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ON EXTREMAL QUASICONFORMAL MAPPINGS COMPATIBLE WITH A FUCHSIAN GROUP

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1. Introduction. Let U be the upper half-plane and let $\hat{\boldsymbol{R}} = \boldsymbol{R} \cup \{\infty\}$ be the extended real line. We denote by $PSL(2, \boldsymbol{R})$ the real Möbius group, that is, the group of all the conformal automorphisms of U. A discrete subgroup G of $PSL(2, \boldsymbol{R})$ is called a Fuchsian group. The limit set $\Lambda(G)$ of a Fuchsian group G is the derived set of the set which consists of all the images $\gamma(i)$ of the point $z = i$ under $\gamma \in G$. We say that a Fuchsian group G is non-elementary whenever $\Lambda(G)$ contains more than two points. A Fuchsian group G is said to be of the first kind if $\Lambda(G) = \hat{\boldsymbol{R}}$; G is said to be of the second kind if $\Lambda(G) \neq \hat{\boldsymbol{R}}$. It is well-known that, if G is a non-elementary Fuchsian group of the second kind, then $\Lambda(G)$ is a nowhere dense perfect subset of $\hat{\boldsymbol{R}}$, which is invariant under G.

Let G be a Fuchsian group and let σ be a closed subset of $\hat{\boldsymbol{R}}$, which is invariant under G and which contains at least three points. We define $\Sigma(G)$ as the family which consists of all such σ. As is known, every σ in $\Sigma(G)$ contains $\Lambda(G)$. Let f be a quasiconformal automorphism of U, which is compatible with G: that is, $fGf^{-1} \subset PSL(2, \boldsymbol{R})$. All such f form a family $F(G)$. It is known that every f in $F(G)$ is extensible to a homeomorphism of $U \cup \hat{\boldsymbol{R}}$, which is also denoted by the same letter f. For $f \in F(G)$ and $\sigma \in \Sigma(G)$, we define $F(G, f, \sigma)$ as the set of all the $g \in F(G)$ satisfying $g|_\sigma = f|_\sigma$, where $g|_\sigma$ means the restriction of g to σ. We put

$$k(G, f, \sigma) = \inf \|\mu_g\| , \tag{1.1}$$

where $\|\mu_g\|$ means the L_∞ norm of the Beltrami coefficient $\mu_g = g_{\bar{z}}/g_z$ of g and the infimum is taken over all $g \in F(G, f, \sigma)$. By means of a normal family argument of quasiconformal mappings, we can check that there exists some $g \in F(G, f, \sigma)$ with $\|\mu_g\| = k(G, f, \sigma)$ (see [6]). Such a mapping g is said to be extremal in the class $F(G, f, \sigma)$.

Let Γ be a subgroup of a Fuchsian group G. By definition, it is obvious that $\Sigma(G) \subset \Sigma(\Gamma)$, $F(G) \subset F(\Gamma)$ and that $F(G, f, \sigma) \subset F(\Gamma, f, \sigma)$ for every $f \in F(G)$ and every $\sigma \in \Sigma(G)$. Thus, by (1.1), clearly we have

$$k(G, f, \sigma) \geqq k(\Gamma, f, \sigma) \tag{1.2}$$

for every $f \in F(G)$ and every $\sigma \in \Sigma(G)$. The fundamental inequality, referred to in the title of Bers [1], plays an important role in characterizing extremal mappings (see Lemma 1 below). As an application, under the hypothesis that the index $[G: \Gamma]$ of Γ in G is finite, we can verify that (1.2) is valid with equality (cf. [8, Theorem 1]).

Ohtake proved in [7] a theorem which implies the following: if a Fuchsian group G is cyclic, then $k(G, f, \hat{\boldsymbol{R}}) = k(1, f, \hat{\boldsymbol{R}})$ for every $f \in F(G)$, where 1 means the trivial group which consists only of the identity transformation of $PSL(2, \boldsymbol{R})$. Strebel [11] says that there exist a Fuchsian group G and some $f \in F(G)$ such that U/G is a compact Riemann surface of genus 2 and which satisfy $k(G, f, \hat{\boldsymbol{R}}) > k(1, f, \hat{\boldsymbol{R}})$.

Now let G be a Fuchsian group of the second kind and let $\sigma \in \Sigma(G)$ be the closure of $\bigcup_{\gamma \in G} \gamma(\delta)$ in $\hat{\boldsymbol{R}}$, where δ is an open interval contained in $\hat{\boldsymbol{R}} \setminus \Lambda(G)$. We denote by Θ_G the Poincaré series operator of G. The precise definition of Θ_G is given in Section 3. In this paper, first we shall show that, for $0 < k_0 < 1$, there exists a quasiconformal mapping $f \in F(G)$ which satisfies $k_0 = \|\mu_f\| = k(G, f, \sigma) = k(1, f, \bar{\delta})$. Next, as applications to the operator Θ_G of G, we shall have some results related to operator norms of restrictions of Θ_G to suitable spaces. Finally, under the further hypothesis that G is non-elementary and finitely generated, we shall ensure the existence of some $g \in F(G)$ such that $k(G, g, \sigma)$ is sufficiently larger than $k(G, g, \Lambda(G))$.

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2. Extremal sequences of holomorphic quadratic differentials. Let G be a Fuchsian group and let $\Omega(G)$ be the region of discontinuity of G. Let $D \subset \Omega(G)$ be an open set which is invariant under G. A meromorphic function ϕ in D is called a meromorphic quadratic differential for G in D if ϕ satisfies $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$ for every $\gamma \in G$. If, in addition, such a differential ϕ is holomorphic in D, and satisfies

$$\phi(z) = O(|z|^{-4}), \quad z \to \infty \quad \text{if} \quad \infty \in D,$$

then ϕ is called a holomorphic quadratic differential for G in D.

