Tôhoku Math. Journ.
34 (1982), 437-483.

ZETA FUNCTIONS IN SEVERAL VARIABLES ASSOCIATED WITH PREHOMOGENEOUS VECTOR SPACES I: FUNCTIONAL EQUATIONS

FUMIHIRO SATO

(Received March 8, 1982)

Introduction. 0.1. Let G be a connected linear algebraic group, V a finite dimensional vector space and ρ a rational representation of G on V. We call a triple (G, ρ, V) a *prehomogeneous vector space* if there exists a proper algebraic subset S of V such that $V-S$ is a single G-orbit. The set S is called the *singular set of* (G, ρ, V). When G and V have structures over a field K such that ρ is defined over K, the triple (G, ρ, V) is said to be *defined over* K. An arithmetic significance of the theory of prehomogeneous vector spaces lies in a conjecture due to M. Sato that one can associate a system of Dirichlet series satisfying certain functional equations with a prehomogeneous vector space defined over an algebraic number field. This conjecture was taken up by Sato himself and Shintani in [14] under the hypothesis that G is reductive, S is an absolutely irreducible hypersurface and (G, ρ, V) is defined over the rational number field $\boldsymbol{Q}$. In this case, according to their results, one can associate with such a triple (G, ρ, V) a system of Dirichlet series in *one* complex variable which satisfies a functional equation similar to those of classical zeta functions such as the Riemann zeta function, the Epstein zeta function, etc. If we remove the assumptions above on the group G and the singular set S, it is natural to consider Dirichlet series in several complex variables. The purpose of this paper is to present a definition of zeta functions in several variables associated with a prehomegeneous vector space satisfying certain mild assumptions and to establish the conjecture of M. Sato for such zeta functions.

Igusa [6] posed a problem to associate a zeta function with a polynomial mapping with coefficients in an algebraic number field. Our result may be regarded as a partial answer to his problem.

0.2. Now we give a summary of this paper. For a prehomogeneous vector space (G, ρ, V) defined over an algebraic number field K, we are able to obtain a prehomogeneous vector space $R_{K/Q}(G, \rho, V)$ defined over $\boldsymbol{Q}$ by restricting the field of definition K to $\boldsymbol{Q}$. The zeta functions associated with (G, ρ, V) should coincide with those associated with

$R_{K/Q}(G, \rho, V)$ for various reasons. Hence, without loss of generality, we may assume that (G, ρ, V) is defined over $\boldsymbol{Q}$. Let $P_1, \cdots, P_n$ be $\boldsymbol{Q}$-irreducible polynomials defining the $\boldsymbol{Q}$-irreducible components of S with codimension 1. It is known that there exist $\boldsymbol{Q}$-rational characters $\chi_1, \cdots, \chi_n$ of G such that

$$P_i(\rho(g)x) = \chi_i(g)P_i(x) \quad (1 \leqq i \leqq n)$$

for all $g \in G$ and for all $x \in V$, namely, the polynomials $P_1, \cdots, P_n$ are relative invariants of (G, ρ, V). Let $G_{\boldsymbol{R}}^+$ be a subgroup of the real Lie group $G_{\boldsymbol{R}}$ containing the connected component of the identity element and let $V_{\boldsymbol{R}} - S_{\boldsymbol{R}} = V_1 \cup \cdots \cup V_\nu$ be the $G_{\boldsymbol{R}}^+$-orbit decomposition. We fix a matrix expression of G and a basis of V compatible with the given $\boldsymbol{Q}$-structure of (G, ρ, V) and such that $\rho(G_{\boldsymbol{Z}})V_{\boldsymbol{Z}} \subset V_{\boldsymbol{Z}}$. Put

$$\Gamma = \{g \in G_{\boldsymbol{Z}} \cap G_{\boldsymbol{R}}^+;\ \chi_i(g) = 1\ (1 \leqq i \leqq n)\}\ .$$

Let L be a Γ-invariant lattice in $V_{\boldsymbol{Q}}$ and set $L_i = L \cap V_i$ $(1 \leqq i \leqq \nu)$. Denote by $\Gamma\backslash L_i$ the set of all Γ-orbits in L_i. Let G_x be the isotropy subgroup of G at a point x in V and denote by G_x° the connected component of the identity element of G_x. Put $G_x^+ = G_x \cap G_{\boldsymbol{R}}^+$ and $\Gamma_x = G_x \cap \Gamma$. We assume that the group of $\boldsymbol{Q}$-rational characters of G_x° is trivial for all x in $V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$. Then, for any X in $V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$, the invariant volume of G_x^+/Γ_x is finite. For any rapidly decreasing function f on $V_{\boldsymbol{R}}$, consider the integrals

$$Z(f, L; s) = \int_{G_{\boldsymbol{R}}^+/\Gamma} \prod_{i=1}^{n} |\chi_i(g)|^{s_i} \sum_{x \in L-S} f(\rho(g)x)dg$$

and

$$\Phi_i(f; s) = \int_{V_i} \prod_{j=1}^{n} |P_j(x)|^{s_j} f(x)dx \quad (1 \leqq i \leqq \nu)$$

where dg is a right invariant measure on $G_{\boldsymbol{R}}^+$ and dx is a Euclidean measure on $V_{\boldsymbol{R}}$. The functions $\Phi_1, \cdots, \Phi_\nu$ have analytic continuations to meromorphic functions of s in $\boldsymbol{C}^n$. By a routine argument, we have at least formally the formula

$$Z(f, L; s) = \sum_{i=1}^{\nu} \xi_i(L; s)\Phi_i(f; s - \delta) \tag{0–1}$$

for some δ in $\boldsymbol{Q}^n$. Here $\xi_1, \cdots, \xi_\nu$ are the Dirichlet series defined by

$$\xi_i(L; s) = \sum_{x \in \Gamma\backslash L_i} \mu(x)|P_1(x)|^{-s_1} \cdots |P_n(x)|^{-s_n} \quad (1 \leqq i \leqq \nu,\ s \in \boldsymbol{C}^n)$$

where $\mu(x)$ is the volume of G_x^+/Γ_x with respect to a suitably normalized Haar measure on G_x^+. The Dirichlet series $\xi_1, \cdots, \xi_\nu$ are called *the zeta functions associated with* (G, ρ, V). We always assume that $\xi_1, \cdots, \xi_\nu$

are absolutely convergent when $\operatorname{Re} s_1, \cdots, \operatorname{Re} s_n$ are all sufficiently large. Then the formula (0-1) is justified in a domain on which both ξ_i's and Φ_i's are absolutely convergent.

