Tôhoku Math. Journ.
34 (1982), 133-139.

ESTIMATES FOR THE ASYMPTOTIC ORDER OF A GRÖTZSCH RING CONSTANT

GLEN D. ANDERSON AND MAVINA K. VAMANAMURTHY

(Received June 9, 1981)

Abstract. Asymptotic approximations in terms of n are obtained for the constant $\log \lambda_n = \lim_{a\to 0} (\operatorname{mod} R_{G,n}(a) + \log a)$ associated with the Grötzsch extremal ring $R_{G,n}$ in euclidean n-space, $n \geqq 3$.

1. **Definitions and notation.** By a *ring* R is meant a domain in finite euclidean n-space whose complement consists of two components C_0 and C_1, where C_0 is bounded. The *conformal capacity* of R (cf. [11]) is

$$\operatorname{cap} R = \inf_{\varphi} \int_R |\nabla \varphi|^n d\omega ,$$

where ∇ denotes the gradient, and where the infimum is taken over all real-valued C^1 functions φ in R with boundary values 0 on ∂C_0 and 1 on ∂C_1. Then the *modulus* of the ring R is defined by

$$\operatorname{mod} R = (\sigma_{n-1}/\operatorname{cap} R)^{1/(n-1)} ,$$

where for each positive integer p we let σ_p denote the p-dimensional measure of the unit sphere

$$S^p = \left\{(x_1, \cdots, x_{p+1}): \sum_{j=1}^{p+1} x_j^2 = 1\right\} .$$

Then

$$(1) \qquad \sigma_p = 2\pi^{(p+1)/2}\Gamma((p+1)/2)^{-1}$$

(cf. [9], [12]), where Γ denotes the classical Gamma function. For later reference we recall that

$$(2) \qquad \int_0^{\pi/2} \cos^p u\, du = \sigma_{p+1}/2\sigma_p$$

for each positive integer p (cf. [2]).

For $n \geqq 2$ and $0 < a < 1$ we let $R_{G,n} = R_{G,n}(a)$ denote the n-dimensional Grötzsch ring, that is, the ring whose complementary components are

$$C_0 = \{(x_1, \cdots, x_n): 0 \leqq x_1 \leqq a,\ x_j = 0,\ 2 \leqq j \leqq n\}$$

1980 Mathematics Subject Classification. Primary 30 C60. Secondary 41 A60.

and

$$C_1 = \left\{(x_1, \cdots, x_n) : \sum_{j=1}^{n} x_j^2 \geqq 1\right\} .$$

In [5] Gehring proved that mod $R_{G,3}(a) + \log a$ is monotone decreasing in the interval $0 < a < 1$. Using analogous methods in higher dimensions, Caraman [4] and Ikoma [9] have shown that the limit

$$\log \lambda_n = \lim_{a \to 0} (\text{mod } R_{G,n}(a) + \log a)$$

exists for each $n \geqq 3$.

Unfortunately, the exact value of λ_n is known only when $n = 2$, in which case $\lambda_2 = 4$. For $n \geqq 3$, some estimates have been given ([1], [2], [3], [4], [6], [9], [10]); in particular, the best known estimates for $n = 3$ and $n = 4$ presently are $9.1942 \cdots \leqq \lambda_3 \leqq 9.9002 \cdots$ and $21.685 \cdots \leqq \lambda_4 \leqq 26.046 \cdots$ ([1], [3]).

Knowledge of the values of λ_n would be helpful in proving other estimates in the theory of quasiconformal mappings (cf. [7]). However, since it is apparently so difficult to determine these constants exactly, it becomes interesting to obtain good estimates for them and to approximate the asymptotic behavior of λ_n as n becomes large. In an earlier paper [2] it was established that

$$\lim_{n \to \infty} \lambda_n^{1/n} = e . \tag{3}$$

In the course of the proof of (3), the upper bound

$$\log \lambda_n \leqq n - 1 + \log 2 , \quad n \geqq 3 , \tag{4}$$

was obtained.

The present authors give an improved upper bound for λ_n, though of the same order as (4), and provide a lower estimate for the asymptotic order of λ_n as a function of n as n becomes large. In particular, we prove the following result.

