near zero can be obtained in this way.

In this paper, we show that the stability properties of the 2π-periodic solutions of (1.2) are the same as the stability properties of the zeros of $G(a, h)$ as equilibriums of the autonomous equation

(1.4) $\dot{a} = G(a, h)$.

This is true regardless of whether $\partial G(a, h)/\partial a$ is $\neq 0$ or $= 0$.

In addition to having interesting implications for nonautonomous equations, the results have immediate applications to Hopf bifurcation. In fact, problems involving Hopf bifurcation lead to equations of the form

$$\dot{\theta} = 1 + \Theta(\theta, x, y, \alpha) \,, \quad \dot{x} = X(\theta, x, y, \alpha) \,, \quad \dot{y} = Ay - Y(\theta, x, y, \alpha)$$

where $x \in \boldsymbol{R}$, $y \in \boldsymbol{R}^{n-1}$, all functions are 2π-periodic in θ, X, Y are second order in x, y when $\alpha = 0$, and Θ vanishes for $(\alpha, x, y) = (0, 0, 0)$. Eliminating θ, one obtains an equation of the form (1.1). The existence and stability of periodic orbits is then determined from the bifurcation function.

The methods are applicable to certain infinite dimensional evolutionary systems. In fact, the only essential facts used are that the linear part of the equation has a spectral decomposition with all roots negative except one root zero if the equation is nonautonomous (and a pair of complex conjugate roots if the equation is autonomous) and exponential estimates on the semigroup generated by the linear part which imply the existence of the center manifold. We briefly illustrate this for certain parabolic systems and retarded functional differential equations. Neutral functional differential equations with a stable D-operator (see [5]) as well as certain hyperbolic systems could also be discussed.

2. A scalar equation. Suppose $v: \boldsymbol{R} \times (-1, 1) \to \boldsymbol{R}$, $f: \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ are continuous functions, $v(t, x)$, $f(t, x)$ have continuos derivatives with respect to x up through order $k \geqq 2$ satisfying the following conditions:
  (i) $v(t, 0) = \partial v(t, 0)/\partial x = 0$.
  (ii) $v(t, x)$, $f(t, x)$ are 2π-periodic in t.
  (iii) $|v|_k$, $|f|_k$ are finite.
  We will suppose $|f|_k$ is small and, thus, be considering the equation

(P$_f$) $\dot{x} = v(t, x) + f(t, x)$

as a perturbation of the equation

(UP) $\dot{x} = v(t, x)$.

Thus, v will be considered fixed and f a parameter. The problem is to determine the 2π-periodic solutions of (P_f) which are close to $x = 0$ and to determine their stability properties.

Suppose $\mathscr{P}_{2\pi} = \{x: \boldsymbol{R} \rightarrow \boldsymbol{R}, \quad x \quad 2\pi$-periodic, continuous$\}$ with $|x| = \sup_{t \in R}|x(t)|$ for $x \in \mathscr{P}_{2\pi}$. The method of Liapunov-Schmidt implies there is an $\varepsilon > 0$ and a neighborhood W of 0 in $\mathscr{P}_{2\pi}$ such that, for any $f \in V(\varepsilon) \overset{\text{def}}{=} \{f: |f|_k < \varepsilon\}$, every 2π-periodic solution of (P_f) in W must be of the form $x(t, a, f)$, $a \in U(\varepsilon) \overset{\text{def}}{=} (-\varepsilon, \varepsilon)$, where $x(t, a, f)$ satisfies

(2.1)
$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} x(t, a, f)dt = a \\[2mm] \dot{x}(t, a, f) = v(t, x(t, a, f)) + f(t, x(t, a, f)) - G(a, f) \end{cases}$$

(2.2) $$G(a, f) \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} \{v(t, x(t, a, f)) + f(t, x(t, a, f))\}dt$$

and a is a zero of the *bifurcation function* $G(a, f)$; that is, satisfies the bifurcation equation

(2.3) $G(a, f) = 0$.

Conversely, any solution of Equation (2.3) gives a 2π-periodic solution of (P_f) in W. The function $G(a, f)$ is C^k in a, f.

In terms of Fourier series, the solution $x(t, a, f)$ of the integro-differential system (2.1) must satisfy

(2.6) $x(t, a, f) = a + \sum_{n \neq 0} (in)^{-1}e^{int}c_n[v(\cdot, x(\cdot, a, f)) + f(\cdot, x(\cdot, a, f))]$

with $x(t, 0, 0) = 0$, where $c_n(g) = \bar{c}_{-n}(g)$ represents the Fourier-coefficient of g corresponding to e^{int}.