The upper half-plane U is invariant under G. For $\sigma \in \Sigma(G)$, we denote by $A(G, \sigma)$ the space consisting of all the holomorphic quadratic differentials ϕ for G in U, which are continuously extensible to $\hat{\boldsymbol{R}} \setminus \sigma$ and are real on $\hat{\boldsymbol{R}} \setminus \sigma$, and satisfy the following conditions:

(1) $\|\phi\|_G \equiv \iint_{U/G} |\phi(z)| dxdy < \infty,$

(2) $\phi(z) = O(|z|^{-4})$, $z \to \infty$ if $\infty \in \hat{\boldsymbol{R}} \setminus \sigma$.

We note that every ϕ in $A(G, \sigma)$ is symmetrically extensible to a holomorphic quadratic differential for G in $\hat{C} \setminus \sigma$, where $\hat{C}$ denotes the extended complex plane. The space $A(G, \sigma)$ is a real Banach space with norm $\| \|_G$. We denote by $A(G, \sigma)_1$ the set of those $\phi \in A(G, \sigma)$ with $\|\phi\|_G = 1$.

Let G be a Fuchsian group and $f \in F(G)$. The Beltrami coefficient μ_f of f induces a bounded real linear functional $L(\mu_f)$ on $A(G, \hat{\boldsymbol{R}})$ which sends $\phi \in A(G, \hat{\boldsymbol{R}})$ into

$$L(\mu_f)(\phi) \equiv \operatorname{Re} \iint_{U/G} \mu_f \phi dxdy . \tag{2.1}$$

On the right hand side of (2.1), Re A denotes the real part of A and the integration is carried out over any fundamental region representing the Riemann surface U/G (see [1] for the precise definition of the fundamental region). Let $\sigma \in \Sigma(G)$. We note that, if $\phi \in A(G, \sigma)$, then $-\phi \in A(G, \sigma)$ and $A(G, \sigma) \subset A(G, \hat{\boldsymbol{R}})$. We denote by $L(\mu_f)|_{A(G,\sigma)}$ the restriction of $L(\mu_f)$ to $A(G, \sigma)$. The functional norm of $L(\mu_f)|_{A(G,\sigma)}$ is

$$\|L(\mu_f)|_{A(G,\sigma)}\| = \sup L(\mu_f)(\phi) ,$$

where the supremum is taken over all $\phi \in A(G, \sigma)_1$. We say, in this paper, that a sequence $\{\phi_n\}$ in $A(G, \sigma)_1$ is an extremal sequence for the triple (μ_f, G, σ) if it satisfies

$$\|L(\mu_f)|_{A(G,\sigma)}\| = \lim_{n\to\infty} L(\mu_f)(\phi_n) . \tag{2.2}$$

A sequence $\{\phi_n\}$ in $A(G, \sigma)_1$ is said to be degenerating if it converges to zero uniformly on every compact subset of U as n tends to ∞. If there exists some $\phi \in A(G, \sigma)_1$ which satisfies

$$\|L(\mu_f)|_{A(G,\sigma)}\| = L(\mu_f)(\phi) ,$$

then we say that ϕ is an extremal differential for the triple (μ_f, G, σ).

The following Lemmas 1 and 2 characterize extremal mappings in an arbitrarily chosen and fixed class $F(G, f, \sigma)$. It is well-known that (2.3) in our Lemma 1 is a necessary condition for g to be extremal in the class $F(G, f, \sigma)$ (see Bers [2, Theorem 7 and Lemma 25]). The reverse implication in Lemma 1 is a by-product of the fundamental inequality in Bers [1, Theorem 2], and is proved in [8, Lemma 6] (cf. Strebel [12, Theorem 5]). By Lemma 1, we easily have our Lemma 2.

LEMMA 1. *Suppose that $g \in F(G, f, \sigma)$. Then g is extremal in the class $F(G, f, \sigma)$ if and only if*

$$\|\mu_g\| = \|L(\mu_g)|_{A(G,\sigma)}\| . \tag{2.3}$$

LEMMA 2. *Suppose that $g \in F(G, f, \sigma)$ and that g is extremal in the class $F(G, f, \sigma)$. In this case, if the triple (μ_g, G, σ) possesses an extremal differential $\phi \in A(G, \sigma)_1$, then μ_g is of the form*

$$\mu_g = \|\mu_g\| |\phi|/\phi . \tag{2.4}$$

Conversely, if μ_g is of the form (2.4) for some $\phi \in A(G, \sigma)_1$, then ϕ is an extremal differential for the triple (μ_g, G, σ) and, moreover, g is a unique extremal mapping in the class $F(G, f, \sigma)$.

Let $\sigma \in \Sigma(G)$. By the mean value property of holomorphic functions, we can easily check that $A(G, \sigma)_1$ is a family whose elements are locally uniformly bounded; that is, for each compact subset K of U, there exists a uniform bound M such that $|\phi(z)| \leqq M$ for all $\phi \in A(G, \sigma)_1$ and all $z \in K$. Hence $A(G, \sigma)_1$ forms a normal family with respect to locally uniform convergence. The following Lemmas 3 and 4 are instrumental in the later discussions, and can be proved in the same way as in Harrington and Ortel [4, Propositions 1.1 and 1.2].

