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HOLOMORPHIC FAMILIES OF RIEMANN SURFACES AND TEICHMÜLLER SPACES III

Bimeromorphic embedding of algebraic surfaces into projective spaces by automorphic forms

YOICHI IMAYOSHI

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Introduction. In this paper, as an application of the results in [4] and [5], we will deal with the bimeromorphic embedding of algebraic surfaces into projective spaces by automorphic forms.

Let X be a two-dimensional, irreducible, non-singular projective algebraic variety over $\boldsymbol{C}$. There exist a non-empty Zariski open subset $\mathscr{S}$ of X, a Riemann surface R of finite type and a holomorphic mapping $\pi: \mathscr{S} \to R$ so that the triple $(\mathscr{S}, \pi, R)$ is a holomorphic family of Riemann surfaces of type (g, n) with $2g-2+n>0$. We may assume that the universal covering space of R is the unit disc. Then the universal covering space $\tilde{\mathscr{D}}$ of $\mathscr{S}$ is a bounded Bergman domain in $\boldsymbol{C}^2$. Let $\tilde{\mathscr{G}}$ be the covering transformation group of the universal covering $\tilde{\Pi}: \tilde{\mathscr{D}} \to \mathscr{S}$. A holomorphic function f is called an automorphic form of weight q on $\tilde{\mathscr{D}}$ for $\tilde{\mathscr{G}}$, if

$$f(T(x)) = f(x)[J_T(x)]^{-q}$$

for all $T \in \tilde{\mathscr{G}}$ and $x \in \tilde{\mathscr{D}}$, where q is an integer and $J_T(x)$ is the Jacobian of T at x. We also say that f is a q-form for $\tilde{\mathscr{G}}$. We assume $q \geqq 2$ throughout this paper.

Our problem is stated as follows: *Can we construct many automorphic q-forms $f_0, \cdots, f_N$ for $\tilde{\mathscr{G}}$ in such a way that $F=(f_0, \cdots, f_N)$ induces a bimeromorphic embedding of X into the N-dimensional complex projective space $\boldsymbol{P}_N(\boldsymbol{C})$?* This problem is solved affirmatively in §8.

At the beginning, in §1, we recall the main results in [4] and [5]. In §2, we construct a domain $\mathscr{D}$ and a discrete subgroup $\mathscr{G}$ of the analytic automorphism group of $\mathscr{D}$ so that our problem for $\tilde{\mathscr{D}}$ and $\tilde{\mathscr{G}}$ can be reduced to that for $\mathscr{D}$ and $\mathscr{G}$. §3 is devoted to constructing some auxiliary domains, which will be used in §7. In §4, we define the behaviour of automorphic forms for $\mathscr{G}$ near boundary points and, in §5, we recall some well-known results on the Poincaré metric and the

Poincaré series, which are used in §6 and §7, where the Poincaré series and the Poincaré-Eisenstein series for $\mathscr{S}$ are constructed and their behaviour near boundary points are studied.

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1. Preliminaries. We shall briefly explain the main results in [4] and [5].

Let $\mathscr{S}$ be a two-dimensional Stein manifold and let R be a Riemann surface of finite type with the universal covering $\rho: D \to R$, where D is the unit disc $|\tau| < 1$ in the complex τ-plane. We assume that a holomorphic mapping $\pi: \mathscr{S} \to R$ satisfies the following two conditions:

(i) π is of maximal rank at every point of $\mathscr{S}$, and

(ii) the fiber $S_t = \pi^{-1}(t)$ of $\mathscr{S}$ is connected and of fixed finite type (g, n) with $2g - 2 + n > 0$ as a Riemann surface for every t in R. Such a triple $(\mathscr{S}, \pi, R)$ is called a holomorphic family of Riemann surfaces of type (g, n) over R.

Take a finitely generated Fuchsian group $\tilde{G}$ of the first kind with no elliptic elements acting on the upper half-plane U such that the quotient space $\tilde{S} = U/\tilde{G}$ is a Riemann surface of finite type (g, n). Let $Q_{\text{norm}}(\tilde{G})$ be the set of all quasiconformal automorphisms w of U leaving $0, 1, \infty$ fixed and satisfying $w \circ \tilde{G} \circ w^{-1} \subset SL'(2; \boldsymbol{R})$, where $SL'(2; \boldsymbol{R})$ is the set of all real Möbius transformations. Two elements w_1 and w_2 of $Q_{\text{norm}}(\tilde{G})$ are called equivalent if $w_1 = w_2$ on the real axis $\boldsymbol{R}$. The Teichmüller space $T(\tilde{G})$ of $\tilde{G}$ is the quotient of $Q_{\text{norm}}(\tilde{G})$ with respect to the above equivalence relation. Let $L^\infty(U, \tilde{G})$ be the complex Banach space of bounded measurable complex-valued functions μ satisfying $\mu(g(z))\overline{g'(z)}/g'(z) = \mu(z)$ for all g in $\tilde{G}$ and let $L^\infty(U, \tilde{G})_1$ be the open unit ball in $L^\infty(U, \tilde{G})$.

Let w_μ be the element of $Q_{\text{norm}}(\tilde{G})$ with a Beltrami coefficient $\mu \in L^\infty(U, \tilde{G})_1$ and let W^μ be a quasiconformal automorphism of the Riemann sphere $\hat{\boldsymbol{C}}$ such that W^μ has the Beltrami coefficient μ on the upper half-plane U, is conformal on the lower half-plane L and

$$W^\mu(z) = 1/(z + i) + O(|z + i|)$$

as z tends to $-i$. This mapping W^μ is uniquely determined by $[w_\mu]$ up to the equivalence relation, that is, $w_\mu = w_\nu$ on $\boldsymbol{R}$ if and only if $W^\mu = W^\nu$ on $L \cup \boldsymbol{R}$. Let ϕ_μ be the Schwarzian derivative of W^μ. Then ϕ_μ in an element of the space $B_2(L, \tilde{G})$ of bounded holomorphic quadratic differentials for $\tilde{G}$ on L. Bers proved that the mapping sending $[w_\mu]$ into ϕ_μ is a biholomorphic mapping of $T(\tilde{G})$ onto a holomorphically convex bounded domain of $B_2(L, \tilde{G})$, which is denoted by the same notation $T(\tilde{G})$. We set $\tilde{G}_{\phi_\mu} = W^\mu \circ \tilde{G} \circ (W^\mu)^{-1}$ and $\tilde{D}_{\phi_\mu} = W^\mu(U)$. Then $\tilde{G}_{\phi_\mu}$ is a quasi-Fuchsian group and Koebe's one-quarter theorem implies that $\tilde{D}_{\phi_\mu} \subset (|w| < 2)$ for every ϕ_μ of $T(\tilde{G})$.

Now, for a holomorphic family of Riemann surfaces $(\mathscr{S}, \pi, R)$ of type (g, n) with $2g - 2 + n > 0$, there exists a holomorphic mapping $\Psi: D \to T(\tilde{G})$ such that the quotient space $\tilde{D}_{\Psi(\tau)}/\tilde{G}_{\Psi(\tau)}$ is conformally equivalent to $S_{\rho(\tau)}$ for every $\tau \in D$. We abbreviate $\tilde{G}_{\Psi(\tau)}$ as $\tilde{G}_\tau$ and $\tilde{D}_{\Psi(\tau)}$ as $\tilde{D}_\tau$. We set

$$\tilde{\mathscr{D}} = \{(\tau, w) \mid \tau \in D, w \in \tilde{D}_\tau\} .$$

This set $\tilde{\mathscr{D}}$ is a bounded Bergman domain in $D \times (|w| < 2)$ and is topologically equivalent to the polydisc $D \times D$. Let F_τ be the conformal mapping of $\tilde{D}_\tau/\tilde{G}_\tau$ onto $S_{\rho(\tau)}$ induced by $\Psi(\tau)$ for every $\tau \in D$. Let $\tilde{\Pi}$ be the holomorphic mapping of $\tilde{\mathscr{D}}$ onto $\mathscr{S}$ sending (τ, w) into $F_\tau([w])$, where $[w]$ is the orbit of w with respect to $\tilde{G}_\tau$. Then $\tilde{\Pi}: \tilde{\mathscr{D}} \to \mathscr{S}$ is a universal covering of $\mathscr{S}$.

Let $\tilde{\mathscr{G}}$ be the covering transformation group of $\tilde{\Pi}: \tilde{\mathscr{D}} \to \mathscr{S}$. We can explicitly express the elements of $\tilde{\mathscr{G}}$ as follows. Let Γ be the covering transformation group of the universal covering $\rho: D \to R$. For each element γ of Γ, the homotopic monodromy $\tilde{\mathscr{M}}_\gamma$ of $(\mathscr{S}, \pi, R)$ is the element of the modular group $\mathrm{Mod}(\tilde{G})$ of $\tilde{G}$ with $\Psi(\gamma(\tau)) = \tilde{\mathscr{M}}_\gamma(\Psi(\tau))$ on D. Denote by $N(\tilde{G})$ the set of all quasiconformal automorphisms $\tilde{\omega}$ of U with $\tilde{\omega} \circ \tilde{G} \circ \tilde{\omega}^{-1} = \tilde{G}$. Take an element $\tilde{\omega}_\gamma$ of $N(\tilde{G})$ which induces $\tilde{\mathscr{M}}_\gamma$, that is, the element $\langle \tilde{\omega}_\gamma \rangle$ of $\mathrm{Mod}(G)$ induced by $\tilde{\omega}_\gamma$ is equal to $\tilde{\mathscr{M}}_\gamma$. We may assume that $\tilde{\omega}_{\gamma \circ \delta} = \tilde{\omega}_\gamma \circ \tilde{\omega}_\delta$ for all $\gamma, \delta \in \Gamma$.

Let $F(\tilde{G})$ be the fiber space over the Teichmüller space $T(\tilde{G})$, that is,

$$F(\tilde{G}) = \{(\phi, w) \mid \phi \in T(\tilde{G}), w \in \tilde{D}_\phi\} .$$

In general, every element $\tilde{\omega}$ of $N(\tilde{G})$ induces an analytic automorphism of $F(\tilde{G})$ as follows. For every element $[w_\mu]$ of $T(\tilde{G})$, we set $w_\nu = \lambda \circ w_\mu \circ \tilde{\omega}^{-1} \in Q_{\mathrm{norm}}(\tilde{G})$, where λ is a real Möbius transformation. If we set

$$\hat{w} = [\tilde{\omega}]_*(\phi_\mu, w) = W^\nu \circ \tilde{\omega} \circ (W^\mu)^{-1}(w)$$

for $w \in \tilde{D}_{\phi_\mu}$, then the mapping $(\langle \tilde{\omega} \rangle_*, [\tilde{\omega}]_*)$ sending (ϕ_μ, w) into $(\phi_\nu, \hat{w})$ is an analytic automorphism of $F(\tilde{G})$. These elements $(\langle \tilde{\omega} \rangle_*, [\tilde{\omega}]_*)$ give rise to the extended modular group $\mathrm{mod}(\tilde{G})$ of $\tilde{G}$.