THEOREM. *For each integer* $n \geqq 3$,

$$\log \lambda_n \leqq n + 1/n - 3/2 + \log 2 \tag{5}$$

and

$$\liminf_{n \to \infty} (\log \lambda_n - n + (1/2) \log n) \geqq -1 + (1/2) \log (8/\pi) . \tag{6}$$

2. An upper bound for λ_n. In our proof of (5) we begin with the estimate (22) in [3]:

$$(7)\qquad \log(\lambda_n/4) \leqq (2\sigma_{n-2}/\sigma_{n-1})\int_0^{\pi/2}\cos^{n-2}u \times \int_0^\infty [(1+\cos^2 u\,\mathrm{csch}^2 v)^{(n-2)/(2n-2)}-1]dvdu\ .$$

Next, we put

$$(8)\qquad I_n = \int_0^\infty (\coth^{(n-2)/(n-1)}v - 1)dv = (2\sigma_{n-2}/\sigma_{n-1})\int_0^{\pi/2}\cos^{n-2}u\int_0^\infty[(1+\mathrm{csch}^2 v)^{(n-2)/(2n-2)}-1]dvdu\ .$$

Now if we apply the Mean value theorem to the function $f(x) = (1 + x\,\mathrm{csch}^2 v)^{(n-2)/(2n-2)}$ on the interval $[\cos^2 u, 1]$, we find that

$$(9)\qquad f(1) - f(\cos^2 u) = ((n-2)/(2n-2))\sin^2 u\,\mathrm{csch}^2 v(1 + c\,\mathrm{csch}^2 v)^{n/(2-2n)}\ ,$$

for some $c \in (\cos^2 u, 1)$. Combining (9) with the integrals in (7) and (8) and using the fact that the right side of (9) is monotonic decreasing as a function of c, we achieve the estimate

$$(10)\qquad \begin{aligned} I_n - \log(\lambda_n/4) &\geqq (2\sigma_{n-2}/\sigma_{n-1})\int_0^{\pi/2}\cos^{n-2}u \times \int_0^\infty[\coth^{(n-2)/(n-1)}v - (1+\cos^2 u\,\mathrm{csch}^2 v)^{(n-2)/(2n-2)}]dvdu \\ &\geqq ((n-2)/(2n-2))(2\sigma_{n-2}/\sigma_{n-1}) \times \int_0^{\pi/2}\cos^{n-2}u\sin^2 u du\int_0^\infty \mathrm{csch}^2 v\tanh^{n/(n-1)}v dv\ . \end{aligned}$$

Next, the substitution $t = \coth v$ gives

$$(11)\qquad \int_0^\infty \mathrm{csch}^2 v\tanh^{n/(n-1)} v dv = \int_1^\infty t^{n/(1-n)}dt = n-1\ ,$$

while (2) together with the reduction formula

$$(12)\qquad \int_0^{\pi/2}\cos^n x dx = (1-1/n)\int_0^{\pi/2}\cos^{n-2}x dx$$

lead to the evaluation

$$(13)\qquad \int_0^{\pi/2}\cos^{n-2}u\sin^2 u du = \sigma_{n-1}/(2n\sigma_{n-2})\ .$$

Thus (10), (11), and (13) yield the inequality

$$I_n - \log(\lambda_n/4) \geqq (n-2)/2n\ .$$

Finally, since $I_n \leqq n - 1 - \log 2$ (cf. [2]), we are led to the upper bound (5) above.

REMARK. If specific manageable upper bounds are needed for each n rather than estimates for the asymptotic order, some improvements are possible, aside from the difficult task of evaluating the integral on the right side of (7) numerically as was done in [3] for $n = 3$ and 4. For example, one may evaluate I_n, the integral in (8), numerically. Or by using the Taylor series with remainder instead of the Mean value theorem in the above proof of (5) one can obtain the slightly better estimate

$$(5') \qquad \log \lambda_n \leqq n + 1/n - 3/2 + \log 2 - (3/4)/(n-2)/((n+2)(2n-1)) - (5/2)(n-2)(3n-2)/((2n-1)(4n-3)(n+2)(n+4)) .$$

3. A lower estimate for the asymptotic order of λ_n. For our proof of (6) we require the following two technical lemmas.

LEMMA 1. *For each positive integer n,*

$$(14) \qquad (\pi/(2n+2))^{1/2} < \int_0^{\pi/2} \cos^n x dx < (\pi/2n)^{1/2} .$$