Under appropriate hypotheses on $G(a, f)$, we show below that the qualitative properties of the solutions of (P_f) in a neighborhood of zero are the same as the qualitative properties of the solutions of the autonomous equation

(B_f) $\dot{a} = G(a, f)$

in $U(\varepsilon)$.

To facilitate the statement of the theorems, we need some definitions.

DEFINITION 2.1. Consider a scalar equation $\dot{x} = g(t, x)$, $g(t + 2\pi, x) = g(t, x)$. A 2π-periodic solution $\phi(t)$ is *stable from the left* if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $-\delta < x_0 - \phi(0) < 0$ implies the solution

$x(t, x_0)$ through x_0 at $t = 0$ satisfies $-\varepsilon < x(t, x_0) - \phi(t) < 0$ for $t \geq 0$. The solution $\phi(t)$ is *asymptotically stable from the left* if it is stable from the left and there is a $b > 0$ such that $-b < x_0 - \phi(0) < 0$ implies $x(t, x_0) - \phi(t) \to 0$ as $t \to \infty$. A similar definition applies to the right with all inequalities reversed and appropriate signs changed. The solution $\phi(t)$ is *asymptotically unstable to the left* (*right*) if it is asymptotically stable to the left (right) with t replaced by $-t$. The solution $\phi(t)$ is *asymptotically stable* if it is asymptotically stable from the left and the right. The solution $\phi(t)$ is *totally unstable* if it is asymptotically stable with t replaced by $-t$. The solution $\phi(t)$ is *asymptotically semistable from the left* (*right*) if it is asymptotically stable from the left (right) and asymptotically unstable from the right (left).

For two scalar equations $\dot{x} = g(t, x)$, $\dot{y} = h(t, y)$ with corresponding 2π-periodic solutions $\phi(t)$, $\psi(t)$, we say ϕ and ψ have the *same stability properties* if ϕ having one of the above concepts of stability is equivalent to ψ having the same type of stability; that is, ϕ being asymptotically stable is equivalent to ψ being asymptotically stable, etc.

DEFINITION 2.2. Two first order equations $\dot{x} = g(t, x)$, $\dot{y} = g(t, y)$ are said to be *topologically equivalent in a neighborhood U of $x = 0$, V of $y = 0$*, if there is a strictly increasing function mapping U onto V which preserves trajectories.

THEOREM 2.1. *Suppose $G(a^*, f) = 0$ and $x(t, a^*, f)$ is the 2π-periodic solution of (P_f) corresponding to a^*. Then the stability properties of a^* as a solution of (B_f) are the same as the stability properties of $x(\cdot, a^*, f)$ as a solution of (P_f).*

REMARK 2.1. There is actually a much closer correspondence between the stability properties of a^* as a solution of (B_f) and $x(\cdot, a^*, f)$ as a solution of (P_f). For example, if $G(a, f) = 0$ for a in an interval, then (P_f) has an interval of initial data corresponding to 2π-periodic solutions. If there is a sequence $a_j \to a^*$ such that $G(a_j, f) = 0$, then there is a sequence $x(t, a_j, f)$ of 2π-periodic solutions of (P_f) converging to $x(t, a^*, f)$. From the proof of Theorem 2.1 below, one will see how to discuss the stability properties of the solutions. There are several other possibilities as well. We have not discussed this in detail since there are too many cases and they may be easily discussed when a particular situation arises with this more complicated structure.

THEOREM 2.2. *If $G(a, f)$ has only a finite number of zeros, then (B_f) is topologically equivalent to (P_f) in neighborhoods of $a = 0$, $x = 0$.*

REMARK 2.2. If $G(a, 0) = a^k + g_1(a)$ where $g_1(a) = o(|a|^k)$ as $|a| \to 0$, then the conditions on $G(a, f)$ in Theorem 2.2 are satisfied. In fact, an easy application of Rolle's theorem shows that $G(a, f)$ has at most k zeros in a sufficiently small neighborhood of $a = 0$ for $f \in V(\varepsilon)$.

REMARK 2.3. The hypothesis that G has only a finite number of zeros in Theorem 2.2 is probably not necessary, but the proof below will not work.

To prove Theorems 2.1, 2.2, we need the following lemma.