LEMMA 3. *Suppose that a sequence $\{\phi_n\}$ in $A(G, \sigma)_1$ converges to ϕ uniformly on every compact subset of U. Then ϕ belongs to $A(G, \sigma)$ and*

$$\lim_{n\to\infty} \|\phi_n - \phi\|_G = 1 - \|\phi\|_G .$$

In particular, $\|\phi\|_G \leqq 1$ and the equality holds if and only if

$$\lim_{n\to\infty} \|\phi_n - \phi\|_G = 0 .$$

LEMMA 4. *Let $f \in F(G)$ and $\sigma \in \Sigma(G)$. Let $\{\phi_n\}$ be an extremal sequence in $A(G, \sigma)_1$ for the triple (μ_f, G, σ), which converges to $\phi \in A(G, \sigma)$ uniformly on every compact subset of U. Suppose that $0 < \|\phi\|_G \leqq 1$. Put $\psi = \phi/\|\phi\|_G$ and $\psi_n = (\phi_n - \phi)/\|\phi_n - \phi\|_G$. Then $\psi \in A(G, \sigma)_1$ is an extremal differential for the triple (μ_f, G, σ). Moreover, in the case $0 < \|\phi\|_G < 1$, the sequence $\{\psi_n\}$ in $A(G, \sigma)_1$ is a degenerating extremal sequence for the triple (μ_f, G, σ).*

COROLLARY 1. *Let $f \in F(G)$ and $\sigma \in \Sigma(G)$. Suppose that the triple (μ_f, G, σ) does not possess any extremal differential which belongs to $A(G, \sigma)_1$. Then every extremal sequence $\{\phi_n\}$ in $A(G, \sigma)_1$ for the triple (μ_f, G, σ) is degenerating.*

3. Certain Teichmüller mappings with infinite norm. Let G be a Fuchsian group and $f \in F(G)$. We say that f is a Teichmüller mapping with infinite norm (resp. with finite norm) for G if $\mu_f = \|\mu_f\| |\phi|/\phi$ for some holomorphic quadratic differential ϕ for G in U with $\|\phi\|_G = \infty$

(resp. with $\|\phi\|_G < \infty$). Lemma 2 says that a Teichmüller mapping with finite norm for G is a unique extremal mapping in a certain class. Sethares [9] gave various conditions, in the case $G = 1$, on the regular function ϕ, which guarantee that a corresponding Teichmüller mapping with infinite norm for $G = 1$ is extremal or uniquely extremal (cf. Strebel [10]). In this section we prove Theorem 1 below. To prove the theorem, first we state some results in [9] and [10] in somewhat modified forms as lemmas.

Let S be a simply connected domain of the w-plane. For any real number v, put $S_v = \{w \in S\colon \operatorname{Im} w > v\}$ and denote by $|\gamma_v|$ the length of $\gamma_v = \{w \in S\colon \operatorname{Im} w = v\}$. Suppose that there exist some v_1 and M, $0 < M < \infty$, such that

$$\gamma_v \neq \emptyset \quad \text{and} \quad |\gamma_v| \leqq M \quad \text{for every} \quad v \geqq v_1 , \tag{3.1}$$

and

$$\text{the area of } S_{v_1} \text{ is infinite} . \tag{3.2}$$

For $v \geqq v_1$, every γ_v consists of a disjoint union of denumerable arcs $\{\gamma_v^j\}$. Let $K_0 > 1$ and let F be the mapping on S which sends $w = u + iv$ into $F(w) = K_0 u + iv$. For $\zeta, \zeta' \in F(S)$, we define $\rho(\zeta, \zeta')$ as the infimum of the lengths of all the curves, in $F(S)$, joining ζ and ζ'. Let H be a K-quasiconformal mapping of S_{v_1} into $F(S)$ which satisfies $H(S_{v_2}) \subset F(S_{v_1})$ for some $v_2 \geqq v_1$. For $v \geqq v_1$ and every j, we consider $H(\gamma_v^j)$ and $F(\gamma_v^j)$ as crossing curves in $F(S)$. Suppose further that, for almost all $v \geqq v_1$ and every j, the ends of $H(\gamma_v^j)$ and those of $F(\gamma_v^j)$ have null distance in the sence of [10], that is, the following hold:

$$\liminf_{w, w' \to a} \rho(F(w), H(w')) = 0 \quad \text{and} \quad \liminf_{w, w' \to b} \rho(F(w), H(w')) = 0 ,$$

where a and b denote the end points of γ_v^j and the inferior limits are taken for $w, w' \in \gamma_v^j$. For $v \geqq v_1$, put

$$d(v) = \sup_{w \in \gamma_v} |\operatorname{Im} H(w) - v| .$$

Then we have the following Lemma 5. The proof of (3.3) is already accomplished in that of [10, Hilfssatz on page 313] (cf. [9, Lemma 2]). By making use of (3.3), we can verify (3.4) in the same way as in [10, Satz 2].