If the representation ρ is irreducible, then we have at most one irreducible relative invariant up to a constant factor. So we are interested in a triple (G, ρ, V) such that ρ is reducible. Especially we consider the case where (G, ρ, V) is decomposed into the form $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, E \oplus F)$ over $\boldsymbol{Q}$ and the singular set S is a hypersurface in V. The invariant subspace F is called a *$\boldsymbol{Q}$-regular subspace of* (G, ρ, V) if there exists a relative invariant $P(x, y)$ $(x \in E, y \in F)$ with coefficients in $\boldsymbol{Q}$ such that the Hessian

$$H_{P,y}(x, y) = \det\left(\frac{\partial^2 P}{\partial y_i \, \partial y_j}(x, y)\right)$$

of P with respect to the variable y in F is not identically zero. We assume the $\boldsymbol{Q}$-regularity of F. Let F^* be the vector space dual to F. Denote by ρ_2^* the representation of G on F^* contragredient to ρ_2. Put $\rho^* = \rho_1 \oplus \rho_2^*$ and $V^* = E \oplus F^*$. As a consequence of the $\boldsymbol{Q}$-regularity of F, the triple (G, ρ^*, V^*) is also a prehomogeneous vector space which has a natural $\boldsymbol{Q}$-structure. The assumptions imposed on (G, ρ, V) are also satisfied by (G, ρ^*, V^*) with the only possible exception of the assumption on the convergence of the zeta functions. Moreover it can be seen that the singular set S^* of (G, ρ^*, V^*) is also a hypersurface in V^* with n $\boldsymbol{Q}$-irreducible components and $V_R^* - S_R^*$ is decomposed into ν orbits under the action of G_R^+. Let $\chi_1^*, \cdots, \chi_n^*$ be the $\boldsymbol{Q}$-rational characters of G corresponding to $\boldsymbol{Q}$-irreducible relative invariants $Q_1(x, y^*), \cdots, Q_n(x, y^*)$ defining the $\boldsymbol{Q}$-irreducible components of S^*. Then there exist an n by n unimodular matrix $U = (u_{ij})$ and an n-tuple λ of half-integers such that

$$\chi_i(g) = \prod_{j=1}^{n} \chi_j^*(g)^{u_{ij}} \quad (1 \leqq i \leqq n) \quad \text{and} \quad \det \rho_2(g)^2 = \prod_{i=1}^{n} \chi_i(g)^{2\lambda_i}.$$

Let M and N be Γ-invariant lattices in E_Q and F_Q respectively. Let N^* be the lattice dual to N. Put $L = M \oplus N$ and $L^* = M \oplus N^*$. For the triple (G, ρ^*, V^*) and a rapidly decreasing function f^* on V_R^*, define $Z^*(f^*, L^*; s)$, $\xi_i^*(L^*; s)$ and $\Phi_i^*(f^*; s)$ $(1 \leqq i \leqq \nu)$ as for (G, ρ, V). Finally we assume the absolute convergence of $\xi_1^*, \cdots, \xi_\nu^*$. Then we have

$$Z^*(f^*, L^*; s) = \sum_{i=1}^{\nu} \xi_i^*(L^*; s) \Phi_i^*(f^*; s - \delta^*) \tag{0-2}$$

where $\delta^* = (\delta - 2\lambda) U$. Denote by B and B^* the domains of absolute convergence of $Z(f, L; s)$ and $Z^*(f^*, L^*; s)$ respectively. Let D (resp.

D^*) be the convex hull of $(B^*U^{-1}+\lambda)\cup B$ (resp. $(B-\lambda)U\cup B^*$) in C^n. Notice that $(D-\lambda)U=D^*$. Set

$$\Phi(f;s)={}^t(\Phi_1(f;s),\cdots,\Phi_\nu(f;s))$$

and

$$\Phi^*(f^*;s)={}^t(\Phi_1^*(f^*;s),\cdots,\Phi_\nu^*(f^*;s))\ .$$

We define the partial Fourier transform of f^* with respect to F^* by the formula

$$\mathscr{F}f^*(x,y)=\int_{F_R^*}f^*(x,y^*)e^{2\pi\sqrt{-1}\langle y,y^*\rangle}dy^*\ .$$

THEOREM 1. *There exist a ν by ν matrix $A(s)$, a Gamma factor $\gamma(s)$ and non-zero complex numbers $c_1,\cdots,c_n$, which are independent of f^*, such that*

$$\Phi(\mathscr{F}f^*;s)=\prod_{i=1}^n c_i^{-s_i}(2\pi\sqrt{-1})^{d^*(s)}\gamma(s)A(s)\Phi^*(f^*;(s+\lambda)U)$$

where $d^*(s)=s_1\deg_{y^*}Q_1(x,y^*)+\cdots+s_n\deg_{y^*}Q_n(x,y^*)$ and all the entries of $A(s)$ are polynomial functions in $\exp(\pi s_1\sqrt{-1})$, $\exp(-\pi s_1\sqrt{-1})$, $\cdots$, $\exp(\pi s_n\sqrt{-1})$, $\exp(-\pi s_n\sqrt{-1})$.

This theorem is a generalization of Sato [11, Theorem 4], Sato and Shintani [14, Theorem 1] and Shintani [16, Theorem 1.1].