LEMMA 2.1. *Let $f \in V(\varepsilon)$, let $a^* \in U(\varepsilon)$ be an equilibrium of Equation* (B$_f$) *and let $x(a^*, f)$ be the corresponding 2π-periodic solution of Equation* (P$_f$). *Then, a^* is hyperbolic (that is, $\partial G(a^*, f)/\partial a \neq 0$) if and only if the characteristic exponent λ of the linear variational equation of $x(a^*, f)$ is nonzero and in this case $\lambda \cdot \partial G(a^*, f)/\partial a > 0$.*

PROOF. The variational equation of Equation (P$_f$) about $x(a^*, f)$ is given by

$$\dot{y}(t) = p(t)y(t)$$

where $p(t) = (\partial v/\partial x)(t, x(t, a^*, f)) + (\partial f/\partial x)(t, x(t, a^*, f))$. Therefore, the characteristic exponent of the above linear equation is given by

$$\lambda = \frac{1}{2\pi} \int_0^{2\pi} p(t)dt .$$

On the other hand, we have, from the definition of G, that

$$\frac{\partial G}{\partial a}(a^*, f) = \frac{1}{2\pi} \int_0^{2\pi} p(t) \frac{\partial x}{\partial a}(t, a^*, f)dt$$

where $z(t) \overset{\text{def}}{=} \partial x(t, a^*, f)/\partial a$ is a solution of

$$\dot{z} = p(t)z + b$$

where $b = -\partial G(a^*, f)/\partial a$. Furthermore, $z(t) = z(t + 2\pi)$, $(2\pi)^{-1} \int_0^{2\pi} z(t)dt = 1$. If $Q(t) = \int_0^t p(s)ds$, then the variation of constants formula and the periodicity condition imply

$$z(t) = e^{Q(t)}\left[z(0) + b \int_0^t e^{-Q(s)}ds \right]$$

$$(1 - e^{2\pi\lambda})z(0) = e^{2\pi\lambda}b \int_0^{2\pi} e^{-Q(s)}ds .$$

Since we know a solution $z(0)$ exists, we have $b \neq 0$ implies $\lambda \neq 0$. If the explicit expression of $z(0)$ is used in the formula for $z(t)$, the condi-

tion $(2\pi)^{-1}\int_0^{2\pi} z(s)ds = 1$ implies

$$b^{-1}(1 - e^{2\pi\lambda}) = \frac{1}{2\pi}\int_0^{2\pi}\left[e^{Q(s)}\int_0^s e^{-Q(u)}du + e^{2\pi\lambda}e^{-Q(s)}\int_0^s e^{Q(u)}du\right]ds > 0 .$$

Thus, $\lambda b < 0$ or $\lambda\partial G(a^*, f)/\partial a > 0$.

Conversely, if $\lambda \neq 0$, then $z(0)$ is uniquely determined and $z(0) = 0$ if $b = 0$. Thus, $b = 0$ implies $z(t) = 0$ which contradicts the fact that $(2\pi)^{-1}\int_0^{2\pi} z(s)ds = 1$.

This proves the lemma.

PROOF OF THEOREM 2.1. Suppose a^* is asymptotically stable from the left. We prove $x(t, a^*, f)$ is asymptotically stable from the left. From the hypotheses, there is an $\eta > 0$ such that $G(a, f) > 0$ on $[a^* - \eta, a^*)$. We claim we may choose an $\eta_1 > 0$ so that the solution $\phi_f(t, x_0)$ of (P_f), $\phi_f(0, x_0) = x_0$, is not 2π-periodic for $-\eta_1 \leqq x_0 - x(0, a^*, f) < 0$. In fact, if this is not the case, for every $\varepsilon > 0$, there is a 2π-periodic solution $x(t, x_{0\varepsilon})$ of (P_f) with $-\varepsilon < x_{0\varepsilon} - x(0, a^*, f) < 0$. Let a_ε^* be the zero of $G(a, f)$ corresponding to $x(t, x_{0\varepsilon})$. Then

$$a_\varepsilon^* = \frac{1}{2\pi}\int_0^{2\pi} x(t, x_{0\varepsilon})dt < \frac{1}{2\pi}\int_0^{2\pi} x(t, a^*, f)dt = a^*$$

and one can choose ε so small that $a_\varepsilon^* \in (a^* - \eta, a^*)$. This is a contradiction since $G(a, f) > 0$ on this interval. Thus, we may assume $-\eta_1 \leqq x_0 - x(0, a^*, f) < 0$ implies $x(t, x_0)$ is not 2π-periodic.