LEMMA 5. *Under the above hypotheses, the following inequalities hold*:

$$d(v) \leqq (KK_0)^{1/2} M \quad \textit{for} \quad v \geqq v_2 , \tag{3.3}$$

$$(3.4) \qquad K \geqq K_0 .$$

REMARK 1. Let S be a simply connected domain of the w-plane. In this paper, we say that S_{v_1} is an upper arm of S if (3.1) and (3.2) are satisfied. Symmetrically, if there exist some v^* and M^*, $0 < M^* < \infty$, such that

$$(3.1)' \qquad \gamma_v \neq \varnothing \quad \text{and} \quad |\gamma_v| \leqq M^* \quad \text{for every} \quad v \leqq v^* ,$$

and

$$(3.2)' \qquad \text{the area of } S \setminus \overline{S_{v^*}} \text{ is infinite} ,$$

then we say that $S \setminus \overline{S_{v^*}}$ is a lower arm of S. Similarly, we can define a horizontal, right or left, arm of a simply connected domain (see [10, § 5]). They are said generically to be arms. For a simply connected domain which has an arm, under obvious modifications of the hypotheses of Lemma 5, we have a result similar to Lemma 5.

The following Lemma 6 is implicitly remarked in [9, Remark on page 117]. Since the lemma plays an important role in the later discussions, we give the proof.

LEMMA 6. *Let ϕ be a holomorphic function in U, which possesses a pole of order two (resp. a zero of order two) at an arbitrarily prescribed point $x_0 \in \boldsymbol{R}$ (resp. $x_0 = \infty$). Let δ be an open interval contained in $\hat{\boldsymbol{R}}$ such that δ contains x_0. Suppose that f is a quasiconformal automorphism of U with $\mu_f = k_0|\phi|/\phi$ for some $0 < k_0 < 1$. Then f is extremal in the class $F(1, f, \bar{\delta})$.*

PROOF. First we assume that $x_0 = 1$. For $\rho > 0$, put $N_\rho = \{z \in U: |z - 1| < \rho\}$. It is known that there exists some $\rho_0 > 0$ such that a single-valued and schlicht branch $w = \Phi(z)$ of $\int (\phi(z))^{1/2} dz$ can be chosen in N_{ρ_0}. Moreover, we may write

$$(3.5) \qquad \Phi(z) = c \log (1 - z) + \eta(z) , \quad z \in N_{\rho_0} ,$$

where $\eta(z)$ is bounded and $c \neq 0$ is a complex number (see [9, Lemma 4]). We may assume that $\partial N_{\rho_0} \cap \hat{\boldsymbol{R}} \subset \delta$, where ∂N_{ρ_0} denotes the boundary of N_{ρ_0} in $U \cup \hat{\boldsymbol{R}}$. Let $S = \Phi(N_{\rho_0})$, $K_0 = (1 + k_0)/(1 - k_0)$ and let F be the mapping on S which sends $w = u + iv$ into $F(w) = K_0 u + iv$. By (3.5), we see that the domain S is contained in a semi-infinite parallel strip and that the area of S is infinite. Therefore S has an arm. We assume that S has an upper arm S_{v_1} and that the conditions (3.1) and (3.2) are satisfied. For $v \geqq v_1$, we have $\gamma_v = \sum_j \gamma_v^j$ as before. Since both f and $F \circ \Phi$ have the same Beltrami coefficient in N_{ρ_0}, the mapping $\Psi = F \circ \Phi \circ f^{-1}$

is a conformal mapping of $f(N_{\rho_0})$ onto $F(S)$. Let $g \in F(1, f, \bar{\delta})$. Choose some ρ_1, $0 < \rho_1 < \rho_0$ such that $g(N_{\rho_1}) \subset f(N_{\rho_0})$. Let H be the quasiconformal mapping of $\Phi(N_{\rho_1})$ into $F(S)$ defined by $H = \Psi \circ g \circ \Phi^{-1}$. By (3.5), we see that the set $\Phi(N_{\rho_0} \setminus N_{\rho_1})$ is bounded. Thus we may assume that

$$S_{v_1} \subset \Phi(N_{\rho_1}) \ . \tag{3.6}$$

Then clearly we have

$$H(S_{v_1}) \subset F(S) \ . \tag{3.7}$$

Similarly, by (3.5), we can choose a sufficiently small $\rho_2 > 0$ and a sufficiently large $v_2 \geqq v_1$ which satisfy

$$(f^{-1} \circ g)(N_{\rho_2}) \subset \Phi^{-1}(S_{v_1}) \ , \tag{3.8}$$

and

$$S_{v_2} \subset \Phi(N_{\rho_2}) \ . \tag{3.9}$$

By (3.8) and (3.9), we have

$$H(S_{v_2}) \subset F(S_{v_1}) \ . \tag{3.10}$$

We assume that the restriction $H|_{S_{v_1}}$ of H to S_{v_1} is a K-quasiconformal mapping. It suffices to prove that $K \geqq K_0$. For $v \geqq v_1$ and every j, we consider $H(\gamma_v^j)$ and $F(\gamma_v^j)$ as crossing curves in $F(S)$. Since both f and g have the same boundary values on $\partial N_{\rho_1} \cap \hat{\boldsymbol{R}}$ and S_{v_1} is an upper arm of S, it follows from (3.6) and the arguments in [10] that, for almost all $v \geqq v_1$ and every j, the ends of $H(\gamma_v^j)$ and those of $F(\gamma_v^j)$ have null distance (see [10, § 6]). Furthermore, H satisfies (3.7) and (3.10). Thus, by Lemma 5, we have $K \geqq K_0$. If S has an arm which is not an upper arm, then, by Remark 1, we can prove the lemma in a similar way.

Next assume that $x_0 \neq 1$. Choose $T \in PSL(2, \boldsymbol{R})$ which satisfies $T^{-1}(x_0) = 1$. Put $\phi_1(z) = \phi(T(z))T'(z)^2$ for $z \in U$, $f_1 = T^{-1}fT$ and $\delta_1 = T^{-1}(\delta)$. Then we have $\mu_{f_1} = k_0|\phi_1|/\phi_1$. By the former part of the proof, f_1 is extremal in the class $F(1, f_1, \bar{\delta}_1)$. In this case, clearly f is extremal in the class $F(1, f, \bar{\delta})$. Thus we have the lemma.