Set

$$\xi(L;s)={}^t(\xi_1(L;s),\cdots,\xi_\nu(L;s))$$

and

$$\xi^*(L^*;s)={}^t(\xi_1^*(L^*;s),\cdots,\xi_\nu^*(L^*;s))\ .$$

The following is the main theorem of the present paper.

THEOREM 2. (i) *The Dirichlet series $\xi_1(L;s),\cdots,\xi_\nu(L;s)$ (resp. $\xi_1^*(L^*;s),\cdots,\xi_\nu^*(L^*;s)$) have analytic continuations to meromorphic functions of s in D (resp. D^*).*

(ii) *There exists a polynomial $b(s)$ (resp. $b^*(s)$) in s such that $b(s-\delta)\xi_1(L;s),\cdots,b(s-\delta)\xi_\nu(L;s)$ (resp. $b^*(s-\delta^*)\xi_1^*(L^*;s),\cdots,b^*(s-\delta^*)\xi_\nu^*(L^*;s)$) are holomorphic functions in D (resp. D^*).*

(iii) *The following functional equation holds for $s\in D$:*

$$v(N^*)\xi^*(L^*;(s-\lambda)U)=\prod_{i=1}^n c_i^{\delta_i-s_i}(-2\pi\sqrt{-1})^{d^*(s-\delta)}\gamma(s-\delta)\,{}^tA(s-\delta)\xi(L;s)$$

where $v(N^*)=\int_{F_R^*/N^*}dy^*$.

Theorem 2 is derived from Theorem 1 and the integral representations

(0-1) and (0-2) of the zeta functions. Under the additional assumptions that G is reductive and V itself is a $\boldsymbol{Q}$-regular subspace, the domains D and D^* coincide with $\boldsymbol{C}^n$, and hence, the associated zeta functions are continued meromorphically in the whole of $\boldsymbol{C}^n$. It is another consequence of Theorem 2 that the zeta functions satisfy at least the same number of functional equations as the number of $\boldsymbol{Q}$-regular subspaces. As is seen in examples, it frequently occurs that (G, ρ, V) has several $\boldsymbol{Q}$-regular subspaces.

0.3. This paper is devided into seven sections. In § 1, § 2 and § 3, we investigate elementary properties of prehomogeneous vector spaces and their $\boldsymbol{Q}$-regular subspaces. The zeta functions are introduced in § 4. Generalizing the method used in [11] and [14], we prove Theorems 1 and 2 in § 5 and § 6 respectively. The final section is devoted to the study of concrete examples. The examples treated in § 7 are rather easy ones. Further applications of Theorem 2 will be seen in subsequent papers ([23], [24]). Another summary of this paper is found in [26].

0.4. The author would like to thank Professor M. Sato and the late Professor T. Shintani for their helpful advises and encouragement.

Notation. As usual, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ are the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For any non-zero real number x, $\operatorname{sgn} x$ is $x/|x|$. For any complex number x, we put $e[x] = \exp 2\pi\sqrt{-1}x$. Let R be a commutative ring with an identity element. We denote by $M(n; R)$ (resp. $M(n, m; R)$) the set of n by n (resp. n by m) matrices with entries in R. For any matrix A, denote by ${}^tA$ the transposed matrix of A. We use the symbols $\operatorname{tr} A$ and $\det A$, respectively, as abbreviations for the trace and the determinant of $A \in M(n; R)$. For an affine algebraic set X defined over a field K, we denote by X_K the set of K-rational points on X. If G is an algebraic group of matrices defined over $\boldsymbol{Q}$, the group of integral matrices with determinant ± 1 contained in G is denoted by $G_{\boldsymbol{Z}}$. For any finite dimensional real vector space V, $\mathscr{S}(V)$ is the space of rapidly decreasing functions on V. For any smooth manifold X, $C_0^\infty(X)$ is the space of smoth functions with compact support on X. Let (X, μ) be a measure space. Denote by $L^1(X, \mu)$ the space of μ-integrable functions on X. We denote by $\Gamma(z)$ the Gamma function and $\zeta(z)$ the Riemann zeta function.

1. *K*-structures on prehomogeneous vector spaces. Let (G, ρ, V) be a triple of a connected complex linear algebraic group G, a finite dimensional vector space V over $\boldsymbol{C}$ and a rational representation ρ of G on V. A triple (G, ρ, V) is called a *prehomogeneous vector space* (briefly a p.v.)

if there exists a proper algebraic subset of V such that $V-S$ is a single G-orbit. Then S is called the *singular set of* (G, ρ, V). By a generic point, we mean a point in $V-S$. For a rational character χ of G, a non-zero rational function P on V is called a *relative invariant of* (G, ρ, V) *corresponding to* χ if $P(\rho(g)x)=\chi(g)P(x)$ (for all $g \in G$ and for all $x \in V$).

For any subfield K of $\boldsymbol{C}$, if G and V admit K-structures such that ρ is defined over K, then (G, ρ, V) is said to be *defined over* K. From now on we fix a K-structure on (G, ρ, V). Identify V with $\boldsymbol{C}^s$ ($s=\dim V$) and G with a closed subgroup of $GL(r)$ defined over K. We may assume that all the entries of $\rho(g) \in \mathrm{Aut}(V)=GL(s)$ ($g \in G$) are rational functions on G with coefficients in K. Let $\mathrm{Gal}(\boldsymbol{C}/K)$ be the Galois group of $\boldsymbol{C}$ over K. The canonical action of σ in $\mathrm{Gal}(\boldsymbol{C}/K)$ on a rational function R on $\boldsymbol{C}^N$ is denoted by R^σ. Let $P(x)$ be a relative invariant corresponding to a rational character χ of G. Then, for any σ in $\mathrm{Gal}(\boldsymbol{C}/K)$, we have

$$P^\sigma(\rho(g)x)=\chi^\sigma(g)P^\sigma(x) \quad (g \in G,\ x \in V)\ . \tag{1-1}$$

LEMMA 1.1. *Let* (G, ρ, V) *be a* p.v. *defined over* K *and* S *be its singular set. Denote by* S' *the union of the irreducible components of* S *with codimension* 1. *Then both* S *and* S' *are defined over* K.