Since $\tilde{\omega}_\gamma \circ \tilde{g}$ is an element of $N(\tilde{G})$ for $\gamma \in \Gamma$ and $\tilde{g} \in \tilde{G}$, an analytic automorphism $(\gamma, \tilde{g})$ of $\tilde{\mathscr{D}}$ is defined by

$$(\gamma, \tilde{g})(\tau, w) = (\gamma(\tau), H_{(\gamma,\tilde{g})}(\tau, w))$$

with $H_{(\gamma,\tilde{g})}(\tau, w) = [\tilde{\omega}_\gamma \circ \tilde{g}]_*(\Psi(\tau), w)$ for $(\tau, w) \in \tilde{\mathscr{D}}$. Then the covering transformation group $\tilde{\mathscr{G}}$ of $\tilde{\Pi}: \tilde{\mathscr{D}} \to \mathscr{S}$ is identical with the set $\{(\gamma, \tilde{g}) | \gamma \in \Gamma, \tilde{g} \in \tilde{G}\}$. By definition, we have the formula

$$(\gamma, \tilde{g}) \circ (\delta, \tilde{h}) = (\gamma \circ \delta, \ \tilde{\omega}_\delta^{-1} \circ \tilde{g} \circ \tilde{\omega}_\delta \circ \tilde{h})$$

for $\gamma, \delta \in \Gamma$ and $\tilde{g}, \tilde{h} \in \tilde{G}$, that is, $\tilde{\mathscr{G}}$ is a semi-direct product of Γ with $\tilde{G}$. The quotient spaces $\tilde{\mathscr{S}} = \tilde{\mathscr{D}}/\tilde{\mathscr{G}}$ is biholomorphically equivalent to $\mathscr{S}$.

Let C be the set of all cusps of Γ, that is, the set of all parabolic fixed points of Γ. For each τ_0 of C, there is an element $\Psi(\tau_0)$ in the closure of $T(\tilde{G})$ such that $\Psi(\tau)$ converges to $\Psi(\tau_0)$ uniformly as $\tau \to \tau_0$ through any cusp region at $\tau = \tau_0$ in D. For each $\tau \in D \cup C$, denote by $\tilde{G}_\tau = \tilde{G}_{\Psi(\tau)}$ the Kleinian group associated with the quadratic differential $\Psi(\tau)$ for $\tilde{G}$, by $\Omega(\tilde{G}_\tau)$ the region of discontinuity of $\tilde{G}_\tau$, and by $\Delta(\tilde{G}_\tau)$ the invariant component corresponding to the lower half-plane. Set $\tilde{D}_\tau = \Omega(\tilde{G}_\tau) - \Delta(\tilde{G}_\tau)$ and let $\tilde{\mathscr{P}}_\tau$ be the set of all fixed points on $\partial\tilde{D}_\tau$ of parabolic transformations of $\tilde{G}_\tau$. We set $\tilde{\mathscr{D}}_c = \{(\tau, w) | \tau \in D \cup C, \ w \in \tilde{D}_\tau \cup \tilde{\mathscr{P}}_\tau\}$. Each point of $\tilde{\mathscr{C}} = \tilde{\mathscr{D}}_c - \tilde{\mathscr{D}}$ is called a cusp of $\tilde{\mathscr{G}}$. A Hausdorff topology on $\tilde{\mathscr{D}}_c$ is defined canonically and every element $(\gamma, \tilde{g})$ of $\tilde{\mathscr{G}}$ is extended to a topological automorphism $(\gamma, \tilde{g})_c$ of $\tilde{\mathscr{D}}_c$. We set

$$\tilde{\mathscr{G}}_c = \{(\gamma, \tilde{g})_c | \gamma \in \Gamma, \tilde{g} \in \tilde{G}\} .$$

Then the quotient space $\tilde{\mathscr{S}}_c = \tilde{\mathscr{D}}_c/\tilde{\mathscr{G}}_c$ is a two-dimensional compact normal space and every compactification of $\mathscr{S}$ is bimeromorphically equivalent to $\tilde{\mathscr{S}}_c$. (See §6 in [5].)

2. Construction of domains $\mathscr{D}$ and $\mathscr{D}'$. Let $\Pi_{\tilde{G}}: U \to \tilde{S}$ be the canonical projection. For a fixed $\nu > 3$, there exists a Fuchsian group G with signature $(g, n; \nu, \cdots, \nu)$ such that the quotient space U'/G is conformally equivalent to $\tilde{S}$, where U' is the complement in U of the set of elliptic fixed points of G. Let $\Pi_G: U' \to \tilde{S}$ be the canonical projection. There is a universal covering $\Pi_{H_0}: U \to U'$ with $\Pi_{\tilde{G}} = \Pi_G \circ \Pi_{H_0}$. The covering transformation group H_0 of $\Pi_{H_0}: U \to U'$ is a normal subgroup of $\tilde{G}$ and we have the relation

$$\tilde{G} = \{\tilde{g} \in \mathrm{SL}'(2; \boldsymbol{R}) | \Pi_{H_0} \circ \tilde{g} = g \circ \Pi_{H_0} \text{ for some } g \in G\} .$$

If $\tilde{S}$ is compact, that is, $n = 0$, then $G = \tilde{G}$, $U' = U$, $\Pi_G = \Pi_{\tilde{G}}$ and Π_{H_0}

is the identity map. The Teichmüller space $T(\widetilde{G})$ is canonically isomorphic to $T(G)$ as follows. (See Bers and Greenberg [3].) For every $\widetilde{\mu} \in L^{\infty}(U, \widetilde{G})_1$, the element $\mu \in L^{\infty}(U, G)_1$ is defined by

$$\mu(\Pi_{H_0}(z)) = \widetilde{\mu}(z)\Pi'_{H_0}(z)/\overline{(\Pi'_{H_0}(z))} .$$

If $w_{\widetilde{\mu}} = w_{\widetilde{\nu}}$ on $\boldsymbol{R}$ for $\widetilde{\mu}$, $\widetilde{\nu} \in L_{\infty}(U, \widetilde{G})_1$, then $w_{\mu} = w_{\nu}$ on $\boldsymbol{R}$. Therefore the mapping m: $T(\widetilde{G}) \to T(G)$ sending $[w_{\widetilde{\mu}}]$ into $[w_{\mu}]$ is well-defined. It can be shown that the mapping m is isomorphic.

Now, the holomorphic mapping Φ: $D \to T(G)$ is defined by $\Phi = m \circ \Psi$. Take a Beltrami coefficient $\widetilde{\mu}_{\tau} \in L^{\infty}(U, \widetilde{G})_1$ such that $\Psi(\tau)$ is the Schwarzian derivative of $W^{\widetilde{\mu}_{\tau}}$. Then $\Phi(\tau)$ is the Schwarzian derivative of $W^{\mu_{\tau}}$. Let $D_{\tau} = D_{\Phi(\tau)} = W^{\mu_{\tau}}(U)$, $D'_{\tau} = D'_{\Phi(\tau)} = W^{\mu_{\tau}}(U')$ and $G_{\tau} = W^{\mu_{\tau}} \circ G \circ (W^{\mu_{\tau}})^{-1}$. We set

$$\mathscr{D} = \{(\tau, w) \mid \tau \in D, w \in D_{\tau}\}$$

and

$$\mathscr{D}' = \{(\tau, w) \mid \tau \in D, w \in D'_{\tau}\} .$$

Since the mapping $M_{\tau}; \widetilde{D}_{\tau} \to D'_{\tau}$ sending w into $W^{\mu_{\tau}} \circ \Pi_{\widetilde{H}_0} \circ (W^{\widetilde{\mu}_{\tau}})^{-1}(w)$ is holomorphic and depends only on $\Psi(\tau)$, we can define a holomorphic mapping M: $\widetilde{\mathscr{D}} \to \mathscr{D}'$ with $M(\tau, w) = (\tau, M_{\tau}(w))$.

For every element $\widetilde{\omega}_{\gamma} \in N(\widetilde{G})$ inducing the homotopic monodromy $\widetilde{\mathscr{M}}_{\gamma} \in \mathrm{Mod}(\widetilde{G})$ for $\gamma \in \Gamma$, there is a unique element $\omega_{\gamma} \in N(G)$ with $\Pi_{H_0} \circ \widetilde{\omega}_{\gamma} = \omega_{\gamma} \circ \Pi_{H_0}$. Hence the element $(\langle\omega_{\gamma} \circ g\rangle_*, [\omega_{\gamma} \circ g]_*)$ of $\mathrm{mod}(G)$ can be defined for $\gamma \in \Gamma$ and $g \in G$. We set

$$(\gamma, g)(\tau, w) = (\gamma(\tau), H_{(\gamma,g)}(\tau, w))$$

with $H_{(\gamma,g)}(\tau, w) = [\omega_{\gamma} \circ g]_*(\Phi(\tau), w)$ for $(\tau, w) \in \mathscr{D}$. Then (γ, g) is an analytic automorphism of $\mathscr{D}$ and all such automorphisms give rise to a properly discontinuous group $\mathscr{G}$ of analytic automorphisms of $\mathscr{D}$. For every element $\widetilde{g} \in \widetilde{G}$ and $g \in G$ with $\Pi_{H_0} \circ \widetilde{g} = g \circ \Pi_{H_0}$, we have the relation $M \circ (\gamma, \widetilde{g}) = (\gamma, g) \circ M$, which implies $M \circ \widetilde{\mathscr{G}} = \mathscr{G} \circ M$.

By the same reasoning as for Ψ, we see the following fact. For each parabolic fixed point τ_0 of Γ, there is an element $\Phi(\tau_0) \in \overline{T(G)}$ such that $\Phi(\tau)$ converges to $\Phi(\tau_0)$ uniformly as $\tau \to \tau_0$ through any cusp region at $\tau = \tau_0$ in D. For each $\tau \in D \cup C$, we denote by $G_{\tau} = G_{\Phi(\tau)}$ the Kleinian group associated with quadratic differential $\Phi(\tau)$ for G, by $\Omega(G_{\tau})$ the region of discontinuity of G_{τ} and by $\Delta(G_{\tau})$ the invariant component corresponding to the lower half-plane. Set $D_{\tau} = \Omega(G_{\tau}) - \Delta(G_{\tau})$ and let $\mathscr{P}_{\tau}$ be the set of all fixed points on ∂D_{τ} of parabolic transformations of G_{τ}. It should be noted that the set $\mathscr{P}_{\tau}$ is empty for $\tau \in D$. We set

$$\hat{\mathscr{D}} = \{(\gamma, w) \mid \gamma \in D \cup C,\ w \in D_\tau \cup \mathscr{P}_\tau\}\ .$$

Each point of $\mathscr{C} = \hat{\mathscr{D}} - \mathscr{D}$ is called a cusp of $\mathscr{G}$. A Hausdorff topology on $\hat{\mathscr{D}}$ is defined canonically and every element of (γ, g) of $\mathscr{G}$ is extended to a topological automorphism $(\gamma, g)_c$ of $\hat{\mathscr{D}}$. We set

$$\hat{\mathscr{G}} = \{(\gamma, g)_c \mid \gamma \in \Gamma,\ g \in G\}\ .$$

Then the quotient space $\hat{\mathscr{S}} = \hat{\mathscr{D}}/\hat{\mathscr{G}}$ is a two-dimensional compact normal space. Moreover, the holomorphic mapping $M\colon \tilde{\mathscr{D}} \to \mathscr{D}'$ is extended to a continuous mapping $\hat{M}\colon \tilde{\mathscr{D}}_c \to \hat{\mathscr{D}}$ with $\hat{M} \circ \tilde{\mathscr{G}}_c = \hat{\mathscr{G}} \circ \hat{M}$, which induces a biholomorphic mapping of $\tilde{\mathscr{S}}_c$ onto $\hat{\mathscr{S}}$.