Now there is a small perturbation g of the vector field f for which the new bifurcation $G(a, f + g)$ satisfies the property that there is an a_g such that $G(a_g, f + g) = 0$, $\partial G(a_g, f + g)/\partial a < 0$, $a_g \in [a^* - \eta, a^*)$, $a_g \to a^*$ as $g \to 0$, $G(a, f + g) > 0$ for $a \in [a^* - \eta, a_g)$. Lemma 3.1 implies the corresponding 2π-periodic solution $x(t, a_g, f + g)$ has a negative characteristic exponent and is uniformly asymptotically stable.

From the same type of argument as above, we may suppose that the solution $\phi_{f+g}(t, x_0)$ of (P_{f+g}) is not 2π-periodic for $x_0 \in [a^* - \eta_1, a_g)$. If $x(t, a^*, f)$ is not asymptotically stable from the left, then $\phi_f(t, x_0) - x^*(t, a^*, f) \to 0$ as $t \to -\infty$ and the mapping $\phi_f(-2\pi, \cdot): [a^* - \eta_1, a^*] \to [a^* - \eta_1, a^*]$ has the property that $\phi_f(-2\pi, a^* - \eta_1) > a^* - \eta_1$. Consequently, for g sufficiently small $\phi_{f+g}(-2\pi, a^* - \eta_1) > a^* - \eta_1$. Therefore, $\phi_{f+g}(-2\pi, \cdot): [a^* - \eta_1, a_g] \to [a^* - \eta_1, a_g]$ with a_g being an unstable fixed point of this map. This implies there is another fixed point ξ in $(a^* - \eta_1, a_g)$ which corresponds to a 2π-periodic solution $\phi_{f+g}(t, \xi)$ of (P_{f+g}). This corresponds to a zero of $G(a, f + g)$ in $[a^* - \eta, a_g]$ which

is a contradiction.

Conversely, suppose $x(t, a^*, f)$ is asymptotically stable from the left. Then there is an $\eta_1 > 0$ such that $\phi_f(t, x_0)$ is not 2π-periodic for $-\eta_1 \leqq x_0 - x(0, a^*, f) < 0$. By an argument similar to the above, there is an $\eta > 0$ such that $G(a, f) \neq 0$ for $a \in [a^* - \eta, a^*)$. To prove $G(a, f) > 0$, one perturbs the vector field f to $f + g$ to make all 2π-periodic solutions of (P_{f+g}) near $x(t, a^*, f)$ have nonzero characteristic exponents. The one with smallest initial value will be asymptotically stable corresponding to a zero a_g of $G(a, f + g)$ with $\partial G(a_g, f + g)/\partial a > 0$.

Assuming that $G(a, f) < 0$ on $[a^* - \eta, a^*)$, one argues by contradiction as before. The other concepts of stability are treated in a similar way to complete the proof of the theorem.

PROOF OF THEOREM 2.2. Let $a_1(f), \cdots, a_p(f)$ be the distinct zeros of $G(a, f)$ and let $b_j(f) = x(0, a_j(f), f)$ be the initial value of the corresponding 2π-periodic solutions. Let $\phi(t, b)$ be the solution of Equation (P_f) with $\phi(0, b) = b$. The $b_j(f)$ are distinct and we may thus assume they are ordered so that $b_1(f) < b_2(f) < \cdots < b_p(f)$. We also have

$$a_j(f) = \frac{1}{2\pi} \int_0^{2\pi} \phi(s, b_j(f)) ds < \frac{1}{2\pi} \int_0^{2\pi} \phi(s, b_{j+1}(f)) ds = a_{j+1}(f)$$

for all j.

To construct a continuous strictly increasing function $h: (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ which preserves trajectories of (P_f) and (B_f), define first a function $\bar{h}: (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon)$ such that $\bar{h}(b_j(f)) = a_j(f)$, $j = 1, 2, \cdots, p$ and extend $\bar{h}$ in any way whatsoever as long as it is strictly increasing and C^1. With the transformation $x \mapsto \bar{h}(x)$, one obtains a new bifurcation function $G(a, f)$ with $b_j(f) = a_j(f)$ for all j. For any $b \in (-\varepsilon, \varepsilon)$, there is a unique $h(b)$ such that $h(\phi(2\pi, 0, b)) = \alpha(2\pi, 0, h(b))$ where $\alpha(t, s, a)$, $\phi(t, s, b)$ are respectively the solutions of (B_f), (P_f) with initial values a, b at $t = s$. The function $h(b)$ is continuous and strictly increasing. For any $t \in [0, 2\pi]$ and c small, there is a unique $\psi(t, c)$ such that $c = \phi(t, 0, \psi(t, c))$. Let $\tilde{h}(t, c) = \alpha(t, 2\pi, h(\phi(2\pi, 0, \psi(t, c))))$. It is easy to verify that $\tilde{h}(t, c)$ is a homeomorphism in c and $\tilde{h}(t, \phi(t, 0, b)) = \alpha(t, 0, \tilde{h}(0, b))$ for all t, s.

