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INVARIANT SUBSPACES FOR SHIFT OPERATORS OF MULTIPLICITY ONE

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Introduction. Let $L^2(T)$ be the Hilbert space consisting of square integrable functions $f = f(e^{ix})$ defined on the unit circle T, and based on the measure $d\sigma(x) = dx/2\pi$. We put $e_n = e_n(e^{ix}) = e^{inx}$ for each n in Z (Z means the set of all integers). Then $L^2(T)$ is the direct sum $\sum_{n\in Z}\bigoplus[e_n]$ of the one-dimensional subspaces $[e_n]$ generated by e_n. We say that a unitary operator U on $L^2(T)$ is a *shift operator* or a shift for short if $U[e_n] = [e_{n+1}]$ for all n in Z. Since each $[e_n]$ is a one-dimensional subspace, each shift U corresponds to a sequence $\{z_n\}_{n\in Z}$ of complex numbers with absolute value 1 by the relation:

$$Ue_n = z_{n+1}e_{n+1} \quad (n \in Z) .$$

Especially, throughout this paper, we denote by S a shift operator defined by

$$Se_n = e_{n+1} \quad (n \in Z) .$$

Our purpose is to analyze the structure of invariant subspaces for a given family $\mathscr{S}$ of shift operators on $L^2(T)$. It is reduced to the works of Beurling [1] and Helson [3] when $\mathscr{S}$ consists of a single shift, because each shift U is unitarily equivalent to the shift S. For a shift operator U, we put $W = US^*$. Then $U = WS$ and W is a unitary operator on $L^2(T)$ such that $W[e_n] = [e_n]$ for all n in Z. For a given family $\mathscr{S}$, we put $W(\mathscr{S}) = \{W; W = US^*, U \in \mathscr{S}\}$. In this paper, it is assumed that $W(\mathscr{S})$ satisfies the following condition $(*)$:

$$(*) \qquad \begin{cases} (1) & W(\mathscr{S}) \text{ is a group} . \\ (2) & S^*W(\mathscr{S})S = W(\mathscr{S}) . \end{cases}$$

The author [4, §1] has shown, in the case of arbiratry multiplicity, that under the condition $(*)$ the invariant subspaces for $\mathscr{S}$ have two fundamental properties for decomposition, that is, (i) every simply invariant

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subspace is decomposed into the pure simply invariant part and the reducing part; (ii) every pure simply invariant subspace is decomposed into the wandering subspaces. If we drop either of the condition $(*)$, the invariant subspaces for $\mathscr{S}$ seem to be complicated (cf. [4, Example 1.5 and 1.6]). However, it is expected that the structure of those spaces for an arbitrary $\mathscr{S}$ is deeply related to that for the smallest family $[\mathscr{S}]$ which contains $\mathscr{S}$ and satisfies the condition $(*)$. In fact, the sets of reducing subspaces for $\mathscr{S}$ and $[\mathscr{S}]$ coincide.

In the paper, we denote by $M(\mathscr{S})$ (resp. $D(\mathscr{S})$) the von Neumann algebra generated by $\mathscr{S}$ (resp. $W(\mathscr{S})$) and by $A(\mathscr{S})$ the algebra generated by $D(\mathscr{S})$ and $\{S^n\}_{n=0}^{\infty}$. Theorem 2.12 in [4] says that if every pure simply invariant subspace for $A(\mathscr{S})$ is of Beurling type then $M(\mathscr{S})$ must be the crossed product (see [5, P. 364]) of a von Neumann algebra $D(\mathscr{S})_0$ on the one-dimensional space $[e_0]$ by $\boldsymbol{Z}$ with respect to a spatial automorphism of $D(\mathscr{S})_0$. Since the trivial algebra $\boldsymbol{C}1$ is the only von Neumann algebra on $[e_0]$, there is no crossed product of the above form on $L^2(T)$ except the von Neumann algebra $M_{L^\infty(T)}$ of all the multiplication operators M_f on $L^2(T)$ by f in $L^\infty(T)$, which is generated by the shift S. Hence there are many of those invariant subspaces for $A(\mathscr{S})$ which are not of Beurling type, if $M(\mathscr{S})$ is distinct from $M_{L^\infty(T)}$.