For an automorphic q-form Ψ on $\mathscr{D}$ for $\mathscr{G}$, we set

$$(M^*\Psi)(\tau, w) = \Psi(M(\tau, w))[J_M(\tau, w)]^q$$

for $(\tau, w) \in \tilde{\mathscr{D}}$. Then $M^*\Psi$ is an automorphic q-form on $\tilde{\mathscr{D}}$ for $\tilde{\mathscr{G}}$.

Therefore, our problem stated in Introduction is reduced to the case for $\mathscr{D}$ and $\mathscr{G}$. So, in the following sections, we will study automorphic forms on $\mathscr{D}$ for $\mathscr{G}$ in place of those on $\tilde{\mathscr{D}}$ for $\tilde{\mathscr{G}}$.

3. Construction of domains $\mathscr{E}_{l,j}$, $\mathscr{E}'_{l,j}$, $\hat{\mathscr{E}}_{l,j}$ and $\hat{\mathscr{E}}'_{l,j}$. In this section, we will use the notations in §2 of [5].

Let $\hat{R}$ be the compactification of R, that is, $\hat{R}$ is a compact Riemann surface of genus g_0 such that the surface obtained from $\hat{R}$ by deleting finitely many points $t_1, \cdots, t_{n_0}$ is conformally equivalent to R. Let $R_{(g,n)}$ be the moduli space of all Riemann surfaces without nodes of signature $(g, n; \nu, \cdots, \nu)$ and let $M_{(g,n)}$ be the moduli space of all Riemann surfaces with nodes of signature $(g, n; \nu, \cdots, \nu)$, where ν is a fixed integer greater than 3. Then the holomorphic mapping $J\colon R \to R_{(g,n)}$ sending t into $[S_t]$ can be extended to a holomorphic mapping $\hat{J}\colon \hat{R} \to M_{(g,n)}$. Let S_l be a Riemann surface with $\hat{J}(t_l) = [S_l]$ for each $l = 1, \cdots, n_0$.

Let $\nu_0 > 3$ be an integer. We set $\nu_{l,l} = \nu_0$ and $\nu_{l,m} = \infty$, $l \neq m$ for $l, m = 1, \cdots, n_0$. Let E be the unit disc $|\zeta| < 1$ in the complex ζ-plane. For each $l = 1, \cdots, n_0$, we take a Fuchsian group $\tilde{\Gamma}_l$ acting on E such that $E/\tilde{\Gamma}_l$ is conformally equivalent to $\hat{R}$ with the given signature $(g, n_0; \nu_{l,1}, \cdots, \nu_{l,n_0})$. Denote by $\tilde{\rho}_l$ the canonical projection of E onto $\hat{R}$ and by E'_l the complement in E of the set of elliptic fixed points of $\tilde{\Gamma}_l$. Let Γ_l be the covering transformation group of the universal covering $\rho_l\colon D \to E'_l$ with $\rho = \tilde{\rho}_l \circ \rho_l$. For each point $\zeta \in E'_l$, we take a point $[\tilde{S}, f_\tau, S_{\rho(\tau)}]$ of $T(\tilde{S})$ corresponding to a point $\Phi(\tau)$ of $T(G)$ with $\rho_l(\tau) = \zeta$. Then there exist an integer ν_0 and a deformation $\alpha_l\colon \tilde{S} \to S_l$ such that the analytic mapping $K_l\colon E'_l \to X(a(S_l))$ sending ζ into $\langle a(S_{\rho(\tau)}), a(\alpha_l \circ f_\tau^{-1}),$

$a(S_l)\rangle$ is single-valued and has a holomorphic extension $\hat{K}_l\colon E \to X(a(S_l))$ for each $l = 1, \cdots, n_0$.

For each $l = 1, \cdots, n_0$ and $\zeta \in E$, we can canonically construct a finitely generated Kleinian group $H_l(\zeta)$ as follows. Let S_l have r_l parts $\Sigma_{l,1}, \cdots, \Sigma_{l,r_l}$ and k_l nodes $P_{l,1}, \cdots, P_{l,k_l}$, and let $a(S_l)$ have r'_l parts $\Sigma_{l,1}, \cdots, \Sigma_{l,r'_l}$ and k'_l nodes $P_{l,1}, \cdots, P_{l,k'_l}$. Assume that each part $\Sigma_{l,j}$ has genus $g_{l,j}$ and $n_{l,j}$ punctures. We choose r'_l Fuchsian groups $H_{l,1}, \cdots, H_{l,r'_l}$ acting on discs $\Delta_{l,1}, \cdots, \Delta_{l,r'_l}$ with disjoint closures such that (i) $H_{l,j}$ has $n_{l,j}$ non-conjugate maximal subgroups with the same fixed order $\nu > 3$, (ii) the Riemann surface $\Delta_{l,j}/H_{l,j}$ with the images of all elliptic vertices removed is conformally equivalent to $\Sigma_{l,j}$ and (iii) $H_{l,1}, \cdots, H_{l,r'_l}$ generate a Kleinian group H_l with an invariant component Δ_0. Let $\Delta'_{l,j}$ be the complement in $\Delta_{l,j}$ of the set of elliptic fixed points of $H_{l,j}$. Let $\Omega(H_l)$ be the region of discontinuity of H_l and let $\Omega'(H_l)$ be the complement in $\Omega(H_l)$ of the set of elliptic fixed points of H_l. We assign to each node $P_{l,i}$ of $a(S_l)$ two non-conjugate maximal elliptic subgroups $\Gamma'_{l,i}, \Gamma''_{l,i}$ of H_l so that, if $P_{l,i}$ joins Σ_{l,j_1} to Σ_{l,j_2}, then $\Gamma'_{l,i} \subset H_{l,j_1}$ and $\Gamma''_{l,i} \subset H_{l,j_2}$. Two elliptic vertices not lying in Δ_0 is called related if they are fixed under elliptic subgroups conjugate to either $\Gamma'_{l,i}$ or to $\Gamma''_{l,i}$. The $\Gamma_{l,i}$ are chosen so that the union of $\Delta_{l,j}/H_{l,j}$ with the images of any two related elliptic vertices identified is isomorphic to $a(S_l)$.

If $s_{l,i} \in \boldsymbol{C}$ is small and is not zero, then there exists a unique loxodromic Möbius transformation $h_{s_l,i}$ which conjugates $\Gamma'_{l,i}$ into $\Gamma''_{l,i}$, has the multiplier $s_{l,i}$ and has fixed points in Δ_{l,j_1} and Δ_{l,j_2}, where j_1 and j_2 are as before. We set $s_l = (s_{l,1}, \cdots, s_{l,k'_l})$. If $|s_l| = \max|s_{l,i}|$ is small, then H_l and $h_{s_l,i}$ generate a Kleinian group H_{l,s_l}. Let s_l be as before and let V be a quasiconformal automorphism of $\hat{\boldsymbol{C}}$ such that $V \circ H_{l,s_l} \circ V^{-1}$ is a Kleinian group, $V|\Delta_0$ is conformal and $V(z) = z + O(1/|z|)$ as $z \to \infty$. Then each $V|\Delta_{l,j}$ defines an element $\xi_{l,j}$ of the Teichmüller space $T(H_{l,j})$. If $s_{l,i} \neq 0$, set $\eta_{l,i} = a_{l,i} - \hat{a}_{l,i}$, where $a_{l,i}$ is the repelling fixed point of $V \circ h_{s_l,i} \circ V^{-1}$ and $\hat{a}_{l,i}$ is the fixed point of $V \circ \Gamma'_{l,i} \circ V^{-1}$ in $V(\Delta_{l,j})$. If $s_{l,i} = 0$, set $\eta_{l,i} = 0$. Then the point

$$(\xi_l, \eta_l) = (\xi_{l,1}, \cdots, \xi_{l,r'_l}, \eta_{l,1}, \cdots, \eta_{l,k'_l})$$

determines the Kleinian group $V \circ H_{l,s_l} \circ V^{-1}$ which is denoted by $H(\xi_l, \eta_l)$. The set of all points (ξ_l, η_l) for which a group $H(\xi_l, \eta_l)$ can be defined, is denoted by $X'(a(S_l))$. We say that such a V is a quasiconformal automorphism associated with (ξ_l, η_l). The deformation space $X(a(S_l))$ is canonically identified with $X'(a(S_l))$. Let $(\xi_l(\zeta), \eta_l(\zeta))$ be the point of $X'(a(S_l))$ corresponding to the point $\hat{K}_l(\zeta)$ of $X(a(S_l))$ for $\zeta \in E$. Denote by $H_l(\zeta)$ the finitely generated Kleinian group determined by the point $(\xi_l(\zeta), \eta_l(\zeta))$. Let $(\tilde{\xi}_l, \tilde{\eta}_l)$ be the point of $X'(a(S_l))$ corresponding to the point $\langle a(\tilde{S}), a(\alpha_l), a(S_l)\rangle$ of $X(a(S_l))$, $\tilde{H}_l$ the Kleinian group determined by $(\tilde{\xi}_l, \tilde{\eta}_l)$ and let $\tilde{V}_l$ be a quasiconformal automorphism of $\hat{C}$ associated with $(\tilde{\xi}_l, \tilde{\eta}_l)$. For each $j = 1, \cdots, r_l$, there is a component $\tilde{\Delta}_{l,j}(\subset \tilde{V}_l(\Delta_{l,j}))$ of the region of discontinuity of $\tilde{H}_l$ such that the Riemann surface $\tilde{\Delta}_{l,j}/\tilde{H}_{l,j}$ is conformally equivalent to $\tilde{S} = U/G$, where $\tilde{H}_{l,j}$ is the component subgroup of $\tilde{H}_l$ for $\tilde{\Delta}_{l,j}$. Hence there exists a holomorphic covering map $\tilde{f}_{l,j}: U \to \tilde{\Delta}_{l,j}$ with $\tilde{f}_{l,j} \circ G = \tilde{H}_{l,j} \circ \tilde{f}_{l,j}$. Let W^{μ_τ} be a quasiconformal automorphism of $\hat{C}$ corresponding to $\Phi(\tau)$ of $T(G)$ for $\tau \in D$ with $\rho_l(\tau) = \zeta \in E_l'$. Then there exists a unique quasiconformal automorphism $\tilde{V}_\zeta$ of $\hat{C}$ and a holomorphic covering map $f_\tau; D_\tau \to \tilde{V}_\zeta(\tilde{\Delta}_{l,j})$ such that $V_\zeta = \tilde{V}_\zeta \circ \tilde{V}_l$ is a quasiconformal automorphism associated with $(\xi_l(\zeta), \eta_l(\zeta))$, $\tilde{V}_\zeta \circ \tilde{H}_l \circ (\tilde{V}_\zeta)^{-1} = H_l(\zeta)$ and $\tilde{V}_\zeta \circ \tilde{f}_{l,j} = f_\tau \circ W^{\mu_\tau}$. We set $\Delta_{l,j}(\zeta) = \tilde{V}_\zeta(\tilde{\Delta}_{l,j})$, $H_{l,j}(\zeta) = \tilde{V}_\zeta \circ \tilde{H}_{l,j} \circ (\tilde{V}_\zeta)^{-1}$ and $h(\zeta, \cdot) = \tilde{V}_\zeta \circ h \circ (\tilde{V}_\zeta)^{-1}$ for $h \in \tilde{H}_l$. Then $\Delta_{l,j}(\zeta)$ is a component of $H_l(\zeta)$ with the component subgroup $H_{l,j}(\zeta)$ and $f_\tau \circ G_\tau = H_{l,j}(\zeta) \circ f_\tau$. Let $\Delta_{l,j}'(\zeta)$ be the complement in $\Delta_{l,j}(\zeta)$ of the set of elliptic vertices of $H_{l,j}(\zeta)$. We set