REMARK 2.3. Suppose $g(t, x)$, $t \in \mathbf{R}$, $x \in \mathbf{R}$, $g(t + 2\pi, x) = g(t, x)$ is a given function, ε is a parameter, and consider the existence of 2π-periodic solutions of the equation $\dot{x} = \varepsilon g(t, x)$. For any given $r > 0$, one can determine an $\varepsilon_0 = \varepsilon_0(r) > 0$, and the bifurcation function

$$G(a, \varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} g(t, x(\cdot, a, \varepsilon)) dt$$

for all $|a| \leqq r$, $|\varepsilon| < \varepsilon_0$ and $x(\cdot, a, \varepsilon)$ satisfying the same integro-differential system as before. The conclusions in Theorems 2.1, 2.2 remain valid for this case.

3. A vector equation. Suppose $v, f: R \times R \times R^n \to R$, $H, g: R \times R \times R^n \to R^n$, $v(t, x, y)$, $H(t, x, y)$, $f(t, x, y)$, $g(t, x, y)$ continuous together with derivatives in x, y up through order $k \geqq 2$, 2π-periodic in t; v and H together with their derivatives in x, y vanishing at $x = 0$, $y = 0$. If A is $n \times n$ matrix whose eigenvalues have negative real parts, we consider the existence of 2π-periodic solutions of the equation

$$
(3.1) \quad
\begin{aligned}
\dot{x} &= v(t, x, y) + f(t, x, y) \stackrel{\text{def}}{=} V(t, x, y, f) \\
\dot{y} &= Ay + H(t, x, y) + g(t, x, y) \stackrel{\text{def}}{=} Ay + W(t, x, y, g)
\end{aligned}
$$

in a neighborhood of $x = 0$, $y = 0$ for $|f|_k$, $|g|_k$ small.

The method of Liapunov-Schmidt implies there is an $\varepsilon > 0$ and a neighborhood W of 0 in $\mathscr{P}_{2\pi} = \{(x(t), y(t)), 2\pi\text{-periodic in } t, \text{ continuous}\}$ such that, for any $(f, g) \in V(\varepsilon) = \{f: |f|_k, |g_k| < \varepsilon\}$, every 2π-periodic solution of (3.1) in W must be of the form $x(t, a, f, g)$, $y(t, a, f, g)$, $a \in U(\varepsilon) = (-\varepsilon, \varepsilon)$, where these functions satisfy the equations

$$
(3.2) \quad \frac{1}{2\pi} \int_0^{2\pi} x(t, a, f, g) dt = a
$$

$$
\dot{x} = V(t, x, y, f) - G(a, f, g), \quad \dot{y} = Ay + W(t, x, y, g)
$$

$$
(3.3) \quad G(a, f, g) = \frac{1}{2\pi} \int_0^{2\pi} \{V(t, x(t, a, f, g), y(t, a, f, g), f)\}
$$

and a is a zero of the *bifurcation function* $G(a, f, g)$; that is, satisfies the bifurcation equation

$$
(3.4) \quad G(a, f, g) = 0 .
$$

Conversely, any solution of Equation (3.4) gives a 2π-periodic solution of (3.1) in W. The function $G(a, f, g)$ is C^k in a, f, g.

In terms of Fourier series, the functions $x(t, a, f, g)$, $y(t, a, f, g)$ must satisfy the equations

$$
(3.5) \quad
\begin{aligned}
x(t) &= a + \sum_{n \neq 0} (in)^{-1} e^{int} c_n[V(\cdot, x(\cdot), y(\cdot), f)] \\
y(t) &= \sum_n (in - A)^{-1} e^{int} d_n[W(\cdot, x(\cdot), y(\cdot), g)]
\end{aligned}
$$

where $c_n[g]$, $d_n[h]$ are Fourier coefficients corresponding to e^{int}.