We obtain the characterization of invariant subspaces for those $\mathscr{S}$ which satisfy the condition $(*)$, in terms of the commutant $M(\mathscr{S})'$ and the shift S. We here note that these subspaces are also invariant under $A(\mathscr{S})$ since the multiplicity is one. For the theory of von Neumann algebras, we refer to the books of Dixmier [2] and Takesaki [5].

1. The von Neumann algebras generated by shift operators. Let $G = \prod_{n\in \boldsymbol{Z}} T_n$, where each T_n is the unit circle in the complex plane. For each $g = (z_n)$ in G, we define a unitary operator W_g on $L^2(T)$ such that $W_g[e_n] = [e_n]$ $(n \in \boldsymbol{Z})$ by the relation $W_g e_n = z_n e_n$ $(n \in \boldsymbol{Z})$. For each natural number k, we denote by G_k the subgroup of G consisting of all periodic sequences $g = (z_n)$ in G with period k (i.e., $z_n = z_{n+k}$ for all n in $\boldsymbol{Z}$). For k in $\boldsymbol{N}$ ($\boldsymbol{N}$ means the set of all natural numbers) or $k = \infty$, let $\mathscr{S}_k = \{U; U = W_g S,\ g \in G_k\}$ where G_∞ means G. Since each G_k is invariant under the shift (i.e., if $g = (z_n)$ belongs to G_k, then $g' = (z_{n+1})$ and $g'' = (z_{n-1})$ belong to G_k), each $\mathscr{S}_k$ satisfies the condition $(*)$ and especially $\mathscr{S}_\infty$ is the family of all shift operators on $L^2(T)$ with respect to the decomposition $\sum_{n\in \boldsymbol{Z}} \oplus [e_n]$. For each k in $\boldsymbol{N}$, $D(\mathscr{S}_k)$ is the von Neumann algebra generated by the finitely many projections $\{P_{k,i}\}_{0\leqq i\leqq k-1}$ where $P_{k,i}$ is the projection of $L^2(T)$ onto the subspace $\sum_{n\in \boldsymbol{Z}} \oplus [e_{nk+i}]$. Therefore, for each k in $\boldsymbol{N}$, $M(\mathscr{S}_k)$ is the von Neumann algebra generated by $\{P_{k,i}\}_{0\leqq i\leqq k-1}$ and the algebra $M_{L^\infty(T)}$. Moreover, we find that $M(\mathscr{S}_\infty)$ is the full operator algebra on $L^2(T)$ because $M(\mathscr{S}_\infty)$ contains the projections $\{P_n\}_{n\in Z}$ and $\{S^n\}_{n\in Z}$ where P_n is the projection of $L^2(T)$ onto $[e_n]$. In this paper, we say that $f = f(e^{ix})$ is a periodic function with period x_0 if the equality $f(e^{ix}) = f(e^{i(x+x_0)})$ holds for almost all x in $[0, 2\pi)$. Then, $P_{k,0}$ is the projection onto the subspace of the periodic functions in $L^2(T)$ with period $2\pi/k$. Moreover, for each k in N, the commutant $M(\mathscr{S}_k)'$ is the algebra of all the multiplication operators M_f by a periodic function f in $L^\infty(T)$ with period $2\pi/k$ and the commutant $M(\mathscr{S}_\infty)'$ is the algebra $C(L^2(T))$ of all the scalar multiples of the identity on $L^2(T)$.

LEMMA 1.1. *Suppose that $\mathscr{S}$ contains the shift S and $S^*W(\mathscr{S})S = W(\mathscr{S})$. Then $D(\mathscr{S}) = D(\mathscr{S}_k)$ for some k in $N \cup \{\infty\}$.*

PROOF. For $\mathscr{S}$, we denote by $G(\mathscr{S})$ the subset $\{g \in G; W_g \in W(\mathscr{S})\}$ of G. Let N_p be the set of all numbers j in N such that $G(\mathscr{S}) \subset G_j$. Let k be the minimum number in N_p if N_p is non-empty, and $k = \infty$ otherwise. If $k = 1$, the von Neumann algebra $D(\mathscr{S})$ is obviously the algebra $C(L^2(T))$.