PROOF. For an $x \in V$, denote by G_x the isotropy subgroup of G at x. It obvious that $G_{x^\sigma}=(G_x)^\sigma$ for all $\sigma \in \mathrm{Gal}(\boldsymbol{C}/K)$. By the proof of [13, § 2, Proposition 2], x is in S if and only if $\dim G_x > \dim G - \dim V$. Since $\dim G_{x^\sigma}=\dim (G_x)^\sigma=\dim G_x$, S is stable under the action of $\mathrm{Gal}(\boldsymbol{C}/K)$. This implies that S is defined over K (cf. [2, Chapter AG, Theorem (14.4)]).

Let $S_1, \cdots, S_n$ be the irreducible components of S with codimension 1. Then $S'=S_1 \cup \cdots \cup S_n$. Since the singular set S is defined over K, S_i^σ ($\sigma \in \mathrm{Gal}(\boldsymbol{C}/K)$) is also an irreducible component of S with codimension 1 for every i. Hence $S'^\sigma=S'$ for any $\sigma \in \mathrm{Gal}(\boldsymbol{C}/K)$. Thus S' is also defined over K.

Let G_1 be the normal closed subgroup of G generated by the commutator group of G and the isotropy subgroup G_x at a generic point x. By the prehomogeneity, the group G_1 is independent of the choice of x. Let $X(G)$ be the group of rational characters of G. Denote by $X_\rho(G)$ the subgroup of $X(G)$ consisting of elements whose restrictions to G_1 are trivial:

$$X_\rho(G)=\{\chi \in X(G);\, \chi|_{G_1} \equiv 1\}\ . \tag{1-2}$$

It is known that the group $X_\rho(G)$ coincides with the group of rational characters corresponding to relative invariants of (G, ρ, V) (cf. [13, § 4, Proposition 19]). Denote by $X_\rho(G)_K$ the subgroup of $X_\rho(G)$ consisting of rational characters defined over K.

LEMMA 1.2. *Let (G, ρ, V) be a* p.v. *defined over K.*

(i) *There exists a finite Galois extension L of K such that any relative invariant coincides with a rational function with coefficients in L up to a constant factor.*

(ii) *Let $P(x)$ be a relative invariant corresponding to a rational character χ of G. Then $P(x)$ coincides with a rational function with coefficients in K up to a constant factor if and only if $\chi \in X_\rho(G)_K$.*

PROOF. (i) As in the proof of Lemma 1.1, let $S_1, \cdots, S_n$ be the irreducible components of S with codimension 1 and put $S' = S_1 \cup \cdots \cup S_n$. Since the algebraic set S' is defined over K, we may take a finite Galois extension L of K as a common field of definition of $S_1, \cdots, S_n$. For each S_i, let $P_i(x)$ be an irreducible polynomial with coefficients in L such that $S_i = \{x \in V; P_i(x) = 0\}$. Then, by [13, § 4, Proposition 5], the polynomials $P_1(x), \cdots, P_n(x)$ are relative invariants and any relative invariant $P(x)$ is of the form

$$P(x) = cP_1(x)^{m_1} \cdots P_n(x)^{m_n} \quad (c \in \boldsymbol{C},\ m_1, \cdots, m_n \in \boldsymbol{Z}) .$$

This proves the first assertion.

(ii) If $P(x)$ has coefficients in K, then, by (1-1),

$$P(\rho(g)x) = \chi^\sigma(g)P(x) \quad (\sigma \in \mathrm{Gal}(\boldsymbol{C}/K)) .$$

Since the characters χ and χ^σ correspond to the same relative invariant $P(x)$, $\chi^\sigma = \chi$ for all $\sigma \in \mathrm{Gal}(\boldsymbol{C}/K)$. Hence $\chi \in X_\rho(G)_K$. Conversely, suppose that χ is defined over K. By (i), we may assume that $P(x)$ has coefficients in a finite Galois extension L over K. Then the equality (1-1) implies that $P^\sigma(\rho(g)x) = \chi(g)P^\sigma(x)$ for arbitrary $\sigma \in \mathrm{Gal}(L/K)$. By [13, § 4, Proposition 3], there exists a non-zero constant $c_\sigma \in L^\times$ such that $P^\sigma = c_\sigma P$. It is obvious that $c_{\sigma\tau} = c_\tau^\sigma c_\sigma$. According to Hilbert-Speiser's theorem, one can find a constant $c \in L^\times$ such that $c_\sigma = (c^\sigma)^{-1}c$ for all $\sigma \in \mathrm{Gal}(L/K)$. Then $(cP)^\sigma = cP$ for all $\sigma \in \mathrm{Gal}(L/K)$. This completes the proof.

Let $S_1, \cdots, S_n$ be the K-irreducible components of S with codimension 1 and $P_1, \cdots, P_n$ be polynomials with coefficients in K defining $S_1, \cdots, S_n$, respectively. Denote by $\chi_1, \cdots, \chi_n$ the rational characters of G corresponding to $P_1, \cdots, P_n$, respectively.

The next lemma follows from [13, § 4, Proposition 5].

LEMMA 1.3. *These $P_1, \cdots, P_n$ are algbraically independent relative invariants and any relative invariant $P(x)$ with coefficients in K is of the form*

$$P(x) = cP_1(x)^{m_1} \cdots P_n(x)^{m_n} \quad (c \in K,\ m_1, \cdots, m_n \in \boldsymbol{Z}) .$$

These polynomials $P_1, \cdots, P_n$ are determined uniquely up to constant factors in K. We call the set $\{P_1, \cdots, P_n\}$ a *complete system of K-irreducible relative invariants of (G, ρ, V).*

The following lemma is an immediate consequence of Lemma 1.2 (ii), Lemma 1.3 and [13, § 4, Lemma 4].