PROOF. For convenience we let $C_n = \int_0^{\pi/2} \cos^n x dx$. Clearly C_n is a strictly decreasing sequence, so that

$$(15) \qquad C_{2n+1} < C_{2n} < C_{2n-1} .$$

Using the exact evaluation of C_n in terms of Gamma functions (cf. (1) and (2) above) we may translate (15) into the statement

$$(16) \qquad \prod_{k=1}^{n} [2k/(2k+1)] < (\pi/2) \prod_{k=1}^{n} [(2k-1)/2k] < \prod_{k=1}^{n-1} [2k/(2k+1)] .$$

If we multiply throughout (16) by the middle term we have

$$(17) \qquad \pi/(4n+2) < (\pi/2)^2 \prod_{k=1}^{n} [(2k-1)/2k]^2 < \pi/4n .$$

Taking square roots throughout (17) then gives (14) for even integers. The proof for odd integers is similar.

We remark that the proof of Lemma 1 is a modification of the standard proof of Wallis' product for π. We also wish to mention that the estimate in Lemma 1 may be used to show that the $L_n[0, \pi/2]$-norm of $\cos x$ has the asymptotic limit

$$\lim_{n \to \infty} (n/\log n)(1 - \|\cos x\|_n) = 1 ,$$

a fact which can also be derived from considerations in [8].

LEMMA 2. *For each* $t \geqq 1$,

$$\left[\int_0^{\pi/2} \cos^n u\,du\right]\left[\int_0^{\pi/2} (1 + t^{-2}\tan^2 u)^{-n/2} du\right]^{-1} \geqq \max\,(1/t,\,((2/\pi)/(n+1))^{1/2}) .$$

PROOF. Making the change of variable $\tan v = t^{-1}\tan u$ in the denominator integral gives

$$\int_0^{\pi/2} (1 + t^{-2}\tan^2 u)^{-n/2} du = t\int_0^{\pi/2} \cos^{n-2} v/(1 + t^2\tan^2 v)dv \leqq t\int_0^{\pi/2} \cos^n v dv ,$$

and the first lower bound follows.

Next, Lemma 1 and the obvious estimate

$$\int_0^{\pi/2} (1 + t^{-2}\tan^2 u)^{-n/2} du \leqq \pi/2$$

together give the second lower bound. The lemma is proved.

To complete the proof of (6) we begin with the lower bound (27) of [1], which, after the change of variable $t = \coth v$ (cf. [2]), may be written as

$$\log\,(\lambda_n/4) \geqq \int_1^\infty \left[((2\sigma_{n-2}/\sigma_{n-1})\int_0^{\pi/2} (t^2 + \tan^2 u)^{1-n/2} du)^{1/(1-n)} - 1\right] \times (t^2 - 1)^{-1} dt . \tag{18}$$

For convenience we now adopt the following notation:

$$\begin{aligned} a_n &= ((2/\pi)/(n+1))^{1/(2n-2)} , \\ \Phi_n(t) &= \left((2\sigma_{n-2}/\sigma_{n-1})\int_0^{\pi/2} (t^2 + \tan^2 u)^{1-n/2} du\right)^{1/(1-n)} . \end{aligned} \tag{19}$$

Finally, let $M > 1$. Then by the first bound in Lemma 2,

$$\int_1^M (\Phi_n(t) - 1)(t^2 - 1)^{-1} dt \geqq \int_1^M (t^{(n-3)/(n-1)} - 1)(t^2 - 1)^{-1} dt . \tag{20}$$

By the Monotone convergence theorem, we have

$$\lim_{n\to\infty} \int_1^M (t^{(n-3)/(n-1)} - 1)(t^2 - 1)^{-1} dt = \int_1^M (t + 1)^{-1} dt = \log\,((M + 1)/2) . \tag{21}$$

Next, the second bound in Lemma 2 gives the estimate

$$\begin{aligned} &\int_M^\infty (\Phi_n(t) - 1)(t^2 - 1)^{-1} dt \\ &\qquad \geqq (n-1)a_n M^{1/(1-n)} + a_n\int_M^\infty t^{(n-2)/(n-1)-2}(t^2 - 1)^{-1}\,dt \\ &\qquad\qquad + (1/2)\log\,((M + 1)/(M - 1)) , \end{aligned} \tag{22}$$

where we have used the fact that $(t^2-1)^{-1}=t^{-2}+t^{-2}(t^2-1)^{-1}$. Again by the Monotone convergence theorem, the last integral above tends to

$$\int_M^\infty t^{-1}(t^2-1)^{-1}dt=\log(M(M^2-1)^{-1/2}) \tag{23}$$

as n tends to ∞, while a_n tends to 1.