$$\mathscr{E}_{l,j} = \{(\zeta, w) \mid \zeta \in E_l', w \in \Delta_{l,j}(\zeta)\} ,$$
$$\mathscr{E}_{l,j}' = \{(\zeta, w) \mid \zeta \in E_l', w \in \Delta_{l,j}'(\zeta)\} ,$$
$$\hat{\mathscr{E}}_{l,j} = \{(\zeta, w) \mid \zeta \in E, w \in \Delta_{l,j}(\zeta)\} ,$$
$$\hat{\mathscr{E}}_{l,j}' = \{(\zeta, w) \mid \zeta \in E, w \in \Delta_{l,j}'(\zeta)\} ,$$

for each $l = 1, \cdots, n_0$ and $j = 1, \cdots, r_l$.

The above holomorphic coverings $f_\tau: D_\tau \to \Delta_{l,j}(\zeta)$ induce a holomorphic covering $F_{l,j}: \mathscr{D} \to \mathscr{E}_{l,j}$ sending (τ, w) into $(\rho_l(\tau), f_\tau(w))$.

For each $h \in \tilde{H}_{l,j}$, the conformal automorphism $h(\zeta, \cdot)$ of $\Delta_{l,j}(\zeta)$ induces an analytic automorphism $\hat{h}$ of $\mathscr{E}_{l,j}$ sending (ζ, w) into $(\zeta, h(\zeta, w))$. Then $\mathscr{H}_{l,j} = \{\hat{h} \mid h \in \tilde{H}_{l,j}\}$ is a properly discontinuous group of analytic automorphisms of $\mathscr{E}_{l,j}$. It is noted that each element $\hat{h}$ of $\mathscr{H}_{l,j}$ has a holomorphic extension on $\hat{\mathscr{E}}_{l,j}$.

Let τ_l be a cusp for Γ with $t_l = \rho(\tau_l)$ and let γ_{τ_l} be a generator of the stabilizer Γ_{τ_l} of τ_l in Γ. Then the element $\gamma_{l,\tau_l} = (\gamma_{\tau_l})^{\nu_0}$ is a generator of the stabilizer Γ_{l,τ_l} of τ_l in Γ_l. We set $\zeta_l = \rho_l(\tau_l)$. We may assume that $\hat{K}_l(\zeta_l) = \langle \mathrm{id} \rangle$, which implies that $H_l(\zeta_l) = H_l$. By a reasoning similar to that in § 4.1 of [5], we can prove that f_τ converges uniformly to a holomorphic covering map f_{τ_l} of a certain component $\Omega_{\tau_l,j}$ of G_{τ_l} onto the component $\Delta_{l,j}$ of H_l on any compact subset of $\Omega_{\tau_l,j}$ as τ tends to τ_l through any cusp region at $\tau = \tau_l$. If a component $\Omega_{\tau_l,i}$ of G_{τ_l} is not $\Omega_{\tau_l,j}$, then f_τ converges to a constant map on any compact subset of $\Omega_{\tau_l,i}$

as τ tends to τ_l through any cusp region at $\tau = \tau_l$. For the component subgroup $G_{\tau_l,j}$ of G_{τ_l} for $\Omega_{\tau_l,j}$, we have $f_{\tau_l} \circ G_{\tau_l,j} = H_{l,j} \circ f_{\tau_l}$. Moreover, we can prove that $\tilde{V}_\zeta \circ (\tilde{V}_l \circ h \circ (\tilde{V}_l)^{-1}) \circ (\tilde{V}_\zeta)^{-1} = V_\zeta \circ h \circ V_\zeta^{-1}$ converges uniformly to h for each $h \in H_l$ and $\tilde{V}_\zeta \circ \tilde{h} \circ (\tilde{V}_\zeta)^{-1}$ converges uniformly to a constant for each $\tilde{h} \in \tilde{H}_l - \tilde{V}_l \circ H_l \circ (\tilde{V}_l)^{-1}$ on any compact subset of $\Omega'(H_l)$ as ζ tends to ζ_l.

Let $\Gamma = \sum_{j=0}^{\infty} \Gamma_{l,\tau_l} \circ \gamma_j$ and $\mathscr{G}_{l,\tau_l} = \{(\gamma, g) \mid \gamma \in \Gamma_{l,\tau_l}, g \in G\}$. Let $\omega_{\gamma_{l,\tau_l}} \in N(G)$ be the quasiconformal automorphism of U with $\langle \omega_{\gamma_{l,\tau_l}} \rangle = \mathscr{M}_{\gamma_{l,\tau_l}}$, where $\mathscr{M}_{\gamma_{l,\tau_l}}$ is the homotopic monodromy of $(\mathscr{S}, \pi, R)$ for γ_{l,τ_l}. Since $K_l \circ \rho_l \circ \gamma_{l,\tau_l} = K_l \circ \rho_l$, we may assume that for a certain positive integer ν_0, $\omega_{\gamma_{l,\tau_l}}$ is induced by a quasiconformal automorphism of $\tilde{S}$ which is homotopic to a product of ν-th powers of Dehn twists about Jordan curves on $\tilde{S}$ mapped by α_l into nodes of S_l for each $l = 1, \cdots, n_0$. Then we have $F_{l,j} \circ \mathscr{G}_{l,\tau_l} = \mathscr{H}_{l,j} \circ F_{l,j}$ and $H_{(\gamma_{l,\tau_l},1)}(\Omega_{\tau_l,j}) = \Omega_{\tau_l,j}$. Hence $F_{l,j}$ induce a biholomorphic mapping of $\mathscr{D}/\mathscr{G}_{l,\tau_l}$ onto $\mathscr{E}_{l,j}/\mathscr{H}_{l,j}$.

By using these facts, we will construct certain automorphic forms on $\mathscr{D}$ for $\mathscr{G}$ in §7.

4. Behaviour of automorphic forms for $\mathscr{G}$ at cusps. We determine the behaviour of a q-form Ψ on $\mathscr{D}$ for $\mathscr{G}$ near a cusp $(\tau_0, w_0) \in \mathscr{C}$ as follows.

(i) If $\tau_0 \in C$, that is, τ_0 is a cusp of Γ, and if $w_0 \in D_{\tau_0}$, then the stabilizer Γ_{τ_0} of τ_0 in Γ is generated by a parabolic transformation γ_{τ_0}. There is a Möbius transformation A of the upper half-plane U onto the unit disc D with $A^{-1} \circ \gamma_{\tau_0} \circ A(\tau) = \tau + c_0$ for a positive constant c_0. Since $\Phi(\tau)$ converges uniformly to $\Phi(\tau_0)$ as τ tends to τ_0 through any cusp region Δ at $\tau = \tau_0$ in D, there is a positive constant δ such that $N_\delta = (|w - w_0| < \delta)$ is contained in D_τ for every $\tau \in \Delta$. (See § 4.1 of [5].) We assume that Δ is the image of the strip region $E_{a,b} = \{t \in U \mid -a < \mathrm{Re}(t) < a, \mathrm{Im}(t) > b\}$ by A, where a and b are positive constants. We set

$$\mathscr{D}^* = \{(t, w) \mid t \in U, w \in D_{A(t)}\} ,$$

and

$$\mathscr{A}(t, w) = (A(t), w) \quad \text{for} \quad (t, w) \in \mathscr{D}^* .$$

Then $\mathscr{A}\colon \mathscr{D}^* \to \mathscr{D}$ is a biholomorphic mapping and

$$(\mathscr{A}^*\Psi)(t, w) = \Psi(\mathscr{A}(t, w))[J_{\mathscr{A}}(t, w)]^q$$

is a q-form on $\mathscr{D}^*$ for $\mathscr{A}^{-1} \circ \mathscr{G} \circ \mathscr{A}$. The behaviour of Ψ near (τ_0, w_0) is determined by that of $\mathscr{A}^*\Psi$ near (∞, w_0) in $E_{a,b} \times N_\delta$.

(ii) Since $\hat{\mathscr{S}}$ is a two-dimensional compact normal complex space and since the cusps for $\mathscr{G}$ except in the case (i) corresponds to a set of finitely many points of $\hat{\mathscr{S}}$, every meromorphic mapping of $\hat{\mathscr{S}}-\{$finitely many points of $\hat{\mathscr{S}}\}$ into a projective space $P_N(C)$ is extended to a meromorphic mapping of $\hat{\mathscr{S}}$ into $P_N(C)$. Thus it is sufficient to study only the behaviour of Ψ near cusps in the case (i).

5. Poincaré metric and Poincaré series. We shall briefly recall some well-known results on the Poincaré metric and on the Poincaré series.

1. Let Ω be a domain on the Riemann sphere whose boundary consists of more than two points. Let $\lambda_\Omega(z)|dz|$ be the Poincaré metric for Ω. We call λ_Ω the Poincaré density of this metric. Then the following proposition is well known. (See Kra [6, Chap. II, Prop. 1.1].)