Next, we consider the case where $1 < k < \infty$. Let m be a natural number such that $1 \leqq m \leqq k-1$. Then, there exists an element $g = (z_n)$ of $G(\mathscr{S})$ such that $z_{i+m} \neq z_i$ for some number i in Z. Since $S^{*i}W_gS^i$ belongs to $W(\mathscr{S})$, we may assume that $i = 0$. Moreover, multiplying g by a suitable complex number z with absolute value 1, $zg = (zz_n)$ becomes an element $zg = (y_n)$ of G such that $y_{nk+m} = y_m = 1$ but $y_{nk} = y_0 \neq 1$ for all n in Z. We put $R_m = (I - zW_g)/(1 - y_0)$. Then R_m becomes an operator in $D(\mathscr{S})$ such that $R_mP_{nk+m} = 0$ and $R_mP_{nk} = P_{nk}$ for all n in Z. Thus, the product of these $k-1$ operators $\{R_m\}_{1\leqq m\leqq k-1}$ is the projection $P_{k,0}$. Hence $P_{k,0}$ belongs to $D(\mathscr{S})$ and the projections $\{P_{k,m}\}_{1\leqq m\leqq k-1}$ also belongs to $D(\mathscr{S})$ because of the hypothesis $S^*W(\mathscr{S})S = W(\mathscr{S})$. Namely we have $D(\mathscr{S}_k) \subset D(\mathscr{S})$. Since $D(\mathscr{S})$ is a subalgebra of $D(\mathscr{S}_k)$ by the definition of k, it follows that $D(\mathscr{S}) = D(\mathscr{S}_k)$.

Finally, we consider the case where N_p is empty. Similarly as in the second case, for each $m \neq 0$, there exists an element $h = (z_n)$ of $G(\mathscr{S})$ such that $z_0 \neq z_m$. This time, for some complex number z with absolute value 1, $zh = (zz_n)$ is an element $zh = (y_n)$ of G such that $y_0 = 1$ and $y_m \neq 1$. We put $S_m = (I + zW_h)/2$. Then S_m is an operator in $D(\mathscr{S})$ such that $S_mP_0 = P_0$, $\|S_mP_m\| < 1$ and $\|S_mP_i\| \leqq 1$ for all $i \neq 0, m$ where $\|\cdot\|$ means the norm of a bounded linear operator on $L^2(T)$. If we put

$$Q_k = S_{-k} \cdots S_{-1}S_1 \cdots S_k$$

for each k in N, then we have that $Q_kP_0 = P_0$ and $\|Q_kP_m\| < 1$ for $m =$

$-k, \cdots, -1, 1, \cdots, k$. Since the sequence

$$\{(Q_k)^n(P_{-k} + \cdots + P_{-1} + P_1 + \cdots + P_k)\}_{n=1}^{\infty}$$

converges uniformly to 0, we get an operator T_k in $D(\mathscr{S}_\infty)$ such that $T_kP_0 = P_0$, $\|T_kP_m\| < 1/k$ for $m = -k, \cdots, -1, 1, \cdots, k$ and $\|T_kP_i\| \leqq 1$ for all $i \neq -k, \cdots, 0, \cdots, k$. Since the sequence $\{T_k\}_{k=1}$ converges strongly to P_0, the von Neumann algebra $D(\mathscr{S})$ contains P_0. Hence $D(\mathscr{S})$ contains the projections $\{P_n\}_{n\in Z}$ by the hypothesis $S^*W(\mathscr{S})S = W(\mathscr{S})$. Therefore, it follows that $D(\mathscr{S}_\infty) \subset D(\mathscr{S})$. Since $G(\mathscr{S})$ is a subset of $G = G_\infty$, we have the conclusion. q.e.d.

PROPOSITION 1.2. *Suppose that $\mathscr{S}$ contains the shift S. Then $M(\mathscr{S}) = M(\mathscr{S}_k)$ for some k in $N \cup \{\infty\}$.*

PROOF. Since $\mathscr{S}$ contains the shift S, $M(\mathscr{S})$ contains the operators $W(\mathscr{S})$, so that it contains the operators $S^{*n}W(\mathscr{S})S^n$ for all n in Z. We put $W(\mathscr{S})_0 = \bigcup_{n\in Z} S^{*n}W(\mathscr{S})S^n$ and $\mathscr{S}_0 = \{U : U = WS,\ W \in W(\mathscr{S})_0\}$. Then we have $M(\mathscr{S}) = M(\mathscr{S}_0)$, and $\mathscr{S}_0$ satisfies the hypothesis of Lemma 1.1. Hence $M(\mathscr{S})$ coincides with the von Neumann algebra $M(\mathscr{S}_k)$, which is generated by $D(\mathscr{S}_k)$ and S, for some k in $N \cup \{\infty\}$. q.e.d.