LEMMA 1.4. *The group $X_\rho(G)_K$ is a free abelian group of rank n generated by $\chi_1, \cdots, \chi_n$.*

2. Direct sum of prehomogeneous vector spaces. 2.1. Let K be a subfield of $\boldsymbol{C}$. Let G be a connected linear algebraic group defined over K. Let ρ_1 and ρ_2 be K-rational representations of G on finite dimensional vector spaces E and F respectively. Put $V = E \oplus F$ and $\rho = \rho_1 \oplus \rho_2$. Here ρ is, by definition, the representation of G on V given by the following formula:

$$\rho(g)(x, y) = (\rho_1(g)x, \rho_2(g)y) \quad (g \in G,\ (x, y) \in E \oplus F = V) .$$

For an $x \in E$, denote by G_x the isotropy subgroup of G at x: $G_x = \{g \in G; \rho_1(g)x = x\}$. Let G_x° be the connected component of the identity element of G_x. If x is a K-rational point in E, then both G_x and G_x° are defined over K (cf. [2, Chapter 1, Proposition (1-2)]).

LEMMA 2.1. *Assume that (G, ρ, V) is a* p.v. *with the singular set S.*

(i) *The triples (G, ρ_1, E) and (G, ρ_2, F) are* p.v.'s *defined over K.*

(ii) *For a K-rational generic point x of (G, ρ_1, E), the triple (G_x°, ρ_2, F) is a* p.v. *defined over K, whose singular set S_x is given by*

$$S_x = \{y \in F; (x, y) \in S\} .$$

For an irreducible component W of S with codimension r, put

$$W_x = \{y \in F; (x, y) \in W\} .$$

If W_x is non-empty, it is of pure codimension r in F.

PROOF. The first part of the lemma is obvious. Let us prove the second part. Denote by $G_{x,y}$ the isotropy subgroup of G_x at $y \in F$. Then

$$G_{x,y} = \{g \in G_x; \rho_2(g)y = y\} = \{g \in G; \rho(g)(x, y) = (x, y)\} .$$

It is clear that the group $G_{x,y} \cap G_x^\circ$ contains the connected component of the identity element of $G_{x,y}$. Hence $\dim(G_{x,y}\cap G_x^\circ)=\dim G_{x,y}$ and $\dim G_x^\circ=\dim G_x$. Therefore, by [13, § 2, Proposition 2], the triple (G_x°, ρ_2, F) is a p.v. and y is a generic point of (G_x°, ρ_2, F) if and only if

$$\dim G_x - \dim G_{x,y} = \dim F\ . \tag{2-1}$$

Since x is a generic point of (G, ρ_1, E) and $\dim G - \dim G_x = \dim E$, the equality (2-1) is equivalent to the following: $\dim G - \dim G_{x,y} = \dim V$. This implies that (G_x°, ρ_2, F) is a p.v. with a generic point y if and only if (x, y) is a generic point of (G, ρ, V). Hence (G_x°, ρ_2, F) is a p.v. and $S_x=\{y\in F; (x, y)\in S\}$. It is obvious that (G_x°, ρ_2, F) is defined over K. Finally let W_x^1 be any irreducible component of W_x. Since x is a generic point, $\dim W_x^1 + \dim E \leqq \dim W$. Hence we have

$$r = \dim V - \dim W \leqq \dim F - \dim W_x^1\ .$$

On the other hand, W_x is the intersection of two irreducible varieties W and $\{x\}\times F$ and each component of W_x is of dimension not smaller than $\dim W + \dim F - \dim V = \dim F - r$. Thus we obtain $r = \dim F - \dim W_x^1$.

Let $(G, \rho, V) = (G, \rho_1\oplus\rho_2, E\oplus F)$ be a p.v. For a relative invariant $Q_1(x)$ of (G, ρ_1, E), put $Q(x, y) = Q_1(x)$. Obviously $Q(x, y)$ is a relative invariant of (G, ρ, V) independent of the second component $y\in F$. The mapping $Q_1\mapsto Q$ gives rise to a natural one to one correspondence between relative invariants of (G, ρ_1, E) and relative invariant of (G, ρ, V) independent of y. In the following we do not distinguish them.

Fix a K-rational generic point x of (G, ρ_1, E). Let P be a relative invariant of (G, ρ, V) which corresponds to $\chi\in X_\rho(G)_K$. Then, as a function of y, $P(x, y)$ is a relative invariant of (G_x°, ρ_2, F) which corresponds to $\chi|_{G_x^\circ}$, the restriction of χ to G_x°. Hence $\chi|_{G_x^\circ}\in X_{\rho_2}(G_x^\circ)_K$ for any $\chi\in X_\rho(G)_K$. Define a homomorphism

$$\alpha\colon X_\rho(G)_K \to X_{\rho_2}(G_x^\circ)_K$$

by the formula $\alpha(\chi)=\chi|_{G_x^\circ}$. A character χ is in the kernel of α if and only if it is a rational character of G which corresponds to a relative invariant $P(x, y)$ with coefficients in K of (G, ρ, V) independent of the second component y. Hence $\operatorname{Ker}\alpha = X_{\rho_1}(G)_K$. Set

$$X_{\rho|\rho_2}(G_x^\circ)_K = \text{the image of } \alpha \text{ in } X_{\rho_2}(G_x^\circ)_K\ . \tag{2-2}$$

This is the group of rational characters of G_x° which correspond to relative invariants of (G_x°, ρ_2, F) with coefficients in K obtained from relative invariants of (G, ρ, V) by restricting them to $\{x\}\times F$.

Put $n = \operatorname{rank} X_\rho(G)_K$ and $r = n - \operatorname{rank} X_{\rho_1}(G)_K$. Let $\{P_1, \cdots, P_n\}$ be a complete system of K-irreducible relative invariants of (G, ρ, V). Then exactly $n - r$ of P_i's are independent of the second component y. We may assume that $P_{r+1}, \cdots, P_n$ are independent of y. The set $\{P_{r+1}, \cdots, P_n\}$ is a complete system of K-irreducible relative invariants of (G, ρ_1, E). Let $\chi_1, \cdots, \chi_n$ be rational characters of G corresponding to $P_1, \cdots, P_n$, respectively.