PROPOSITION A.
(a) *If $f: \Omega \to \Omega_1$ is a conformal mapping, then*

$$\lambda_{\Omega_1}(f(z))|f'(z)| = \lambda_\Omega(z)\ , \quad z \in \Omega\ .$$

(b) *If $\Omega_1 \subset \Omega$, then $\lambda_\Omega(z) \leqq \lambda_{\Omega_1}(z)$ for $z \in \Omega_1$.*
(c) *Let $\delta_\Omega(z) = \inf\{|z-\zeta|;\ \zeta \in \partial\Omega\}$. Then*

$$\lambda_\Omega(z)\delta_\Omega(z) \leqq 1\ , \quad z \in \Omega\ .$$

(d) *If Ω is connected and simply connected and if $\infty \in \Omega$, then*

$$\lambda_\Omega(z)\delta_\Omega(z) \geqq 1/4\ .$$

2. Let Γ be a finitely generated Fuchsian group of the first kind with translations acting on the upper half-plane U. Let Γ_∞ be the stabilizer of ∞ for Γ. Then Γ_∞ is generated by a parabolic element $\gamma_\infty(z) = z + c$ with $c > 0$. Writing $\Gamma_\infty\backslash\Gamma = \Gamma_\infty\gamma_0 + \Gamma_\infty\gamma_1 + \cdots$, we have a system $(\Gamma_\infty\backslash\Gamma) = \{\gamma_i | i = 0, 1, 2, \cdots\}$ of representatives of the cosets $\Gamma_\infty\backslash\Gamma$. The following proposition is also known. (See Lehner [7, Chap. 2, Prop. 1.B and Prop. 1.E].)

PROPOSITION B. *For any integer $q > 1$, the series*

$$\sum_{i=0}^{\infty} |\gamma_i'(z)|^q$$

converges for each $z \in U$ and converges uniformly on each closed region

$$E_a = \{z = x + iy \mid |x| \leqq a^{-1},\ y \geqq a > 0\}\ .$$

Let x_0 be a parabolic fixed point for Γ on the real axis which is not equivalent to ∞ under Γ and take the real Möbius transformation $\alpha(z) = (zx_0 - 1)/z$ sending x_0 into ∞. Then the series

$$\sum_{j=0}^{\infty} |(\gamma_j \circ \alpha)'(z)|^q$$

converges to zero uniformly as z *tends to* ∞ *through* E_a.

3. Let X be a bounded domain in C^n and let H be a discrete subgroup of the analytic automorphism group of X. For any bounded holomorphic function f on X, we set

$$P_f(x) = \sum_{h \in H} f(h(x)) J_h(x)^q$$

for $x \in X$. This series is called the Poincaré series of weight q for H. The following proposition holds. (See Baily [1, Chap. 5, Prop. 1].)

PROPOSITION C. *The Poincaré series* P_f *converges absolutely and uniformly on each compact subset of* X *for* $q \geqq 2$ *and is a holomorphic* q*-form on* X *for* H.

We denote by H_a the stabilizer of $a \in X$ in H. Let $\mathscr{N}$ be a neighbourhood of the origin O in C^n and let λ_a be a biholomorphic mapping of $\mathscr{N}$ onto a neighbourhood $\mathscr{U}$ of a stable under H_a with $\lambda_a(O) = a$ and $|J_{\lambda_a}(O)| = 1$. We may assume that $\mathscr{N}$, $\mathscr{U}$ and λ_a are chosen in such a way that (1) $h \in H$, $h(\mathscr{U}) \cap \mathscr{U} \neq \emptyset$ imply $h \in H_a$, and (2) $\tilde{H}_a = \lambda_a^{-1} \circ H_a \circ \lambda_a$ acts on $\mathscr{N}$ by linear transformations.

If f is a holomorphic function on a neighbourhood of a satisfying $f(h(x))J_h(x)^q = f(x)$ for all $h \in H_a$ when $h(x)$ is contained in the domain of definition $\operatorname{def}(f)$ of f, then we say f is a local automorphic form of weight q with respect to H_a. For such a function f, define $\lambda_a^* f$ by

$$(\lambda_a^* f)(\zeta) = f(\lambda_a(\zeta)) J_{\lambda_a}(\zeta)^q$$

for $\zeta \in \mathscr{N} \cap \lambda_a^{-1}(\operatorname{def}(f))$. Then we have

$$(\lambda_a^* f)(\tilde{h}(\zeta)) = (\lambda_a^* f)(\zeta) J_{\tilde{h}}(\zeta)^{-q}$$

for each $\tilde{h} \in \tilde{H}_a$. Since $\tilde{h} \in \tilde{H}_a$ is linear, $J_{\tilde{h}}(\zeta)$ is a constant N-th root of unity, where N is the order of H_a. Let $\mathscr{A}(q)_a$ denote the linear space of germs of local automorphic forms of weight q for H_a at a. For each q devisible by N, the mapping λ_a^* is an isomorphism of $\mathscr{A}(q)_a$ onto the ring $\mathscr{O}(\tilde{H}_a)$ of germs of $\tilde{H}_a$-invariant holomorphic functions at O. Let $\mathscr{M}_0$ be the maximal ideal in the ring $\mathscr{O}_0$ of germs of holomorphic functions at O in C^n, that is,

$$\mathscr{M}_0 = \{f \in \mathscr{O}_0 \mid f(O) = 0\} .$$

Then we know that the following proposition holds. (See Baily [1, Chap. 5, Theorem 10].)

PROPOSITION D. *Let $a_1, \cdots, a_k \in X$ belong to distinct orbits of H and let a positive integer l be given. Let $f_i \in \mathcal{O}(\tilde{H}_{a_i})$ be given for $i = 1, \cdots, k$. Then there exists a positive integer q and a Poincaré series P_f of weight q for H such that*

$$\lambda_{a_i}^* P_f \equiv f_i \qquad \mathrm{mod}\, \mathcal{M}_0{}^{l+1}$$

in a neighborhood of O for each $i = 1, \cdots, k$.

6. Poincaré series on $\mathscr{D}$ for $\mathscr{G}$. Let f be an arbitrary bounded holomorphic function on the domain $\mathscr{D}$ defined in §2. Assume that $|f| \leqq M$ on $\mathscr{D}$. We set

$$\begin{aligned} P_f(\tau, w) &= \sum_{(\gamma, g)} f[(\gamma, g)(\tau, w)][J_{(\gamma,g)}(\tau, w)]^q \\ &= \sum_{(\gamma, g)} f[(\gamma(\tau), H_{(\gamma,g)}(\tau, w)]H'_{(\gamma,g)}(\tau, w)^q \gamma'(\tau)^q \end{aligned}$$

for $(\tau, w) \in \mathscr{D}$, where (γ, g) runs through $\Gamma \times G$, $H'_{(\gamma,g)}(\tau, w) = \partial H_{(\gamma,g)}(\tau, w)/\partial w$ and $q \geqq 2$ is an arbitrary integer. By Proposition C, this Poincaré series P_f converges absolutely and uniformly on any compact subset of $\mathscr{D}$ and is a holomorphic q-form for $\mathscr{G}$.

We study the behaviour of P_f near a cusp for $\mathscr{G}$. Let (τ_0, w_0) be a cusp for $\mathscr{G}$ such that τ_0 is a cusp for Γ and $w_0 \in D_{\tau_0}$. We use the notations of §4 and §5. Let $\Gamma^* = A^{-1} \circ \Gamma \circ A$ and $\gamma^* = A^{-1} \circ \gamma \circ A$ for each $\gamma \in \Gamma$. The stabilizer Γ_∞^* of ∞ in Γ^* is generated by $\gamma_0^* = A^{-1} \circ \gamma \circ A$ which is a translation $\gamma_0^*(\tau) = \tau + c_0$ with a positive constant c_0. Let $\{\gamma_i^* \mid i = 0, 1, 2, \cdots\}$ be a system of representatives of the left cosets $\Gamma_\infty^* \backslash \Gamma^*$.

LEMMA 1. *There exists a positive constant C_1 such that*

$$\sum_{g \in G} |H'_{(\gamma,g)}(\tau, w)|^q \leqq C_1$$

on $\underset{\cdot}{A}(E_{a,b}) \times N_\delta$ for each $\gamma \in \Gamma$.

PROOF. Let λ_τ be the Poincaré density of D_τ and F_τ a fundamental domain for G_τ. We set $g_\tau(w) = H_{(1,g)}(\tau, w)$ for each $g \in G$. Since $(\gamma, g) = (1, \omega_\gamma \circ g \circ \omega_\gamma^{-1})(\gamma, 1)$, we have $H_{(\gamma,g)}(\tau, w) = (\omega_\gamma \circ g \circ \omega_\gamma^{-1})_{\gamma(\tau)} \circ H_{(\gamma,1)}(\tau, w)$. Hence,

$$\begin{aligned} \sum_{g \in G} |H'_{(\gamma,g)}(\tau, w)|^q &= \sum_{g \in G} |\{(\omega_\gamma \circ g \circ \omega_\gamma^{-1})_{\gamma(\tau)} \circ H_{(\gamma,1)}\}'(\tau, w)|^q \\ &= \sum_{g \in G} |\{g_{\gamma(\tau)} \circ H_{(\gamma,1)}\}'(\tau, w)|^q \\ &= \sum_{g \in G} |g'_{\gamma(\tau)}(H_{(\gamma,1)}(\tau, w))|^q |H'_{(\gamma,1)}(\tau, w)|^q \end{aligned}$$

and

$$\iint_{F_\tau} \lambda_\tau(w)^{2-q} \sum_{g \in G} |H'_{(\gamma,g)}(\tau, w)|^q |dw \wedge d\bar{w}|$$

$$= \sum_{g \in G} \iint_{F_\tau} \lambda_\tau(w)^{2-q} |g'_{\gamma(\tau)}(H_{(\gamma,1)}(\tau, w))|^q |H'_{(\gamma,1)}(\tau, w)|^q |dw \wedge d\bar{w}|$$

$$= \sum_{g \in G} \iint_{F_{\gamma(\tau)}} \lambda_{\gamma(\tau)}(z)^{2-q} |g'_{\gamma(\tau)}(z)|^q |dz \wedge d\bar{z}|$$

$$= \sum_{g \in G} \iint_{g_{\gamma(\tau)}(F_{\gamma(\tau)})} \lambda_{\gamma(\tau)}(z)^{2-q} |dz \wedge d\bar{z}|$$

$$= \iint_{D_{\gamma(\tau)}} \lambda_{\gamma(\tau)}(z)^{2-q} |dz \wedge d\bar{z}| \leqq \iint_{\Omega} \lambda_\Omega(z)^{2-q} |dz \wedge d\bar{z}| = K_1$$

for each $\tau \in D$, where $z = H_{(\gamma,1)}(\tau, w)$, $F_{\gamma(\tau)} = H_{(\gamma,1)}(\tau, F_\tau)$ and $\Omega = (|z| < 2)$ which contains $D_{\gamma(\tau)}$ for each $\tau \in D$. We may assume that $N_{3\delta} = (|w - w_0| < 3\delta)$ is contained in F_τ for each $\tau \in A(E_{a,b})$. Since $\lambda_\tau(w)\delta_{D_\tau}(w) \leqq 1$ by Proposition A, we have $\delta_{D_\tau}(w)^{q-2} \leqq \lambda_\tau(w)^{2-q}$ for $w \in D_\tau$. Hence

$$\sum_{g \in G} \iint_{N_{2\delta}} |H'_{(\gamma,g)}(\tau, w)|^q |dw \wedge d\bar{w}|$$

$$\leqq \delta^{2-q} \sum_{g \in G} \iint_{N_{2\delta}} \delta_{D_\tau}(w)^{q-2} |H'_{(\gamma,g)}(\tau, w)|^q |dw \wedge d\bar{w}|$$

$$\leqq \delta^{2-q} \sum_{g \in G} \iint_{F_\tau} \lambda_\tau(w)^{2-q} |H'_{(\gamma,g)}(\tau, w)|^q |dw \wedge d\bar{w}|$$

$$\leqq \delta^{2-q} K_1 .$$

Therefore, there exists a positive constant C_1 such that

$$\sum_{g \in G} |H'_{(\gamma,g)}(\tau, w)|^q \leqq C_1$$

for each $(\tau, w) \in A(E_{a,b}) \times N_\delta$ and for each $\gamma \in \Gamma$.