The equation for $y(t)$ can also be written as

$$
(3.6) \quad y(t) = \int_{-\infty}^t e^{A(t-s)} W(s, x(s), y(s), g) ds .
$$

For Equation (3.1), there is a $\delta > 0$ and a center manifold $M_{f,g}$,

$$(3.7) \quad M_{f,g} = \{(t, x, y): y = \phi(t, x, f, g), \; t \in R, \; |x| < \delta, \; |f|_k < \delta, \; |g|_k < \delta\}$$

where $\phi(t, x, f, g)$ is 2π-periodic in t, $\phi(t, 0, 0, 0) = 0$ and is a solution of the integral equation

$$(3.8) \quad \psi(t, x) = \int_{-\infty}^{t} e^{A(t-s)} W(s, \zeta(s, t, x, \psi, f), \psi(s, \zeta(s, t, x, \psi, f)), g) ds$$

where $\zeta(s, t, x, \psi, f)$ is the solution of the equation

$$(3.9) \qquad \frac{d\zeta}{ds} = V(s, \zeta(s), \psi(s, \zeta(s)), f)$$

with the initial value $\zeta(t) = x$.

This center manifold is asymptotically stable and the flow on the center manifold is given as $(x(t), \phi(t, x(t), f, g))$ where x satisfies the equation

$$(3.10) \qquad \qquad \dot{x} = V(t, x, \phi(t, x, f, g), f) .$$

Any 2π-periodic solution of (3.1) in W is determined as $(x(t), \phi(t, x(t), f, g))$ where $x(t)$ is a 2π-periodic solution of (3.10). We can apply the method of Liapunov-Schmidt to Equation (3.10) to obtain the bifurcation function $\widetilde{G}(a, f, g)$. The stability properties of the 2π-periodic solutions are determined from the stability properties of the zeros of $\widetilde{G}(a, f, g)$ as a solution of $\dot{a} = \widetilde{G}(a, f, g)$ from Theorem 2.1. The zeros of $\widetilde{G}(a, f, g)$ and $G(a, f, g)$ are the same from the way they are constructed and the fact that they give the 2π-periodic solutions of (3.1). If they have the same sign between zeros, then the stability properties of the zeros of $\widetilde{G}(a, f, g)$ as solutions of $\dot{a} = \widetilde{G}(a, f, g)$ are the same as the stability properties of the zeros of $G(a, f, g)$ as solutions of $\dot{a} = G(a, f, g)$. To prove they have the same sign between zeros, we may assume by making a small perturbation in f in the differential equation that the zeros of both functions are simple. Suppose this has been done and there is an a^* such that $G(a^*, f, g) = 0$, $\widetilde{G}(a^*, f, g) = 0$ and the derivatives have opposite sign. Now replace $f(x, y)$ by $f(x, y) + \varepsilon$ and obtain new bifurcation functions

$$G(a, f, g, \varepsilon) = G(a, f, g) + \varepsilon ,$$
$$\widetilde{G}(a, f, g, \varepsilon) = \widetilde{G}(a, f, g) + \delta(\varepsilon)\varepsilon + O(|\varepsilon|^2) , \quad \delta(\varepsilon) \geq \delta_0 > 0$$

as $\varepsilon \to 0$. These new functions must have a common zero near a^* if ε is small. But this is a contradiction since the derivatives have opposite signs. Thus, $G(a, f, g)$, $\widetilde{G}(a, f, g)$ have the same sign between zeoros

and we have proved

THEOREM 3.1. *Suppose* $G(a^*, f, g) = 0$ *and* $x(t, a^*, f, g)$, $y(t, a^*, f, g)$ *is the* 2π-*periodic solution of* (3.1) *corresponding to* a^*. *Then the stability properties of the solution* a^* *of*

$$(3.11) \qquad\qquad \dot{a} = G(a, f, g)$$

are the same as the stability properties of the solution $x(t, a^*, f, g)$, $y(t, a^*, f, g)$ *of* (3.1).

REMARK 3.1. Theorem 3.1 eliminates the consideration of averaging and normal forms for the determination of the stability of periodic solutions of nonautonomous equations. All of the information on stability is determined from the bifurcation function $G(a, f, g)$. This function is very easy to calculate since it involves only determining the Fourier coefficients of a periodic function. Of course, the method of averaging gives more quantitative information about the rate at which solutions decay or grow in t. On the other hand, for a differential equation of the form

$$\dot{x} = \varepsilon f(t, x, y, \varepsilon)\,, \quad \dot{y} = Ay + \varepsilon g(t, x, y, \varepsilon)$$

with ε a small parameter, Flockerzi [4] has shown that the bifurcation function $G(a, \varepsilon)$ has the following property. If one averages on the center manifold up through terms of order ε^k to obtain a polynomial of degree $p_k(x, \varepsilon)$ approximating the vector field on the center manifold up through terms of order ε^k, then

$$G(x, \varepsilon) - p_k(x, \varepsilon) = O(|\varepsilon|^{k+1})$$

as $|\varepsilon| \to 0$. These additional results of Flockerzi [4] show that the quantitative information on the flow is also contained in $G(a, \varepsilon)$.