LEMMA 2.2. *Fix a K-rational generic point x of (G, ρ_1, E). Then, as functions of y, $P_1(x, y), \cdots, P_r(x, y)$ are algebraically independent.*

PROOF. Assume that $P_1(x, y), \cdots, P_r(x, y)$ are not algebraically independent for a generic point x of (G, ρ_1, E). Then, by [13, § 4, Lemma 4], there exists an $(m_1, \cdots, m_r) \in \boldsymbol{Z}^r - \{(0, \cdots, 0)\}$ such that $\chi_1^{m_1} \cdots \chi_r^{m_r} \equiv 1$ on G_x°. Hence $\chi_1^{m_1} \cdots \chi_r^{m_r} \in \operatorname{Ker} \alpha = X_{\rho_1}(G)_K$. Since $X_{\rho_1}(G)_K$ is generated by $\chi_{r+1}, \cdots, \chi_n$, we have a non-trivial relation

$$\chi_1^{m_1} \cdots \chi_r^{m_r} = \chi_{r+1}^{m_{r+1}} \cdots \chi_n^{m_n} .$$

This contradicts the fact that $P_1, \cdots, P_n$ are algebraically independent (cf. Lemma 1.3).

COROLLARY. *The group $X_{\rho|\rho_2}(G_x^\circ)_K$ is a free abelian group of rank r generated by $\chi_1|_{G_x^\circ}, \cdots, \chi_r|_{G_x^\circ}$.*

LEMMA 2.3. *The following three assertions are equivalent.*

(i) *α is surjective, namely, $X_{\rho|\rho_2}(G_x^\circ)_K = X_{\rho_2}(G_x^\circ)_K$.*

(ii) *Any relative invariant $Q(y)$ of (G_x°, ρ_2, F) with coefficients in K is of the form*

$$c \prod_{i=1}^{r} P_i(x, y)^{m_i} \quad (c \in K,\ m_1, \cdots, m_r \in \boldsymbol{Z}) .$$

(iii) *For any $i = 1, \cdots, r$, $P_i(x, y)$ is a K-irreducible polynomial in y.*

PROOF. The equivalence of the first and the second assertions is quite obvious. We shall show that the second assertion implies the third. Assume that $P_i(x, y) = Q_1(y)Q_2(y)$ for some polynomials Q_1 and Q_2 in y with coefficients in K. Then, by Lemma 1.3, Q_1 and Q_2 are relative invariants of (G_x°, ρ_2, F) and, by the assumption, they are written as follows:

$$Q_k(y) = c_k \prod_{j=1}^{r} P_j(x, y)^{m_{kj}} \quad (c_k \in K,\ m_{kj} \in \boldsymbol{Z},\ k = 1, 2,\ j = 1, \cdots, r) .$$

Since Q_1 and Q_2 are polynomial functions in y and x is a generic point of (G, ρ_1, E), the rational functions $\prod_{j=1}^{r} P_j^{m_{1j}}$ and $\prod_{j=1}^{r} P_j^{m_{2j}}$ have no poles in $(E - S_E) \times F$ where we denote by S_E the singular set of (G, ρ_1, E).

Hence the exponents m_{kj} are all non-negative integers. As a function of y, $(\prod_{j=1}^{r} P_j(x, y)^{m_{1j}+m_{2j}})\cdot P_i(x, y)^{-1}$ is a constant and we have

$$\left(\prod_{j=1}^{r} P_j^{m_{1j}+m_{2j}}\right)\cdot P_i^{-1} = c \prod_{j=r+1}^{n} P_j^{u_j}$$

for some $c \in K$ and some $u_{r+1}, \cdots, u_n \in \boldsymbol{Z}$. Since m_{kj} are non-negative, this equality implies that

$$u_j = 0\ , \quad m_{kj} = 0 \quad (j \neq i)\ , \qquad \{m_{1i}, m_{2i}\} = \{1, 0\}\ .$$

Therefore the polynomial $P_i(x, y)$ of y is K-irreducible for any i. Finally, it follows from Lemma 2.1 (ii) that any irreducible component of the singular set of (G_x°, ρ_2, F) with codimension 1 is contained in the set of zero-points of $\prod_{j=1}^{r} P_i(x, y)$. Hence, by Lemma 1.3, the third condition implies the second.

Let F^* be the vector space dual to F and ρ_2^* be the representation of G on F^* contragredient to ρ_2. Put

$$V^* = E \oplus F^*\ , \qquad \rho^* = \rho_1 \oplus \rho_2^*\ .$$

We call ρ^* the partial contragredient representation of ρ with respect to F. Fix K-structures of (G, ρ_1, E) and (G, ρ_2, F) and identify F with $\boldsymbol{C}^m$ $(m = \dim F)$. We identify F^* with $\boldsymbol{C}^m$ via the symmetric bilinear form

$$\langle y, y^*\rangle = y_1y_1^* + \cdots + y_my_m^*\ .$$

Then ρ_2^* and ρ^* are K-rational representations of G on F^* and V^*, respectively. For any relative invariant P of (G, ρ, V), define a rational mapping ϕ_P of $V - S$ into V^* by

$$\phi_P(x, y) = (x, \operatorname{grad}_y \log P(x, y)) \tag{2-3}$$

where

$$\operatorname{grad}_y \log P(x, y) = \left(\frac{1}{P(x, y)}\frac{\partial}{\partial y_1}P(x, y), \cdots, \frac{1}{P(x, y)}\frac{\partial}{\partial y_m}P(x, y)\right)\ .$$

The mapping ϕ_P is independent of the choice of a basis in F. If P has coefficients in K, ϕ_P is defined over K. Moreover we have

$$\phi_P(\rho(g)(x, y)) = \rho^*(g)\phi_P(x, y) \quad ((x, y) \in V - S,\ g \in G)\ . \tag{2-4}$$

Put

$$H_{P,y}(x, y) = \det\left(\frac{\partial^2 P}{\partial y_i \partial y_j}(x, y)\right)\ .$$

If there exists a relative invariant P of (G, ρ, V) such that $H_{P,y}(x, y)$ is not identically zero, then F is called a *regular subspace of* (G, ρ, V).