Let G be a Kleinian group, $\Omega(G)$ the region of discontinuity of G and $\Lambda(G)$ the limit set of G. Let $D \subset \Omega(G)$ be an open set which is invariant under G. Let Φ be a meromorphic function in D. The Poincaré series $\Theta_G\Phi$ of Φ is defined by

$$(\Theta_G\Phi)(z) = \sum_{\gamma \in G} \Phi(\gamma(z))\gamma'(z)^2 \ , \quad z \in D \ , \tag{3.11}$$

whenever, for each compact subset S of D, the right hand side of (3.11), from which a possible finite number of terms are removed, converges absolutely and uniformly on S. In this case, the series $\Theta_G\Phi$ converges to a meromorphic quadratic differential for G in D uniformly on every compact subset of D with respect to the spherical metric.

The following lemma is easily concluded by Kra [5, Chap. III, Theorem 3.3] and is implicitly established in the proof of [1, Theorem 2].

LEMMA 7. *Let G be a Fuchsian group and $\sigma \in \Sigma(G)$. Then, for every Φ in $A(1, \hat{\boldsymbol{R}})$, the series $\Theta_G\Phi$ defined by* (3.11) *converges absolutely and uniformly on every compact subset of U. Moreover, the restriction $\Theta_G|_{A(1,\sigma)}$ of Θ_G to $A(1, \sigma)$ gives a bounded real linear mapping of $A(1, \sigma)$ onto $A(G, \sigma)$, and the operator norm $\|\Theta_G|_{A(1,\sigma)}\|$ is less than or equal to* 1.

The following lemma is a slightly generalized form of [5, Chap. III, Corollary to Lemma 9.2].

LEMMA 8. *Let G be a Kleinian group and let Φ be a rational function with its poles in $\Omega(G)$. In the case $\infty \in \Lambda(G)$, suppose further that Φ satisfies*

$$(3.12) \qquad \Phi(z) = O(|z|^{-4}) , \quad z \to \infty .$$

Denote by E the set of all the points where Φ possesses its poles. Then the series $\Theta_G\Phi$ converges to a meromorphic quadratic differential for G in $\Omega(G)$ and is holomorphic in $\Omega(G) \setminus \bigcup_{\gamma\in G} \gamma(E \cup \{\infty\})$. Suppose further that there exists some $z_0 \in E \setminus \{\gamma(\infty): \gamma \in G\}$ which is not fixed by any elliptic element of G and which satisfies $\{\gamma(z_0): \gamma \in G\} \cap E = \{z_0\}$. Then, if Φ possesses a pole of order $n > 0$ at z_0, then so does $\Theta_G\Phi$.

PROOF. First assume that $\infty \in \Omega(G)$. In this case, the corollary in [5] quoted above says that our lemma holds whenever G is a non-elementary Kleinian group. Examination of the proof of the corollary, however, shows that our lemma is valid even if G is an elementary Kleinian group, too.

Next assume that $\infty \in \Lambda(G)$. Let $x_0 \in \Omega(G) \setminus \bigcup_{\gamma\in G} \gamma(E)$ and let $T(z) = (az + b)/(cz + d)$, $ad - bc = 1$, be a Möbius transformation which satisfies $T(x_0) = \infty$. Put $G^* = TGT^{-1}$ and let Ψ be the mapping defined by $\Psi(z) = \Phi(T^{-1}(z))(T^{-1})'(z)^2 = \Phi(T^{-1}(z))/(-cz + a)^4$. Then, by (3.12), we can easily check that Ψ is holomorphic at the point $a/c = T(\infty) \in \Lambda(G^*)$ and that Ψ is a rational function with its poles in $T(E) \subset \Omega(G^*)$. Since $\infty \in \Omega(G^*)$, it follows from the former part of the proof that $\Theta_{G^*}\Psi$ converges to a meromorphic quadratic differential for G^* in $\Omega(G^*)$. Since we can easily check that $(\Theta_{G*}\Psi)(T(z))T'(z)^2$ is none other than $(\Theta_G\Phi)(z)$ for $z \in \Omega(G)$, we see that the series $\Theta_G\Phi$ converges to a meromorphic quadratic differential for G in $\Omega(G)$. Now we easily have the lemma.

Now we prove the following theorem.

THEOREM 1. *Let G be a Fuchsian group of the second kind and let δ be an open interval contained in $\hat{\boldsymbol{R}} \setminus \Lambda(G)$. Let σ be the closure of $\bigcup_{\gamma \in G} \gamma(\delta)$ in $\hat{\boldsymbol{R}}$. Then, for $0 < k_0 < 1$, there exists a quasiconformal mapping $f \in F(G)$ which satisfies*

$$k_0 = \|\mu_f\| = k(G, f, \sigma) = k(1, f, \bar{\delta}) . \tag{3.13}$$

PROOF. Let $x_0 \in \delta$, $x_1 \in \Omega(G) \setminus (U \cup \{\infty\})$ be two distinct points which satisfy $\{\gamma(x_0)\colon \gamma \in G\} \cap \{x_1\} = \varnothing$ and $\{\gamma(\infty)\colon \gamma \in G\} \cap \{x_0\} = \varnothing$. Put $\Phi(z) = 1/(z - x_0)^2(z - x_1)^2$. Since $x_0 \in \hat{\boldsymbol{R}} \setminus \Lambda(G)$, as is known, the point x_0 is not fixed by any elliptic element of G. Thus, by Lemma 8, $\Theta_G\Phi$ is holomorphic in U and possesses a pole of order two at x_0. It is well-known that there exists a quasiconformal automorphism f of U with $\mu_f = k_0|\Theta_G\Phi|/\Theta_G\Phi$ (see [6]). By Lemma 6, f is extremal in the class $F(1, f, \bar{\delta})$. In other words, we have