THEOREM 1.2. *Let $\mathscr{S}$ be a family of shift operators. Then $M(\mathscr{S})$ is spatially isomorphic to $M(\mathscr{S}_k)$ for some k in $N \cup \{\infty\}$ and k is uniquely determined by $\mathscr{S}$.*

PROOF. We take a shift operator U in $\mathscr{S}$. Then $U = W_gS$ for some $g = (z_n)$ in $G(\mathscr{S})$. For this sequence, we define a unitary operator W such that $W[e_n] = [e_n]$ $(n \in Z)$ as follows; $We_n = \overline{z_nz_{n-1}\cdots z_1}e_n$ if $n \geqq 1$, $We_0 = e_0$ and $We_n = z_{n+1}z_{n+2}\cdots z_0e_n$ if $n \leqq -1$. Then $W\mathscr{S}W^*$ is a family of shift operators containing S. Hence, by Proposition 1.2, $WM(\mathscr{S})W^* = M(W\mathscr{S}W^*) = M(\mathscr{S}_k)$ for some k in $N \cup \{\infty\}$. For each n in N, $M(\mathscr{S}_n)$ is spatially isomorphic to the von Neumann algebra $M_{L^\infty(T)} \otimes B(H_n)$ on $L^2(T) \otimes H_n$ where H_n is an n-dimensional Hilbert space. Namely, for each n in $N \cup \{\infty\}$, $M(\mathscr{S}_n)$ is a von Neumann algebra of type I_n, so that $\{M(\mathscr{S}_n)\}_{n\in N\cup\{\infty\}}$ are mutually non-isomorphic von Neumann algebras [2, Chapter III, §3, Proposition 1]. Hence k is uniquely determined by $\mathscr{S}$. q.e.d.

2. The structure of invariant subspaces. Let $\mathscr{S}$ be a family of shift operators which satisfies the condition $(*)$. Then the structure of non-reducing invariant subspaces for $\mathscr{S}$ is essentially the same as that of non-reducing invariant subspaces for $A(\mathscr{S})$ (cf. [4, Proposition 1.7]). Hence, Lemma 1.1 reduces the study of these subspaces to that of non-reducing invariant subspaces for $\mathscr{S}_k$'s. For an arbitrary $\mathscr{S}$ of shift operators, each reducing subspace for $\mathscr{S}$ corresponds to a projection in the commutant $M(\mathscr{S})'$. Hence, Theorem 1.2 reduces the study of these subspaces to that of reducing subspaces for $\mathscr{S}_k$'s. Since the structure of $M(\mathscr{S}_k)'$ is plain, we easily get the following theorem.

THEOREM 2.1. (1) ($1 \leqq k < \infty$). *A subspace $\mathfrak{R}$ of $L^2(T)$ reduces $\mathscr{S}_k$ if and only if $\mathfrak{R}$ is of the form $\mathfrak{R} = M_{\chi_E}L^2(T)$ where χ_E is a periodic characteristic function in $L^\infty(T)$ with period $2\pi/k$.*

(2) ($k = \infty$). *$L^2(T)$ is the only non-zero reducing subspace for $\mathscr{S}_\infty$.*

For a subset $\mathfrak{N}$ of $L^2(T)$, $[\mathfrak{N}]$ means the closed linear span of $\mathfrak{N}$. A subspace $\mathfrak{M}$ is said to be simply invariant if $\mathfrak{M}$ is invariant under $\mathscr{S}$ and $[\mathscr{S}\mathfrak{M}]$ is a proper subspace of $\mathfrak{M}$. Moreover, a simply invariant subspace $\mathfrak{M}$ is said to be pure if $\bigcap_{n=1}^{\infty} [\mathscr{S}^n\mathfrak{M}] = \{0\}$. If $\mathscr{S}$ satisfies the condition (*), then every non-reducing invariant subspace for $\mathscr{S}$ is simply invariant [4, Proposition 1.12]. We now show the following main theorem, in which $H^2(T)$ means the Hardy space in $L^2(T)$ (i.e., $H^2(T) = \sum_{n=0}^{\infty} \oplus [e_n]$). Though the assertion (2) of the theorem is well-known in the general theory of operators, we give a proof for the sake of completeness.