LEMMA 2. *There exists a positive constant C_2 such that*

$$\sum_{n=-\infty}^{\infty} |A'[(\gamma_0^*)^n \circ \gamma_j^*(t)]|^q \leqq C_2$$

on $E_{a,b}$ for each $j = 0, 1, 2, \cdots$.

PROOF. Let $A(t) = e^{i\theta}(t - i\alpha)/(t + i\alpha)$ and $\gamma_0^*(t) = t + c_0$, where α and c_0 are positive real numbers and θ is real. Set $\gamma_j^*(t) = u + iv$ with $v > 0$. Then we have

$$\sum_{n=-\infty}^{\infty} |A'[(\gamma_0^*)^n \circ \gamma_j^*(t)]|^q = (2\alpha)^q \sum_{n=-\infty}^{\infty} \frac{1}{\{(u + c_0 n)^2 + (\alpha + v)^2\}^q}$$

$$\leqq (2\alpha)^q \left\{ \int_{-\infty}^{\infty} \frac{dx}{\{(u + c_0 x)^2 + (\alpha + v)^2\}^q} + \frac{2}{(\alpha + v)^{2q}} \right\}$$

$$\leqq (2\alpha)^q \left\{ \frac{\pi}{c_0(\alpha + v)^{2q-1}} + \frac{2}{(\alpha + v)^{2q}} \right\}$$

$$\leqq \frac{2^q \pi}{c_0 \alpha^{q-1}} + \frac{2^{q+1}}{\alpha^q} = C_2 .$$

Now we have the following:

THEOREM 1. *Let* (τ_0, w_0) *be a cusp for* $\mathscr{G}$ *such that* τ_0 *is a cusp for* Γ *and* $w_0 \in D_{\tau_0}$. *Then* $(\mathscr{A}^* P_f)(t, w)$ *converges to zero uniformly as* (t, w) *tends to* (∞, w_1) *with* $w_1 \in N_\delta$ *through* $E_{a,b} \times N_\delta$.

PROOF.

$$\begin{aligned} |\mathscr{A}^* P_f(t, w)| &= |\sum_{\gamma, g} f[(\gamma, g)(A(t), w)][J_{(\gamma,g)}(A(t), w)]^q A'(t)^q| \\ &\leqq \sum_{\gamma, g} | f[(\gamma, g)(A(t), w)]| |H'_{(\gamma,g)}(A(t), w)(A \circ \gamma^*)'(t)|^q \\ &\leqq M \sum_{j=0}^{\infty} [\sum_{n=-\infty}^{\infty} \{ \sum_{g \in G} |H'_{(\gamma_0^n \circ \gamma_j, g)}(A(t), w)|^q \} |A'[(\gamma_0^*)^n \gamma_j^*(t)]|^q] |(\gamma_j^*)'(t)|^q \end{aligned}$$

and the series on the right hand side converges to zero uniformly as (t, w) tends to (∞, w_1) through $E_{a,b} \times N_\delta$ by Proposition B and Lemmas 1, 2. This proves our Theorem 1.

Let $a = (\tau_0, w_0)$ be a point of $\mathscr{D}$ and let G_{τ_0, w_0} be the stabilizer of w_0 in G_{τ_0}. We use the notations of § 5.3.

Case 1. $G_{\tau_0, w_0} = \{\mathrm{id}\}$. We set $(x, y) = (\tau - \tau_0, w - w_0)$ and $\lambda_a(x, y) = (x + \tau_0, y + w_0)$. Then (x, y) are local coordinates of $\mathscr{D}/\mathscr{G}$ in a neighbourhood of $[\tau_0, w_0]$. Since the stabilizer $\mathscr{G}_a$ of a in $\mathscr{G}$ is trivial, the group $\tilde{\mathscr{G}}_a = \lambda_a^{-1} \circ \mathscr{G}_a \circ \lambda_a$ is also trivial. Therefore, each element of $\mathscr{O}(\tilde{\mathscr{G}}_a)$ is a convergent power series

$$\sum_{n,m=0}^{\infty} a_{nm} x^n y^m .$$

Case 2. G_{τ_0, w_0} is generated by an elliptic transformation g_{τ_0}. The transformation $\hat{w} = g_\tau(w)$ is given by the relation

$$(\hat{w} - \xi_1(\tau))/(\hat{w} - \xi_2(\tau)) = \exp(2\pi i/\nu)(w - \xi_1(\tau))/(w - \xi_2(\tau)) ,$$

where $\xi_1(\tau) \neq \xi_2(\tau)$ are holomorphic functions of $\tau \in D$ and $\xi_1(\tau_0) = w_0$. We set

$$(t, z) = ((\xi_1(\tau_0) - \xi_2(\tau_0))(\tau - \tau_0), (w - \xi_1(\tau))/(w - \xi_2(\tau))) ,$$

$\lambda_a(t, z) = (\tau, w)$ and $(x, y) = (t, z^\nu)$. For the stabilizer $\mathscr{G}_a$ of a in $\mathscr{G}$, the group $\tilde{\mathscr{G}}_a = \lambda_a^{-1} \circ \mathscr{G}_a \circ \lambda_a$ is generated by the linear transformation sending (t, z) into $(t, (\exp 2\pi i/\nu) z)$. Since each element ϕ of $\mathscr{O}(\tilde{\mathscr{G}}_a)$ is a convergent power series

$$\sum_{n,m=0}^{\infty} a_{nm} t^n z^{\nu m} ,$$

the function ϕ is regarded as a holomorphic function of $(x, y) = (t, z^{\nu})$ and (x, y) are local coordinates of $\mathscr{D}/\mathscr{G}$ in a neighbourhood of $[\tau_0, w_0]$.

Thus, for any point $a = (\tau_0, w_0) \in \mathscr{D}$ and for any automorphic form f_0 of weight q_0 for $\mathscr{G}$ with $f_0(a) \neq 0$, Proposition D implies that there exist two Poincaré series f_1 and f_2 for $\mathscr{G}$ of the same weight q such that

$$\partial((\lambda_a^* f_1)^d/(\lambda_a^* f_0)^{d_0}, (\lambda_a^* f_2)^d/(\lambda_a^* f_0)^{d_0})/\partial(x, y) \neq 0$$

at $(x, y) = 0$ for all positive integers d_0 and d with $d_0 q_0 = dq$.

Now we have the following.

PROPOSITION 1. *Let $a = (\tau_0, w_0)$ be a point of $\mathscr{D}$ and let f_0 be an automorphic form of weight q_0 on $\mathscr{D}$ for $\mathscr{G}$ with $f_0(a) \neq 0$. Then there exist two Poincaré series f_1, f_2 for $\mathscr{G}$ of the same weight q such that*

$$\partial((\lambda_a^* f_1)^d/(\lambda_a^* f_0)^{d_0}, (\lambda_a^* f_2)^d/(\lambda_a^* f_0)^{d_0})/\partial(x, y) \neq 0$$

at $(x, y) = 0$ for all positive integers d_0, d with $d_0 q_0 = dq$, where (x, y) are local coordinates of $\mathscr{D}/\mathscr{G}$ in a neighbourhood of $[\tau_0, w_0]$ so that $[\tau_0, w_0]$ is given by $(x, y) = 0$.

7. **Poincaré-Eisenstein series on $\mathscr{D}$ for $\mathscr{G}$.** We use the notations in § 3 and § 4, but for the sake of simplicity, let us simply denote B, f_t, σ and σ_j instead of $\rho_l \circ A$, $f_{A(t)}$, γ^* and γ_j^*, respectively.

For any bounded holomorphic function f on $\hat{\mathscr{E}}_{l,j}$, set

$$Q_f(\zeta, w) = \sum f[\zeta, h(\zeta, w)][h'(\zeta, w)]^q$$

for $(\zeta, w) \in \hat{\mathscr{E}}_{l,j}$, where $h(z, \cdot)$ runs through $H_l(\zeta)$ and $h'(\zeta, w) = \partial h(\zeta, w)/\partial w$. Proposition C implies that this Poincaré series Q_f is a holomorphic q-form on $\mathscr{E}_{l,j}$ for $\mathscr{H}_{l,j}$.

Let τ_l be a cusp for Γ with $t_l = \rho(\tau_l)$ and let γ_{τ_l} be a generator of the stabilizer Γ_{τ_l} of τ_l in Γ. The element $\gamma_{l,\tau_l} = (\gamma_{\tau_l})^{\nu_0}$ is a generator of the stabilizer Γ_{l,τ_l} of τ_l in Γ_l. Take a Möbius transformation $A: U \to D$ such that $A^{-1} \circ \gamma_{\tau_l} \circ A(t) = t + c$ for a positive constant c. Let $\Gamma_l^* = A^{-1} \circ \Gamma \circ A$, $\Gamma_{l,\tau_l}^* = A^{-1} \circ \Gamma_{l,\tau_l} \circ A$ and $\sigma = A^{-1} \circ \gamma \circ A$ for $\gamma \in \Gamma$. We set

$$\mathscr{D}_l^* = \{(t, w) \mid t \in U,\ w \in D_{A(t)}\} ,$$
$$H_{(\sigma,g)}(t, w) = H_{(\gamma,g)}(A(t), w) ,$$
$$\mathscr{A}(t, w) = (A(t), w) .$$

Then

$$R_f(t, w) = Q_f[B(t), f_t(w)][f_t'(w)]^q$$

is a q-form on $\mathscr{D}_l^*$ for $\mathscr{G}_{l,\tau_l}^* = \mathscr{A}^{-1} \circ \mathscr{G}_{l,\tau_l} \circ \mathscr{A}$. In fact, for each $(\sigma, g) \in \mathscr{G}_{l,\tau_l}^*$

with $\sigma = A^{-1}\circ\gamma\circ A$, we have $R_f[(\sigma, g)(t, w)][J_{(\sigma,g)}(t, w)]^q = \dot{Q}_f[B\circ\sigma(t), f_{\sigma(t)}\circ H_{(\sigma,g)}(t, w)][(f_{\sigma(t)}\circ H_{(\sigma,g)})'(t, w)]^q = Q_f[\rho_l\circ\gamma\circ A(t), f_{\gamma\circ A(t)}\circ H_{(\gamma,g)}(A(t), w)]\times [(f_{\gamma\circ A(t)}\circ H_{(\gamma,g)})'(A(t), w)]^q$. Since $\omega_{\gamma_{l,\tau_l}}$ of $N(G)$ is induced by a quasiconformal automorphism of $\tilde{S}$ which is homotopic to a product of ν-th powers of Dehn twists about Jordan curves on $\tilde{S}$ mapped by α_l into nodes of S_l, we see that, if ν_0 is sufficiently large, then there exists an element $h(B(t), \cdot)$ of $H_l(B(t))$ with

$$f_{\gamma\circ A(t)}\circ H_{(\gamma,g)}(A(t), w) = h(B(t), f_t(w))$$

for an element $\gamma\in\Gamma_{l,\tau_l}$. Since $\rho_l\circ\gamma = \rho_l$ for $\gamma\in\Gamma_{l,\tau_l}$ and $\sigma' = 1$ for $\sigma\in\Gamma_{l,\tau_l}$, we get $R_f[(\sigma, g)(t, w)][J_{(\sigma,g)}(t, w)]^q = Q_f[B(t), h(B(t), f_t(w))][\{h(B(t), f_t(w))\}']^q = Q_f[B(t), f_t(w)][f_t'(w)]^q = R_f(t, w)$. Hence R_f is a q-form on $\mathscr{D}_l^*$ for $\mathscr{G}_{l,\tau_l}^*$.