However, it should be remarked that the stability Theorem 3.1 applies even when averaging is not applicable. In fact, one can apply the results of averaging only when zeros of the corresponding polynomial $p_k(x, \varepsilon)$ are simple. As a consequence, it generally never permits one to discuss directly what happens at bifurcation points, whereas Theorem 3.1 gives the complete qualitative picture for a uniform neighborhood of $\varepsilon = 0$.

REMARK 3.2. Theorem 3.1 not only has applications to nonautonomous equations of the form (3.1), but gives very interesting results for Hopf bifurcation. In fact, the problem of Hopf bifurcation leads to a system of equations of the form

$$\dot{\theta} = 1 + \Theta(\theta, x, y, \alpha)$$
(3.12)
$$\dot{x} = \bar{f}(\theta, x, y, \alpha) \,, \quad \dot{y} = Ay + \bar{g}(\theta, x, y, \alpha)$$

where α is a small real parameter, all functions are 2π-periodic in θ, (θ, x) represent polar coordinates in the two-dimensional subspace corresponding to the imaginary roots for $\alpha = 0$ of the original problem and y represents the vector corresponding to the roots with negative real parts. The functions $\bar{f}$, $\bar{g}$ vanish for $(x, y) = (0, 0)$ and are second order in these variables for $\alpha = 0$, the function Θ vanishes for $(\alpha, x, y) = (0, 0, 0)$.

For (α, x, y) sufficiently small, we can replace t by θ in the differentiations of x, y to obtain equations

(3.13) $$\frac{dx}{d\theta} = f(\theta, x, y, \alpha) \,, \quad \frac{dy}{d\theta} = Ay + g(\theta, x, y, \alpha)$$

with f, g being 2π-periodic in θ, vanishing for $(x, y) = (0, 0)$, and second order in (x, y) for $\alpha = 0$.

The previous theory is now directly applicable to Equation (3.13) to obtain the bifurcation function $G(a, \alpha)$. The zeros of $G(a, \alpha)$ correspond to 2π-periodic solutions in θ of (3.13) and thus periodic solutions with period $\omega(\alpha)$ close to 2π of the original equations. Theorem 3.1 gives the stability properties of the solutions of (3.13) and thus the orbital stability properties of the original equations.

Again, we remark that the computations are easier than the ones used in averaging since one need only equate Fourier coefficients. We also remark that α being a vector parameter will not change the theory.

REMARK 3.3. Theorem 3.1 remains valid for certain infinite dimensional problems. In fact, suppose A is a sectorial operator on a Banach space X, X^{α} is the Banach space corresponding to the graph norm of A^{α}, $h: \boldsymbol{R} \times X^{\alpha} \to X$ is sufficiently smooth and sufficiently small, $h(t, u) = h(t + 2\pi, u)$, and A has zero as a simple eigenvalue with the other elements of the spectrum satisfying $\mathrm{Re}\,\lambda \geqq \delta > 0$. Let us determine the small 2π-periodic solutions of the equation

(3.14) $$\dot{u} + Au = h(t, u) \,.$$

If $X = X_0 \oplus X_1$ where $\dim X_0 = 1$, $X_0, X_1 \cap D(A)$ are invariant under A, $A\phi = 0$ for $\phi \in X_0$ and $u = x + y$, $x \in X_0$, $y \in X_1$, then Equation (3.14) is equivalent to

(3.15) $$\dot{x} = f(t, x, y) \,, \quad \dot{y} = -By + g(t, x, y)$$

where B is sectorial and the spectrum of $-B$ on X_1 satisfies $\mathrm{Re}\,\lambda \leqq -\delta < 0$ and f, g are smooth and small.

One can now apply the method of Liapunov-Schmidt to obtain the bifurcation function $G(a, h)$. Applying the theory in Henry [6] on the center manifold, we obtain a generalization of Theorem 3.1. We do not state the precise conditions on h since they are technical and the reader may consult Henry [6] or a recent paper of Kielhöfer [7].

We remark that the problem of Hopf bifurcation can also be treated for similar equations taking into account Remark 3.2. Using this result one obtains stability of the solutions discussed in Kielhöfer [7].