THEOREM 2.2. (1) ($1 \leqq k < \infty$). *A subspace $\mathfrak{M}$ of $L^2(T)$ is a non-reducing invariant subspace for $\mathscr{S}_k$ if and only if $\mathfrak{M}$ is of the form $\mathfrak{M} = M_u S^m H^2(T)$ where $u = u(e^{ix})$ is a periodic unitary function in $L^\infty(T)$ with period $2\pi/k$ and m is an integer such that $0 \leqq m \leqq k - 1$.*

(2) ($k = \infty$). *The subspaces $\{S^n H^2(T); n \in \boldsymbol{Z}\}$ are the set of all non-trivial non-reducing invariant subspaces for $\mathscr{S}_\infty$.*

PROOF. (1) The subspaces $\mathfrak{M}$ of the form $\mathfrak{M} = M_u S^m H^2(T)$ are invariant under $\mathscr{S}_k$ because M_u commute with $A(\mathscr{S}_k)$ and $S^m H^2(T)$ is obviously invariant under all shift operators. Moreover $\mathfrak{M}$ is simply invariant because $\mathfrak{M} \ominus [\mathscr{S}_k\mathfrak{M}] = [ue_m]$ where $[ue_m]$ is the one-dimensional subspace generated by the vector ue_m in $L^2(T)$. Hence $\mathfrak{M}$ does not reduce $\mathscr{S}_k$.

We conversely assume that $\mathfrak{M}$ is a pure simply invariant subspace. We put $\mathfrak{M}_0 = \mathfrak{M} \ominus [\mathscr{S}_k\mathfrak{M}]$. By Proposition 1.7 in [4], $\mathfrak{M}$ has a decomposition

$$\mathfrak{M} = \mathfrak{M}_0 \oplus S[W(\mathscr{S}_k)\mathfrak{M}_0] \oplus S^2[W(\mathscr{S}_k)\mathfrak{M}_0] \oplus \cdots .$$

The subspace $\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{M}_0$ is also an invariant subspace for the shift S such that $\mathfrak{N}_0 = \mathfrak{N} \ominus [S\mathfrak{N}] = S[W(\mathscr{S}_k)\mathfrak{M}_0]$. By Beurling's theorem [3, Lecture II, Theorem 3], we find that the wandering subspace $\mathfrak{N}_0$ for S is one-dimensional. Then $[W(\mathscr{S}_k)\mathfrak{M}_0]$ is one-dimensional since S is a unitary operator, and it contains the subspace $\mathfrak{M}_0$. Hence we have $[D(\mathscr{S}_k)\mathfrak{M}_0] = [W(\mathscr{S}_k)\mathfrak{M}_0] = \mathfrak{M}_0$. Thus $\mathfrak{M}_0$ is invariant under the mutually orthogonal projections $\{P_{k,i}\}_{0\leqq i\leqq k-1}$, whose sum equals the identity. Hence $\mathfrak{M}_0 = P_{k,m}\mathfrak{M}_0$ for some m, $0 \leqq m \leqq k-1$, and $P_{k,i}\mathfrak{M}_0 = \{0\}$ for all $i \neq m$. We now consider the invariant subspace $\mathfrak{L} = S^{-m}\mathfrak{M}$ for $\mathscr{S}_k$ and we put $\mathfrak{L}_n = [\mathscr{S}_k^n\mathfrak{L}] \ominus [\mathscr{S}_k^{n+1}\mathfrak{L}]$ $(n = 0, 1, 2, \cdots)$. Then we have that $\mathfrak{L}_0 = P_{k,0}\mathfrak{L}_0$ and $P_{k,i}\mathfrak{L}_0 = \{0\}$ for all $i = 1, 2, \cdots, k-1$. For each i, $0 \leqq i \leqq k-1$, we put $\mathfrak{K}_i = \sum_{n=0}^{\infty} \oplus S^{nk}\mathfrak{L}_i$. Then each $\mathfrak{K}_i$ is a subspace of $P_{k,i}L^2(T) = \sum_{n\in Z} \oplus [e_{nk+i}]$ respectively. Moreover, we have $\mathfrak{L} = \sum_{i=0}^{k-1} \oplus \mathfrak{K}_i$ and $\mathfrak{K}_i = S^i\mathfrak{K}_0$ for each i, $0 \leqq i \leqq k-1$.