Let $\Gamma_l^* = \sum_{i=0}^{\infty}\Gamma_{l,\tau_l}^*\circ\sigma_i$. Set

$$E_f^*(t, w) = \sum_{i=0}^{\infty} R_f[(\sigma_i, 1)(t, w)][J_{(\sigma_i,1)}(t, w)]^q$$

for $(t, w)\in\mathscr{D}_l^*$. This series E_f^* is called a Poincaré-Eisenstein series for $\mathscr{G}_l^*$. Explicitly, E_f^* is given by

$$E_f^*(t, w) = \sum_{i=0}^{\infty}\{f[B\circ\sigma_i(t), h(B\circ\sigma_i(t), f_{\sigma_i(t)}\circ H_{(\sigma_i,1)}(t, w))]$$
$$\times[\{h(B\circ\sigma_i(t), f_{\sigma_i(t)}\circ H_{(\sigma_i,1)}(t, w))\}']^q\}\sigma_i'(t)^q ,$$

where $h(B\sigma_i(t), \cdot)$ runs through $H_l(B\circ\sigma_i(t))$ for $i = 0, 1, 2, \cdots$.

LEMMA 3. *Let t_0 be a point in U or a parabolic fixed point of Γ_l^* and let w_0 be a point in $D_{A(t_0)}$. Take a neighbourhood Δ of t_0 or a cusp region Δ at t_0 such that a neighbourhood N_δ of w_0 is contained in $D_{A(t)}$ for each t in Δ. Then there exists a positive constant not depending on $i = 0, 1, 2, \cdots$ such that*

$$\sum|\{h(B\circ\sigma_i(t), f_{\sigma_i(t)}\circ H_{(\sigma_i,1)}(t, w))\}'|^q \leqq C_3$$

on $\Delta\times N_\delta$, where $h(B\sigma_i(t), \cdot)$ runs through $H_l(B\circ\sigma_i(t))$.

PROOF. Let $\tau = A(t)$, $\zeta = B\circ\sigma_i(t)$, $\phi(w) = f_{\sigma_i(t)}\circ H_{(\sigma_i,1)}(t, w)$ and let λ_τ be the Poincaré density of D_τ. Let λ_h be the Poincaré density of the domain $\Delta_h = h(\zeta, \Delta_{l,j}(\zeta))$ for each $h\in H_l(\zeta)$. Since $h(\zeta, \phi(w)): D_\tau\to\Delta_h$ is a universal covering, by definition, $\lambda_h[h(\zeta, \phi(w))]|\{h(\zeta, \phi(w))\}'| = \lambda_\tau(w)$. Hence, for a fundamental domain F for G_τ,

$$\sum_h\iint_F\lambda_\tau(w)^{2-q}|\{h(\zeta, \phi(w))\}'|^q|dw\wedge d\bar{w}| = \sum_h\iint_{F_h}\lambda_h(z)^{2-q}|dz\wedge d\bar{z}| ,$$

where $F_h = h(\zeta, \phi(F))$ and $z = h(\zeta, \phi(w))$. Since $V_\zeta(z) = z + O(1/|z|)$ as z tends to ∞, Koebe's one-quarter theorem implies there is a positive constant r_0 such that Δ_h is contained in $D_0 = (|z| < r_0)$ for each $\zeta \in E$ and each $h(\zeta, \cdot) \in H_l(\zeta)$. If λ_0 is the Poincaré density of D_0, then $\lambda_h(z) \geqq \lambda_0(z)$ for $z \in \Delta_h$. Therefore,

$$\sum_h \iint_{F_h} \lambda_h(z)^{2-q} |dz \wedge d\bar{z}| \leqq \sum_h \iint_{F_h} \lambda_0(z)^{2-q} |dz \wedge d\bar{z}| \leqq \iint_{D_0} \lambda_0(z)^{2-q} |dz \wedge d\bar{z}| \leqq K_2$$

for each $t \in U$, where K_2 is a positive constant not depending on $i = 0, 1, 2, \cdots$. Hence, by the same reasoning as in the proof of Lemma 1, we can prove Lemma 3.

THEOREM 2. *The Poincaré-Eisenstein series E_f^* for $\mathscr{G}_l^*$ converges absolutely and uniformly on any compact subset of $\mathscr{D}_l^*$ and is a holomorphic q-form for $\mathscr{G}_l^*$.*

PROOF. Proposition B and Lemma 3 imply that E_f^* converges absolutely and uniformly on any compact subset of $\mathscr{D}_l^*$.

For each $(\sigma, g) \in \mathscr{G}_l^*$, we have

$$E_f^*[(\sigma, g)(t, w)][J_{(\sigma,g)}(t, w)]^q = \sum_{i=0}^{\infty} R_f[(\sigma_i \circ \sigma, g)(t, w)][J_{(\sigma_i \circ \sigma, g)}(t, w)]^q .$$

Since there exists an integer α_i and a non-negative integer k_i with $\sigma_i \circ \sigma = (\gamma_{l,\tau_l}^*)^{\alpha_i} \circ \sigma_{k_i}$ for each i, we have $(\sigma_i \circ \sigma, g) = ((\gamma_{l,\tau_l})^{\alpha_i}, g_i) \circ (\sigma_{k_i}, 1)$ with $g_i = \omega_{\gamma_{k_i}} \circ g \circ \omega_{\gamma_{k_i}}^{-1}$ and $\gamma_{k_i} = A \circ \sigma_{k_i} \circ A^{-1}$. Hence, $E_f^*[(\sigma, g)(t, w)] [J_{(\sigma,g)}(t, w)]^q = \sum_{i=0}^{\infty} R_f[(\sigma_{k_i}, 1)(t, w)][J_{(\sigma_{k_i},1)}(t, w)]^q = E_f^*(t, w)$. Therefore, E_f^* is a q-form for $\mathscr{G}_l^*$. This completes the proof of Theorem 2.

Now, we set

$$E_f(\tau, w) = ((\mathscr{A}^{-1})^* E_f^*)(\tau, w) .$$

Then E_f is a q-form on $\mathscr{D}$ for $\mathscr{G}$, which is called a Poincaré-Eisenstein series on $\mathscr{D}$ for $\mathscr{G}$.

We study the behaviour of E_f near cusps of $\mathscr{G}$.

THEOREM 3. *If $w_l \in D_{\tau_l}$, then $\mathscr{A}_l^* E_f$ is bounded in the domain $E_{a,b} \times N_\delta$ for (τ_l, w_l). If $w_l \in \Omega_{\tau_l, j}$, then $(\mathscr{A}_l^* E_f)(t, w)$ converges uniformly to*

$$E_f^0(\tau_l, w) = \sum_{i=0}^{\nu_0 - 1} \{ \sum_{h \in H_{l,j}} f[\rho_l(\tau_l), h \circ f_{\tau_l} \circ H_{(\gamma_i,1)}(\tau_l, w_1)][(h \circ f_{\tau_l} \circ H_{(\gamma_i,1)})'(\tau_l, w_1)]^q \}$$

as (t, w) tends to (∞, w_1) through $E_{a,b} \times N_\delta$, where $w_1 \in N_\delta$ and $\gamma_i = (\gamma_{\tau_l})^i$. Moreover, E_f^0 is a holomorphic q-form on $\Omega_{\tau_l, j}$ for the group generated by $G_{\tau_l, j}$ and $H_{(\gamma_i,1)}$, $n = 1, 2, \cdots, \nu_0 - 1$. On the other hand, if a parabolic fixed point τ_0 for Γ is not equivalent to τ_l under Γ and if $w_0 \in D_{\tau_0}$, then

$(\mathscr{A}^*E_f)(t, w)$ *converges to zero uniformly as* (t, w) *tends to* (∞, w_1) *with* $w_1 \in N_\delta$ *through* $E_{a,b} \times N_\delta$.

PROOF. By Proposition B and Lemma 3, it is clear that $\mathscr{A}_l^*E_f = E_f^*$ is bounded in $E_{a,b} \times N_\delta$ and it is also clear that E_f^* converges uniformly on any compact subset of $\Omega_{\tau_l,j}$ as (t, w) tends to (∞, w_1) through $E_{a,b} \times N_\delta$. Each covering $f_{\sigma_i(t)} \circ H_{(\sigma_i,1)}(t, w)$ converges uniformly to the covering $f_{\tau_l} \circ H_{(\gamma_i,1)}(\tau_l, w_1)$ as (t, w) tends to (∞, w_1) through $E_{a,b} \times N_\delta$. As stated in § 3, $V_\zeta \circ h \circ V_\zeta^{-1}$ converges uniformly to h for each $h \in H_l$, and $\tilde{V}_\zeta \circ \tilde{h} \circ (\tilde{V}_\zeta)^{-1}$ converges uniformly to a constant mapping for each $\tilde{h} \in \tilde{H}_l - \tilde{V}_l \circ H_l \circ (\tilde{V}_l)^{-1}$ on any compact subset of $\Omega'(H_l)$ as ζ tends to ζ_l. Therefore, if $w_l \in \Omega_{\tau_l,j}$, then $\lim_{(t,w)\to(\infty,w_1)}(\mathscr{A}_l^*E_f)(t, w) = E_f^*(\infty, w_1) = \sum_{i=0}^{\nu_0-1}\{\sum_{h\in H_{l,j}} f[\rho_l(\tau_l), h \circ f_{\tau_l} \circ H_{(\gamma_i,1)}(\tau_l, w_1)][(h \circ f_{\tau_l} \circ H_{(\gamma_i,1)})'(\tau_l, w_1)]^q\}$.