$$k_0 = \|\mu_f\| = k(1, f, \bar{\delta}) . \tag{3.14}$$

On the other hand, we can easily check that f is compatible with G. Thus we may consider the class $F(G, f, \sigma)$. Since $f \in F(G, f, \sigma) \subset F(1, f, \bar{\delta})$, it follows from definition that

$$\|\mu_f\| \geqq k(G, f, \sigma) \geqq k(1, f, \bar{\delta}) . \tag{3.15}$$

By (3.14) and (3.15), we have (3.13).

REMARK 2. Let G, δ and σ satisfy the hypotheses of Theorem 1. Then it is obvious by definition that $k(G, f, \sigma) \geqq k(1, f, \sigma) \geqq k(1, f, \bar{\delta})$. Thus (3.13) implies

$$k(1, f, \sigma) = k(1, f, \bar{\delta}) . \tag{3.16}$$

4. **Operator norm of Poincaré series.** Let G be a Fuchsian group. In this section we shall consider whether operator norms of restrictions of Θ_G to suitable spaces is equal to 1. First we prove the following theorem.

THEOREM 2. *Let G be a Fuchsian group and $\sigma \in \Sigma(G)$. Suppose that there exists $f \in F(G)$ which satisfies*

$$\|\mu_f\| = k(1, f, \sigma) > 0 . \tag{4.1}$$

Then the operator norm $\|\Theta_G|_{A(1,\sigma)}\|$ *of the restriction* $\Theta_G|_{A(1,\sigma)}$ *of* Θ_G *to* $A(1, \sigma)$ *is equal to* 1.

PROOF. Let $\{\Phi_n\}$ be an extremal sequence in $A(1, \sigma)_1$ for the triple $(\mu_f, 1, \sigma)$. Put $\phi_n = \Theta_G \Phi_n$. Our hypothesis (4.1) means that f is extremal in the class $F(1, f, \sigma)$. Thus, in view of (2.2) and (2.3), we have

$$\|\mu_f\| = \lim_{n\to\infty} \mathrm{Re} \iint_U \mu_f \Phi_n dxdy \ . \tag{4.2}$$

Since $f \in F(G)$, we can easily check that

$$\iint_U \mu_f \Phi_n dxdy = \iint_{U/G} \mu_f \phi_n dxdy \ . \tag{4.3}$$

By Lemma 7, $\|\phi_n\|_G \leqq 1$. By (4.1), (4.2) and (4.3), we may assume that $\|\phi_n\|_G \neq 0$ for every $n = 1, 2, \cdots$. It suffices to prove that the sequence $\{\|\phi_n\|_G\}$ converges to 1 as n tends to ∞. Assume the contrary. Then there exist some $\varepsilon > 0$ and a subsequence $\{\phi_{n_k}\}$ such that

$$\|\phi_{n_k}\|_G \leqq 1 - \varepsilon \quad \text{for every} \quad k = 1, 2, \cdots \ . \tag{4.4}$$

By (4.2), (4.3) and (4.4), we have

$$\|\mu_f\| = \lim_{k\to\infty} \mathrm{Re} \iint_{U/G} \mu_f \phi_{n_k} dxdy < \lim_{k\to\infty} \mathrm{Re} \iint_{U/G} \mu_f \phi_{n_k} / \|\phi_{n_k}\|_G dxdy \leqq \|\mu_f\| \ ,$$

which is absurd. Thus we have the theorem.

In view of (3.16), we deduce the following as an immediate corollary of our Theorems 1 and 2.

COROLLARY 2. *Let G and σ satisfy the hypotheses of Theorem* 1. *Then the operator norm* $\|\Theta_G|_{A(1,\sigma)}\|$ *is equal to* 1.

REMARK 3. To the author's knowledge, it is unknown whether there exists a non-elementary Fuchsian group G such that $\|\Theta_G|_{A(1,\Lambda(G))}\| < 1$ (cf. Theorem 3 below).

Using the following Lemma 9, which is proved in [8, Lemma 9], we shall prove Theorem 3 below.

LEMMA 9. *Let G and Γ ($\subsetneqq G$) be Fuchsian groups and $\sigma \in \Sigma(G)$. Let Φ be an arbitrary element of $A(1, \sigma)$. Put $\phi = \Theta_G \Phi$ and $\psi = \Theta_\Gamma \Phi$. Then*

$$\|\phi\|_G \leqq \|\psi\|_\Gamma \leqq \|\Phi\|_1 \ .$$

Furthermore, suppose that $\psi \neq 0$. Then the following three conditions are equivalent to each other:

(1) $\|\phi\|_G = \|\psi\|_\Gamma$,

(2) $\psi \in A(G, \sigma)$,
(3) $n \equiv [G: \Gamma] < \infty$ *and* $\phi = n\psi$.

THEOREM 3. *Let G be a Fuchsian group which has a non-elementary finitely generated subgroup Γ of G such that $[G: \Gamma] = \infty$. Then the operator norm of the restriction $\Theta_G|_{A(1, \Lambda(\Gamma))}$ of Θ_G to $A(1, \Lambda(\Gamma))$ is less than* 1.