Let V be the canonical isometric isomorphism from $P_{k,0}L^2(T)$ onto $L^2(T)$, that is, $Ve_{nk} = e_n$ for each n in $\boldsymbol{Z}$. Since $\mathfrak{K}_0$ is invariant under S^k, $V\mathfrak{K}_0$ is invariant under S on $L^2(T)$. We apply Beurling's theorem again to find a unitary function $v = v(e^{ix})$ in $L^\infty(T)$ such that $V\mathfrak{K}_0 = M_vH^2(T)$. Thus, for each i, $0 \leqq i \leqq k-1$. We have

$$\mathfrak{K}_i = S^i\mathfrak{K}_0 = S^iV^*M_vH^2(T) = S^iV^*M_vVV^*H^2(T) = S^iV^*M_vVP_{k,0}H^2(T) .$$

We put $u(e^{ix}) = v(e^{ikx})$. Then it follows that $V^*M_vV = M_u$ and u is a periodic function in $L^\infty(T)$ with period $2\pi/k$. Thus we have $\mathfrak{K}_i = M_uS^iP_{k,0}H^2(T) = M_uP_{k,i}H^2(T)$. Therefore we have

$$\begin{aligned}\mathfrak{L} &= \mathfrak{K}_0 \oplus S\mathfrak{K}_0 \oplus \cdots \oplus S^{k-1}\mathfrak{K}_0 \\ &= M_uP_{k,0}H^2(T) \oplus M_uP_{k,1}H^2(T) \oplus \cdots \oplus M_uP_{k,k-1}H^2(T) \\ &= M_uH^2(T) .\end{aligned}$$

Consequently, $\mathfrak{M}$ is of the form $\mathfrak{M} = S^m\mathfrak{L} = S^mM_uH^2(T)$.

Next we shall show that every simply invariant subspace $\mathfrak{M}$ for $\mathscr{S}_k$ is pure. By Theorem 1.7 in [4], $\mathfrak{M}$ has a decomposition $\mathfrak{M} = \mathfrak{M}_p \oplus \mathfrak{M}_r$ where $\mathfrak{M}_p$ is a pure simply invariant subspace and $\mathfrak{M}_r$ reduces $\mathscr{S}_k$. By what we have shown above, the subspace $\mathfrak{M}_p$ contains a unitary function w $(=ue_m)$ in $L^2(T)$ and $\mathfrak{M}_r = \mathfrak{M}_{\chi_E}L^2(T)$ for some characteristic function χ_E (Theorem 2.1, (1)). Hence two vectors w and $w\chi_E$ are mutually orthogonal. But this phenomenon does not occur except the case where the measure of E is zero.

(2) For a pure simply invariant subspace $\mathfrak{M}$, the wandering subspace $\mathfrak{M}_0$ is invariant under the projections $\{P_n\}_{n\in Z}$ in $D(\mathscr{S}_\infty)$. As we showed in the preceding case, $\mathfrak{M}_0$ is one-dimensionl, so that $\mathfrak{M}_0 = [e_n]$ for some integer n. In this case, $\mathfrak{M}$ is of the form $\mathfrak{M} = S^nH^2(T)$. Similarly we find that every simply invariant subspace has no reducing part. q.e.d.

REMARK. Let Φ be the canonical spatial isomorphism of $M(\mathscr{S}_k)$ onto

$M_{L^\infty(T)} \otimes B(H_k)$, which is implemented by the isometry V defined by the relation $Ve_{nk+i} = e_n \otimes e_i$ ($n \in \boldsymbol{Z}$, $0 \leqq i \leqq k-1$). The author described the structure of invariant subspaces for the non-commutative Hardy space $M_{H^\infty(T)} \otimes B(H_k)$ [4, Corollary 2.13]. However we cannot apply this result to the case of $A(\mathscr{S}_k)$, because $\Phi(A(\mathscr{S}_k))$ is a non-self adjoint algebra which is distinct from $M_{H^\infty(T)} \otimes B(H_k)$. Indeed, we have the following inclusion:

$$M_{H^\infty(T)} \otimes J_k \subsetneqq \Phi(A(\mathscr{S}_k)) \subsetneqq M_{H^\infty(T)} \otimes B(H_k)$$

where J_k is the lower triangular algebra on H_k with respect to the base $\{e_i\}_{0 \leqq i \leqq k-1}$.

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