Moreover, if P can be taken so that P has coefficients in K, we say that the subspace F is *regular over* K or K-*regular*. We call a p.v. (G, ρ, V) *regular over* K if V is a K-regular subspace. When $K = \boldsymbol{C}$, (G, ρ, V) is simply called regular instead of regular over $\boldsymbol{C}$. This terminology is consistent with [13, § 4, Definition 7]. If F is a K-regular subspace, the p.v. (G_x°, ρ_2, F) is regular over K for any K-rational generic point x of (G, ρ_1, E).

Basic properties of p.v.'s with a regular subspace are summarized in the next lemma which follows from Lemma 1.2 (ii) and [13, § 4, Proposition 10, Remark 11].

LEMMA 2.4. *Let* $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, E \oplus F)$ *be a* p.v. *with a* K-*regular subspace* F.

(i) *The triple* $(G, \rho^*, V^*) = (G, \rho_1 \oplus \rho_2^*, E \oplus F^*)$ *is a* p.v. *with a* K-*regular subspace* F^*.

(ii) *For an* $(x, y) \in V - S$, *put* $(x, y^*) = \phi_P(x, y)$. *If* $H_{P,y}$ *does not vanish identically, then* $G_{x,y} = G_{x,y^*}$.

(iii) $X_\rho(G)_K = X_{\rho^*}(G)_K$ *and* $X_{\rho_2}(G_x^\circ)_K = X_{\rho_2^*}(G_x^\circ)_K$ *for any* K-*rational generic point* x *of* (G, ρ_1, E).

(iv) *Let* S *and* S^* *be the singular sets of* (G, ρ, V) *and* (G, ρ^*, V^*) *respectively. For a* $\chi \in X_\rho(G)_K = X_{\rho^*}(G)_K$, *let* P *and* Q *be relative invariants of* (G, ρ, V) *and* (G, ρ^*, V^*) *corresponding to* χ *and* χ^{-1} *respectively. If* $H_{P,y}$ *does not vanish identically,* ϕ_P *is a biregular rational mapping defined over* K *of* $V - S$ *onto* $V^* - S^*$ *and the inverse mapping of* ϕ_P *is given by* ϕ_Q.

(v) *The singluar set* S *of* (G, ρ, V) *is a hypersurface if and only if the singular set* S^* *of* (G, ρ^*, V^*) *is a hypersurface.*

LEMMA 2.5. *Denote by* $\det \rho_2(g)$ *the determinant of* $\rho_2(g)$ *in* F. *If* F *is a* K-*regular subspace, then* $\det \rho_2(g)^2 \in X_\rho(G)_K$.

PROOF. Let P be a relative invariant of (G, ρ, V) such that $H_{P,y}$ is not identically zero. Then an easy computation shows that $P^m H_{P,y}^{-1}$ ($m = \dim F$) is a relative invariant of (G, ρ, V) corresponding to $\det \rho_2(g)^2$. Since ρ_2 is assumed to be K-rational, it is clear that $\det \rho_2(g)^2$ is a K-rational character of G.

2.2. Let (G, ρ, V) be a p.v. defined over a subfield K of $\boldsymbol{C}$. Assume that (G, ρ, V) is decomposed into the form $(G, \rho_1 \oplus \rho_2 \oplus \rho_3, V_1 \oplus V_2 \oplus V_3)$ over K and V_3 is a K-regular subspace. Denote by V_3^* the dual space of V_3 and by ρ_3^* the representation contragradient to ρ_3. Then the triple $(G, \rho^*, V^*) = (G, \rho_1 \oplus \rho_2 \oplus \rho_3^*, V_1 \oplus V_2 \oplus V_3^*)$ is also a p.v. Let S and S^*

be the singular sets of (G, ρ, V) and (G, ρ^*, V^*) respectively.

LEMMA 2.6. *The following assertions are equivalent.*

(1) $V_2 \oplus V_3$ *is a K-regular subspace of* (G, ρ, V).

(2) V_2 *is a K-regular subspace of* (G, ρ^*, V^*).

PROOF. Put $V^\# = V_1 \oplus V_2^* \oplus V_3^*$ where V_2^* is the vector space dual to V_2. For a relative invariant P of (G, ρ, V), we define two mappings $\phi_P\colon V - S \to V^\#$ and $\phi'_P\colon V - S \to V^*$ by

$$\phi_P(x, y, z) = \left(x, \frac{1}{P}\operatorname{grad}_y P(x, y, z), \frac{1}{P}\operatorname{grad}_z P(x, y, z)\right)$$

and

$$\phi'_P(x, y, z) = \left(x, y, \frac{1}{P}\operatorname{grad}_z P(x, y, z)\right).$$

By the K-regularity of V_3, we can find a P with coefficients in K such that ϕ'_P is a biregular rational mapping of $V - S$ onto $V^* - S^*$. Put $Q(x, y, z^*) = P(\phi'^{-1}_P(x, y, z^*))$. The function Q is a rational function on V^* defined over K and is a relative invariant of (G, ρ^*, V^*). Let ϕ_Q be the mapping of $V^* - S^*$ into $V^\#$ defined by

$$\phi_Q(x, y, z^*) = \left(x, \frac{1}{Q}\operatorname{grad}_y Q(x, y, z^*), z^*\right).$$