PROOF. Assume the contrary. Then there exists a sequence $\{\Phi_n\}$ in $A(1, \Lambda(\Gamma))_1$ such that

$$\lim_{n\to\infty} \|\phi_n\|_G = 1 , \tag{4.5}$$

where $\phi_n = \Theta_G \Phi_n$. Put $\psi_n = \Theta_\Gamma \Phi_n$. By Lemma 7, we have

$$\psi_n \in A(\Gamma, \Lambda(\Gamma)) . \tag{4.6}$$

Since $A(1, \Lambda(\Gamma)) \subset A(1, \hat{\boldsymbol{R}})$ and $\hat{\boldsymbol{R}} \in \Sigma(G)$, we may apply Lemma 9 choosing $\hat{\boldsymbol{R}}$ as σ in the lemma. Then we have

$$\|\phi_n\|_G \leqq \|\psi_n\|_\Gamma \leqq \|\Phi_n\|_1 = 1 . \tag{4.7}$$

Since Γ is non-elementary and finitely generated, the dimension of the space $A(\Gamma, \Lambda(\Gamma))$ is finite (see [2]). Thus, in view of (4.6) and (4.7), we may assume that for some subsequence, which is also denoted by $\{\psi_n\}$, we have

$$\lim_{n\to\infty} \|\psi_n - \psi\|_\Gamma = 0 \quad \text{for some} \quad \psi \in A(\Gamma, \Lambda(\Gamma)) . \tag{4.8}$$

By (4.5), (4.7) and (4.8), we have $\|\psi\|_\Gamma = 1$. By Lemma 7, there exists some $\Phi \in A(1, \Lambda(\Gamma))$ such that $\psi = \Theta_\Gamma \Phi$. Put $\phi = \Theta_G \Phi$. Then, by Lemma 9, we have

$$\|\phi_n - \phi\|_G \leqq \|\psi_n - \psi\|_\Gamma . \tag{4.9}$$

By (4.5), (4.8) and (4.9), we see that $\|\phi\|_G = 1$. Thus we have

$$\|\phi\|_G = \|\psi\|_\Gamma = 1 . \tag{4.10}$$

But, by Lemma 9, (4.10) implies $[G: \Gamma] < \infty$. This contradiction proves the theorem.

REMARK 4. For a non-elementary Fuchsian group H, we know that the hyperbolic area of U/H is non-zero and that it is finite if and only if H is finitely generated and of the first kind (see [5]). Let G and Γ satisfy the hypotheses of Theorem 3. Then the hyperbolic area of U/Γ is not finite, because it is equal to $[G: \Gamma]$ times the hyperbolic area of U/G. From these considerations, we see that, under the hypotheses of

Theorem 3, Γ is necessarily of the second kind.

5. Comparison of $k(G, f, \sigma)$ and $k(G, f, \Lambda(G))$. Let T be a conformal mapping of the unit disk onto the upper half-plane U. For every $m = 1, 2, \cdots$, put

$$D_m = \{w: |w| < 1 - (1/2m)\}, \qquad K_m = T(D_m).$$

Let G be a Fuchsian group, $f \in F(G)$ and ω a fundamental region representing the Riemann surface U/G. Let $\mu = \mu_f$ be the Beltrami coefficient of f. For every $m = 1, 2, \cdots$, we define μ_m by the following properties, where $\mathrm{Int}\,\omega$ means the interior of ω:

(1) $\mu_m(z) = 0$ for $z \in K_m \cap \mathrm{Int}\,\omega$,

(2) $\mu_m(z) = \mu(z)$ for $z \in \bar{\omega} \setminus (K_m \cap \mathrm{Int}\,\omega)$,

and

(3) $\mu_m(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu_m(z)$ for all $z \in U$ and all $\gamma \in G$.

For every $m = 1, 2, \cdots$, we denote by f_m a quasiconformal automorphism of U with its Beltrami coefficient $\mu_{f_m} = \mu_m$ and which leaves the points 0, 1 and ∞ fixed; it is well-known that f_m exists and belongs to $F(G)$ and that f_m is uniquely determined by μ_m (see [6]).

LEMMA 10. *Let G be a Fuchsian group, $\sigma \in \Sigma(G)$ and $f \in F(G)$. Let $\mu = \mu_f$ be the Beltrami coefficient of f. Suppose that f is extremal in the class $F(G, f, \sigma)$ and that the triple (μ, G, σ) possesses a degenerating extremal sequence $\{\phi_n\}$ in $A(G, \sigma)_1$. Then, for every $m = 1, 2, \cdots$, f_m is extremal in the class $F(G, f_m, \sigma)$ and $\|\mu\| = \|\mu_m\|$. Furthermore, for every $m = 1, 2, \cdots$, the sequence $\{\phi_n\}$ is a degenerating extremal sequence for the triple (μ_m, G, σ), too.*

PROOF. By our hypothesis, the sequence $\{\phi_n\}$ converges to zero uniformly on every compact subset of U as n tends to ∞. Thus, in view of (2.2) and (2.3), we have, for every $m = 1, 2, \cdots$,

$$\|\mu\| = \lim_{n\to\infty} \mathrm{Re} \iint_{U/G} \mu\phi_n dxdy = \lim_{n\to\infty} \mathrm{Re} \iint_{U/G} \mu_m\phi_n dxdy \leqq \|\mu_m\| \leqq \|\mu\|.$$

Hence, for every $m = 1, 2, \cdots$,

$$\|\mu\| = \|\mu_m\| = \|L(\mu_m)|_{A(G,\sigma)}\| = \lim_{n\to\infty} \mathrm{Re} \iint_{U/G} \mu_m\phi_n dxdy. \tag{5.1}$$

By Lemma 1, we see that (5.1) implies our Lemma 10.