Next, by the Mean value theorem

$$\begin{aligned}(n-1)a_nM^{1/(1-n)}&=(n-1)\exp[(1/(2-2n))\log((\pi/2)(n+1)M^2)]\\&=n-(1/2)\log n-1-(1/2)\log(\pi/2)\\&\quad-\log M+o(1)\end{aligned} \tag{24}$$

as n tends to ∞.

Finally, combining all of the formulas (18) through (24), we have

$$\liminf_{n\to\infty}(\log(\lambda_n/4)-n+(1/2)\log n)\geqq-1-(1/2)\log(2\pi)\ .$$

Simplification then yields (6).

REMARK. The lower bound for $\log\lambda_n$ which leads to (6) in the preceding argument is rather intractable and therefore not reported separately above. A simpler (though crude) lower bound for λ_n in terms of n may be obtained from (18) by using the first bound in Lemma 2 in the following way.

$$\begin{aligned}\log(\lambda_n/4)&\geqq\int_1^\infty(\Phi_n(t)-1)(t^2-1)^{-1}dt\\&\geqq\int_1^\infty(t^{(n-3)/(n-1)}-1)(t^2-1)^{-1}dt\\&=\int_1^\infty t^{-2}(t^{(n-3)/(n-1)}-1)dt+\int_1^\infty(t^{(n-3)/(n-1)}-1)t^{-2}(t^2-1)^{-1}dt\\&\geqq(n-3)/2+((n-3)/(n-1))\int_1^\infty t^{-3}(t+1)^{-1}dt\end{aligned}$$

by exact evaluation of the first integral in the preceding line and by application of the Mean value theorem to the function $t^{(n-3)/(n-1)}-1$ in the next integral. But the final integral above may be evaluated exactly as $\log 2-1/2$. Thus we are led to the estimate

$$\log\lambda_n\geqq(n-3)(n-2)/(2n-2)+((3n-5)/(n-1))\log 2\ .$$

REFERENCES

[1] G. D. ANDERSON, Extremal rings in n-space for fixed and varying n, Ann. Acad. Sci. Fenn. Ser. AI Math. No. 575 (1974), 1-21.

[2] G. D. ANDERSON, Dependence on dimension of a constant related to the Grötzsch ring, Proc. Amer. Math. Soc. 61 (1976), 77-80.

[3] G. D. ANDERSON, Limit theorems and estimates for extremal rings of high dimension, Proc. of the Romanian-Finnish Seminar on Complex Analysis 1976, Lecture Notes in Mathematics 743, Springer Verlag, Berlin, Heidelberg, New York, 1979, 10-34.
[4] P. CARAMAN, On the equivalence of the definitions of the n-dimensional quasiconformal homeomorphisms (QCfH), Rev. Roumaine Math. Pures Appl. 12 (1967), 889-943.
[5] F. W. GEHRING, Symmetrization of rings in space, Trans. Amer. Math. Soc. 101 (1961), 499-519.
[6] F. W. GEHRING, Inequalities for condensers, hyperbolic capacity, and extremal lengths, Michigan Math. J. 18 (1971), 1-20.
[7] F. W. GEHRING, A remark on domains quasiconformally equivalent to a ball, Ann. Acad. Sci. Fenn. Ser. AI Math. Vol. 2 (1976), 147-155.
[8] R. A. HANDELSMAN AND J. S. LEW, On the convergence of the L^p norm to the L^∞ norm, Amer. Math. Monthly 79 (1972), 618-622.
[9] K. IKOMA, An estimate for the modulus of the Grötzsch ring in n-space, Bull. Yamagata Univ. Natur. Sci. 6 (1967), 395-400.
[10] N. V. LAM, Eine Abschätzung des Modulus des Grötzschschen Ringes im Raum, Bull. Math. Soc. Sci. Math. R. S. Roumaine (N.S.) 23 (71) (1979), 359-366.
[11] C. LOEWNER, On the conformal capacity in space, J. Math. Mech. 8 (1959), 411-414.
[12] J. VÄISÄLÄ, Lectures on n-Dimensional Quasiconformal Mappings, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

MICHIGAN STATE UNIVERSITY
DEPARTMENT OF MATHEMATICS
EAST LANSING, MICHIGAN 48824
U.S.A.

AND

UNIVERSITY OF AUCKLAND
DEPARTMENT OF MATHEMATICS
AUCKLAND
NEW ZEALAND