Let τ_0 be a cusp of Γ which is not equivalent to τ_l under Γ. We set

$$B(t) = (tA_l^{-1}(\tau_0) - 1)/t \ ,$$

$A = A_l \circ B$, $\mathscr{B}(t, w) = (B(t), w)$ and $\mathscr{A}(t, w) = (A(t), w)$. Then

$$(\mathscr{A}^*E_f)(t, w) = (\mathscr{B}^*E_f^*)(t, w) = E_f^*(B(t), w)B'(t)^q \ .$$

Hence, Proposition B and Lemma 3 imply that $(\mathscr{A}^*E_f)(t, w)$ converges to zero uniformly as (t, w) tends to (∞, w_1) through $E_{a,b} \times N_\delta$. This completes the proof of Theorem 3.

Now, by Propositions B, D and Theorem 3, it can be proved that for each $l = 1, \cdots, n_0$, there exist finitely many Poincaré-Eisenstein series $E_{f_{l,1}}, \cdots, E_{f_{l,\alpha_l}}$ on $\mathscr{D}$ for $\mathscr{G}$ of the same weight such that they have finitely many common zeros on the compactification of D_{τ_l}/G_{τ_l}. Therefore, $E_{f_{1,1}}, \cdots, E_{f_{1,\alpha_1}}, \cdots, E_{f_{n_0,1}}, \cdots, E_{f_{n_0,\alpha_{n_0}}}$ have finitely many common zeros on $\hat{\mathscr{D}}/\hat{\mathscr{G}} - \mathscr{D}/\mathscr{G}$.

Thus, we have the following.

COROLLARY. *Let* $\Sigma = \hat{\mathscr{D}}/\hat{\mathscr{G}} - \mathscr{D}/\mathscr{G}$. *Then there exist finitely many Poincaré-Eisenstein series* $E_1, \cdots, E_m$ *on* $\mathscr{D}$ *for* $\mathscr{G}$ *of the same weight* q_0 *such that they have finitely many common zeros on* Σ.

Now, we have the following.

THEOREM 4. *If* f_1 *and* f_2 *are non-zero Poincaré or Poincaré-Eisenstein series for* $\mathscr{G}$ *of the same weight* q, *then the quotient* $f = (f_1/f_2)^d$ *is a meromorphic function on* $\hat{\mathscr{S}}$ *for all positive integers* d_0, d *with* $d_0q_0 = dq$.

PROOF. Let Z_1 be the set of common zeros of $E_1, \cdots, E_m$ on Σ and let Z_2 be the set of points on Σ which correspond to cusps (τ_0, w_0) for

$\mathscr{G}$ with $w_0 \in \mathscr{P}_{\tau_0}$. Set $Z = Z_1 \cup Z_2$, which consists of finitely many points. Since f_1, f_2 are holomorphic on $\mathscr{D}$, it is clear that f is meromorphic on $\mathscr{D}/\mathscr{G}$. For each point $p \in \Sigma - Z$, there exists a Poincaré-Eisenstein series E_i for some $i = 1, \cdots, m$ such that $E_i(p) \neq 0$. By Theorems 1 and 3, the functions $(f_1^d)/(E_i^{d_0})$ and $(f_2^d)/(E_i^{d_0})$ are holomorphic and bounded in $U_p - \Sigma$, where U_p is a neighbourhood of p in $\hat{\mathscr{S}}$. Since $\hat{\mathscr{S}}$ is normal and Σ is a one-dimensional analytic subset of $\hat{\mathscr{S}}$, $(f_1^d)/(E_i^{d_0})$ and $(f_2^d)/(E_i^{d_0})$ are holomorphic in U_p, which implies that f is meromorphic on $\hat{\mathscr{S}} - Z$. Therefore, by Levi's extension theorem, f is meromorphic on $\hat{\mathscr{S}}$. This completes the proof of Theorem 4.

8. Bimeromorphic embedding of algebraic surfaces into projective spaces by automorphic forms.

THEOREM 5. *There exist holomorphic automorphic forms $\phi_0, \cdots, \phi_N$ of the same weight on $\mathscr{D}$ for $\mathscr{G}$ so that $\Phi = (\phi_0, \cdots, \phi_N)$ induces a bimeromorphic embedding of $\hat{\mathscr{S}} = \hat{\mathscr{D}}/\hat{\mathscr{G}}$ into the N-dimensional complex projective space $P_N(C)$.*

PROOF. Set $\Sigma = \hat{\mathscr{D}}/\hat{\mathscr{G}} - \mathscr{D}/\mathscr{G}$. There exist finitely many Poincaré-Eisenstein series $E_1, \cdots, E_m$ of the same weight q_0 on $\mathscr{D}$ for $\mathscr{G}$ such that the set Z_1 of their common zeros on Σ consists of finitely many points. Let Z_2 be the set of all points on Σ which correspond to cusps (τ_0, w_0) for $\mathscr{G}$ with $w_0 \in \mathscr{P}_{\tau_0}$.

For arbitrary non-zero Poincaré series f_0, f_1 for $\mathscr{G}$ of the same weight q, Theorem 4 implies that $F_1 = (f_1/f_0)^d$ is a meromorphic function on $\hat{\mathscr{S}}$ for all positive integers d_0, d with $d_0 q_0 = dq$. Let $I(F_1)$ be the set of points of indeterminacy of F_1. Set

$$\Delta(F_1) = \{(p, q) \mid F_1(p) = F_1(q),\ p, q \in \hat{\mathscr{S}} - I(F_1)\} .$$

Since $\Delta(F_1)$ is a three-dimensional analytic subset of $(\hat{\mathscr{S}} - I(F_1)) \times (\hat{\mathscr{S}} - I(F_1)) - (\hat{\mathscr{S}} \times I(F_1)) \cup (I(F_1) \times \hat{\mathscr{S}})$ and since $(\hat{\mathscr{S}} \times I(F_1)) \cup (I(F_1) \times \hat{\mathscr{S}})$ is a two-dimensional analytic subset of $\hat{\mathscr{S}} \times \hat{\mathscr{S}}$, Remmert-Stein's extension theorem implies that the closure of $\Delta(F_1)$ in $\hat{\mathscr{S}} \times \hat{\mathscr{S}}$ is a three-dimensional analytic subset of $\hat{\mathscr{S}} \times \hat{\mathscr{S}}$. Therefore, by Proposition D and Theorem 4, there exist finitely many Poincaré series $f_{i,0}, f_{i,1}$ for $\mathscr{G}$ of the same weight q_i for each $i = 1, \cdots, \alpha$ such that the mapping $F = (f_{1,0}, f_{1,1}; \cdots; f_{\alpha,0}, f_{\alpha,1})$ of $\hat{\mathscr{S}}$ into the product of α copies of $P_1(C)$ is meromorphic on $\hat{\mathscr{S}}$ and is injective on $\hat{\mathscr{S}} - \Sigma$.

For arbitrary non-zero Poincaré series g_0, g_1, g_2 for $\mathscr{G}$ of the same weight q', Theorem 4 implies that $G_1 = (g_1/g_0)^d$ and $G_2 = (g_2/g_0)^d$ are meromorphic functions on $\hat{\mathscr{S}}$ for all positive integers d_0, d with $d_0 q_0 = dq'$. Let $I(G_1, G_2)$ be the set of points of indeterminacy of G_1 or G_2 and let $\mathrm{Sing}(\hat{\mathscr{S}})$ be the set of singular points of $\hat{\mathscr{S}}$. Since $\hat{\mathscr{S}}$ is a normal complex space, $I(G_1, G_2)$ and $\mathrm{Sing}(\hat{\mathscr{S}})$ are analytic subsets of $\hat{\mathscr{S}}$ of codimension 2. The set $D(G_1, G_2)$ of points on $\hat{\mathscr{S}} - I(G_1, G_2) \cup \mathrm{Sing}(\hat{\mathscr{S}})$, where the mapping (G_1, G_2) is degenerate, is a one-dimensional analytic subset of $\hat{\mathscr{S}} - I(G_1, G_2) \cup \mathrm{Sing}(\hat{\mathscr{S}})$. By Remmert-Stein's extension theorem, the closure of $D(G_1, G_2)$ in $\hat{\mathscr{S}}$ is a one-dimensional analytic subset of $\hat{\mathscr{S}}$. Therefore, by Proposition 1 and Theorem 4, there exist finitely many Poincaré series $g_{j,0}$, $g_{j,1}$, $g_{j,2}$ for $\mathscr{G}$ of the same weight q'_j for each $j = 1, \cdots, \beta$ such that the mapping $G = (g_{1,0}, g_{1,1}, g_{1,2}; \cdots; g_{\beta,0}, g_{\beta,1}, g_{\beta,2})$ of $\hat{\mathscr{S}}$ into the product of β copies of $\boldsymbol{P}_2(\boldsymbol{C})$ is meromorphic on $\hat{\mathscr{S}}$ and is of maximal rank at every point of $\hat{\mathscr{S}} - \Sigma$.

We now use the well-known Segre mapping, that is, for any two projective spaces $\boldsymbol{P}_n(\boldsymbol{C})$ and $\boldsymbol{P}_m(\boldsymbol{C})$, the Segre mapping is an injective holomorphic mapping of $\boldsymbol{P}_n(\boldsymbol{C}) \times \boldsymbol{P}_m(\boldsymbol{C})$ into $\boldsymbol{P}_M(\boldsymbol{C})$, where $M = ((n+1) \times (m+1) - 1)$. By this Segre mapping, the above mappings F and G induce a meromorphic mapping Φ of $\hat{\mathscr{S}}$ into $\boldsymbol{P}_N(\boldsymbol{C})$, where $N = 2^\alpha 3^\beta - 1$. This mapping Φ is injective on $\hat{\mathscr{S}} - \Sigma$ and is of maximal rank at every point of $\hat{\mathscr{S}} - \Sigma$. We set

$$G_\Phi = \{(p, x) \mid x \in \Phi(p),\ p \in \hat{\mathscr{S}}\} .$$

Since Φ is a meromorphic mapping of $\hat{\mathscr{S}}$ into $\boldsymbol{P}_N(\boldsymbol{C})$, the graph G_Φ of Φ is a two-dimensional analytic subset of $\hat{\mathscr{S}} \times \boldsymbol{P}_N(\boldsymbol{C})$ and the projection p_1 of G_Φ onto $\hat{\mathscr{S}}$ is a proper modification. Let p_2 be the projection of G_Φ into $\boldsymbol{P}_N(\boldsymbol{C})$ and let $Y = p_2(G_\Phi)$. Then, by the proper mapping theorem, Y is an analytic subset of $\boldsymbol{P}_N(\boldsymbol{C})$. If p_Y is the projection of G_Φ onto Y, then p_Y induces a biholomorphic mapping of $G_\Phi - p_Y^{-1}(p_Y(\Sigma))$ onto $Y - p_Y(\Sigma)$, which implies that p_Y is a proper modification. Therefore $\Phi\colon \hat{\mathscr{S}} \to Y$ is a bimeromorphic mapping. This completes the proof of Theorem 5.

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DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
OSAKA UNIVERSITY
TOYONAKA, OSAKA 560
JAPAN