REMARK 3.4. Theorem 3.1 can also be generalized to functional differential equations. The determination of the bifurcation equations by the method of Liapunov-Schmidt has already been given in Hale [5]. Since the center manifold theory is also available (see Chafee [2]), one can complete the proof of Theorem 3.1 for this case.

To apply the results to the Hopf bifurcation for functional differential equations is a little more difficult since the elimination of t as in Remark 3.2 is not possible. However, some results of Chow and Mallet-Paret [3] allow one to proceed almost exactly as in Remark 3.2.

Consider the functional differential equation

$$(3.16) \qquad \dot{x}(t) = Lx_t + f(x_t)$$

where $L\colon C \to R^n$, $f\colon C \to R^n$, $C = C([-r, 0], R^n)$, $r > 0$, $x_t(\theta) = x(t + \theta)$, $-r \leqq \theta \leqq 0$, L is continuous and linear.

Let $Z = C \oplus \langle X_0 \rangle$, be the space of functions uniformly continuous on $[-r, 0)$ with a jump discontinuity at 0. Here X_0 is the $n \times n$ matrix function $X_0(\theta) = 0$, $\theta < 0$, $X_0(0) = I$. Define the map

$$(3.17) \qquad A\colon C^1 \to Z , \quad A\phi = \dot{\phi} + X_0[L\phi - \dot{\phi}(0)] .$$

Now consider the abstract evolutionary equation

$$(3.18) \qquad \frac{dx_t}{dt} = Ax_t + X_0 f(x_t)$$

with $x_0 = \phi$, $\phi \in C^1$.

If x is a solution of (3.18) with $x_0 = \phi$, $\phi \in C^1$, $\dot{\phi}(0) = L\phi + f(\phi)$, then (see [3]) x is a solution of (3.16). The converse is also easy to prove. Therefore, all smooth solutions of (3.16) are obtained through (3.18) and one can therefore use (3.18) to obtain all qualitative properties of solutions (see [3]).

If $L = L(\alpha)$, $f = f(x, \alpha)$ and the characteristic equation

(3.19) $$\det [\lambda I - L(0)e^{\lambda \cdot}] = 0$$

has two simple eigenvalues on the imaginary axis, say $\pm i$, then we can introduce coordinates using the decomposition theory in Hale [5]. In fact, the space C can be decomposed as

$$C = P \oplus Q , \quad \dim P = 2 , \quad T(t)P \subset P , \quad T(t)Q \subset Q ,$$

such that if $\Phi = (\phi_1, \phi_2)$ is a basis for P and $A\Phi = \Phi B$, then there is a 2×2 matrix C such that,

$$x_t \in C^1 , \quad \dot{x}(t) = Lx_t + f(x_t) ,$$

then

$$x_t = \Phi y(t) + z(t) , \quad z \in Q$$
$$\dot{y} = By + Cf(\Phi y + z) , \quad \dot{z} = Az + X_0^Q f(\Phi y + z)$$

where $X_0^Q = X_0 - \Phi C$. Furthermore, B may be assumed to have the form $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

If we introduce polar coordinates for y as θ, u and eliminate t we have

(3.20) $$\frac{du}{d\theta} = \bar{f}(\theta, u, z) , \quad \frac{dz}{d\theta} = Az + \bar{g}(\theta, u, z)$$

where f, g are 2π-periodic in θ.

Under the hypothesis that all other solutions of (3.19) have $\text{Re } \lambda < 0$, one can now proceed as in Remark 3.2 to obtain the bifurcation function $G(a, f)$ for 2π-periodic solutions of (3.16). Using the center manifold theorem, one also obtains the stability properties of the periodic solution from the bifurcation function.

REMARK 3.5. It is not necessary to assume the eigenvalues of A have negative real parts to derive the bifurcation equations. One needs only to assume a nonresonance condition for the eigenvalues on the imaginary axis. However, one cannot prove a stability result as in Theorem 3.1 by using the bifurcation function.

REMARK 3.6. One can also consider questions analogous to the ones studied in Chafee [1] concerning a classification of the perturbations according to the number of small periodic solutions of the perturbed system. As noted recently by Chafee himself, the division theorem for smooth functions in Michor [9] makes the classification in [1] more complete. The generalization of these results to more general evolutionary equations should present no essentially new ideas since the analysis is concerned only with the bifurcation function $G(a, f)$ under the hypoth-

esis that $G(a, 0) = a^{2k+1} + o(|a|^{2k+1})$ as $a \to 0$. The results in this paper give the additional information about the stability properties of the periodic solutions for any given f in a neighborhood of zero in C^k.

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