THEOREM 4. *Let G be a non-elementary finitely generated Fuchsian group of the second kind and let δ be an open interval contained in $\hat{\boldsymbol{R}} \setminus \Lambda(G)$. Let σ be the closure of $\bigcup_{\gamma \in G} \gamma(\delta)$ in $\hat{\boldsymbol{R}}$. Let k_0 and ε be arbi-*

trarily chosen and fixed positive numbers which satisfy $0 < k_0 < 1$ *and* $0 < \varepsilon < k_0$. *Then there exists a quasiconformal mapping* $g \in F(G)$ *which satisfies*

$$\|\mu_g\| = k_0 = k(G, g, \sigma) \quad \text{and} \quad k(G, g, \Lambda(G)) < \varepsilon \,. \tag{5.2}$$

PROOF. Let $f \in F(G)$ be the quasiconformal mapping which is mentioned in the proof of Theorem 1 and put $\mu = \mu_f$. The mapping f has the following properties: f is extremal in the class $F(G, f, \sigma)$, $\|\mu\| = k_0$ and f is not a Teichmüller mapping with finite norm for G. Thus, by Lemma 2, the triple (μ, G, σ) does not possess any extremal differential which belongs to $A(G, \sigma)_1$. Hence, by Corollary 1, there exists a degenerating extremal sequence $\{\phi_n\}$ in $A(G, \sigma)_1$ for the triple (μ, G, σ). Consequently, by Lemma 10, f_m is extremal in the class $F(G, f_m, \sigma)$ and

$$k_0 = \|\mu\| = \|\mu_m\| = k(G, f_m, \sigma) \,. \tag{5.3}$$

Since the sequence $\{\mu_m\}$ converges to 0 as m tends to ∞, it follows that the sequence $\{f_m\}$ converges to the identity automorphism of U uniformly on every compact subset of U as m tends to ∞ (see [6]). Let $\gamma_1, \gamma_2, \cdots, \gamma_j$ be a system of generators for G. Put $\gamma_{i,m} = f_m \circ \gamma_i \circ f_m^{-1}$ for every i, $1 \leqq i \leqq j$. Then, for every i, $1 \leqq i \leqq j$, the sequence $\{\gamma_{i,m}\}$ converges to γ_i as m tends to ∞. As is known, a non-elementary finitely generated Fuchsian group G is symmetrically quasi-stable (see Gardiner and Kra [3, Theorem 10.2]). Thus the following holds: there exists a sequence $\{g_m\}$ in $F(G)$ which satisfies

$$g_m \circ \gamma_i \circ g_m^{-1} = f_m \circ \gamma_i \circ f_m^{-1} \quad \text{for every } i\,, \quad 1 \leqq i \leqq j\,, \tag{5.4}$$

and

$$\lim_{m \to \infty} \|\mu_{g_m}\| = 0 \,. \tag{5.5}$$

It is easily checked that (5.4) implies

$$g_m|_{\Lambda(G)} = f_m|_{\Lambda(G)} \,. \tag{5.6}$$

By definition and (5.6), clearly we have

$$k(G, f_m, \Lambda(G)) \leqq \|\mu_{g_m}\| \,. \tag{5.7}$$

By (5.5), we can choose a sufficiently large m^* such that

$$\|\mu_{g_{m^*}}\| < \varepsilon \,. \tag{5.8}$$

Put $g = f_{m^*}$. Then, by (5.3), (5.7) and (5.8), we have the desired conclusion (5.2).

REFERENCES

[1] L. BERS, A new proof of a fundamental inequality for quasiconformal mappings, J. Analyse Math. 36 (1979), 15-30.

[2] L. BERS, Extremal quasiconformal mappings, in Advances in the Theory of Riemann Surfaces, Stony Brook 1969, (Ed. by L. V. Ahlfors et al.), Ann. of Math. Studies 66, Princeton Univ. Press, 1971, 27-52.

[3] F. P. GARDINER AND I. KRA, Stability of Kleinian groups, Indiana Univ. Math. J. 21 (1972), 1037-1059.

[4] A. HARRINGTON AND M. ORTEL, The dilatation of an extremal quasiconformal mapping, Duke Math. J. 43 (1976), 533-544.

[5] I. KRA, Automorphic forms and Kleinian groups, W. A. Benjamin, Reading, Massachusetts, 1972.

[6] O. LEHTO AND K. I. VIRTANEN, Quasiconformal mappings in the plane, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[7] H. OHTAKE, Lifts of extremal quasiconformal mappings of arbitrary Riemann surfaces, to appear.

[8] K. SAKAN, On quasiconformal mappings compatible with a Fuchsian group, to appear in Osaka J. Math. 19 (1982).

[9] G. C. SETHARES, The extremal property of certain Teichmüller mappings, Comment. Math. Helv. 43 (1968), 98-119.

[10] K. STREBEL, Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises, Comment. Math. Helv. 36 (1962), 306-323.

[11] K. STREBEL, On lifts of extremal quasiconformal mappings, J. Analyse Math. 31 (1977), 191-203.

[12] K. STREBEL, On quasiconformal mappings of open Riemann surfaces, Comment. Math. Helv. 53 (1978), 301-321.

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