We shall prove the equality $\phi_P = \phi_Q \circ \phi'_P$. Since $P = Q \circ \phi'_P$, we have

$$\frac{\partial P}{\partial y_i}(x, y, z) = \left(\frac{\partial Q}{\partial y_i}\right)(\phi'_P(x, y, z)) + \sum_k \left(\frac{\partial Q}{\partial z_k^*}\right)(\phi'_P(x, y, z)) \times \frac{\partial}{\partial y_i}\left(\frac{1}{P}\cdot\frac{\partial P}{\partial z_k}\right)(x, y, z). \tag{2-5}$$

It is clear that the character of G corresponding to Q coincides with that corresponding to P. By Lemma 2.3 (iv), the mapping

$$\phi'_Q(x, y, z^*) = \left(x, y, -\frac{1}{Q}\operatorname{grad}_{z^*} Q(x, y, z^*)\right)$$

is the inverse mapping of ϕ'_P. Hence we have

$$\left(\frac{\partial Q}{\partial z_k^*}\right)(\phi'_P(x, y, z)) = -P(x, y, z)\cdot z_k$$

and the second term of the right hand side of (2-5) is equal to

$$-P\sum_k z_k\frac{\partial}{\partial y_i}\left(\frac{1}{P}\cdot\frac{\partial P}{\partial z_k}\right) = -P\frac{\partial}{\partial y_i}\left(\frac{1}{P}\sum_k z_k\frac{\partial P}{\partial z_k}\right).$$

The function P is homogeneous in z (cf. [13, § 4, Proposition 3]). By

Euler's identity, we obtain

$$\frac{\partial P}{\partial y_i}(x, y, z) = \left(\frac{\partial Q}{\partial y_i}\right)(\phi'_P(x, y, z)) \quad (1 \leqq i \leqq \dim V_2) .$$

The equality $\phi_P = \phi_Q \circ \phi'_P$ follows immediately from this identity. Either of the conditions (1) and (2) yields that $(G, \rho^\#, V^\#) = (G, \rho_1 \oplus \rho_2^* \oplus \rho_3^*, V_1 \oplus V_2^* \oplus V_3^*)$ is a p.v. Denote by $S^\#$ the singular set of this p.v. If the condition (1) is satisfied, there exists a relative invariant P of (G, ρ, V) such that ϕ_P and ϕ'_P are biregular rational mapping of $V - S$ onto $V^\# - S^\#$ and $V^* - S^*$ respectively. Then $\phi_Q = \phi_P \circ \phi_P'^{-1}$ is a biregular rational mapping of $V^* - S^*$ onto $V^\# - S^\#$. This implies the condition (2) (cf. [13, § 4, Proposition 10]). Similarly we are able to prove that the condition (2) implies the condition (1).

3. Partial b-functions. We keep the notation in § 2.1 and assume that F is a K-regular subspace. By Lemma 2.4 (iii), rank $X_\rho(G)_K =$ rank $X_{\rho^*}(G)_K$. Let $P_1, \cdots, P_n$ (resp. $Q_1, \cdots, Q_n$) be a complete system of K-irreducible relative invariants of (G, ρ, V) (resp. (G, ρ^*, V^*)) where $n = \operatorname{rank} X_\rho(G)_K = \operatorname{rank} X_{\rho^*}(G)_K$. For every i, the K-rational character of G corresponding to P_i (resp. Q_i) is denoted by χ_i (resp. χ_i^*). For a character χ in $X_\rho(G)_K = X_{\rho^*}(G)_K$, let $\delta(\chi) = (\delta(\chi)_1, \cdots, \delta(\chi)_n)$ and $\delta^*(\chi) = (\delta^*(\chi)_1, \cdots, \delta^*(\chi)_n)$ be the elements in $\boldsymbol{Z}^n$ such that

$$\chi = \prod_{i=1}^n \chi_i^{\delta(\chi)_i} = \prod_{i=1}^n \chi_i^{*\delta^*(\chi)_i} .$$

Since $\{\chi_1, \cdots, \chi_n\}$ and $\{\chi_1^*, \cdots, \chi_n^*\}$ form two system of generators of the free abelian group $X_\rho(G)_K$, there exists a unimodular matrix $U \in GL(n; \boldsymbol{Z})$ such that

$$\delta(\chi) U = \delta^*(\chi) \quad (\chi \in X_\rho(G)_K) . \tag{3-1}$$

In particular,

$$\chi_i = \prod_{j=1}^n \chi_j^{*u_{ij}} \quad (1 \leqq i \leqq n)$$

where u_{ij} is the (i, j)-entry of the matrix U.

For a $\chi \in X_\rho(G)_K = X_{\rho^*}(G)_K$, we put

$$P^\chi(x, y) = \prod_{i=1}^n P_i(x, y)^{\delta(\chi)_i} \quad \text{and} \quad Q^\chi(x, y^*) = \prod_{i=1}^n Q_i(x, y^*)^{\delta^*(\chi)_i} .$$

[13, § 4, Proposition 3] implies that P_i (resp. Q_i) are homogeneous with respect to the variable y (resp. y^*) in F (resp. F^*). Denote by $d(\chi)$ and $d^*(\chi)$, respectively, the homogeneous degrees of P^χ and Q^χ with respect to y and y^*:

$$d(\chi) = \deg_y P^\chi = \sum_{i=1}^n \delta(\chi)_i \deg_y P_i\ , \qquad d^*(\chi) = \deg_{y^*} Q^\chi = \sum_{i=1}^n \delta^*(\chi)_i \deg_{y^*} Q_i\ .$$

If $\delta^*(\chi)_i \geqq 0$ (resp. $\delta(\chi)_i \geqq 0$) for all i, we can define a partial differential operator $Q^\chi(x, \mathrm{grad}_y)$ (resp. $P^\chi(x, \mathrm{grad}_{y^*})$) in $K[x, \partial/\partial y]$ (resp. $K[x, \partial/\partial y^*]$) such that

$$(3\text{-}2)\qquad \begin{cases} Q^\chi(x, \mathrm{grad}_y)e^{\langle y, y^*\rangle} = Q^\chi(x, y^*)e^{\langle y, y^*\rangle}\ , \\ P^\chi(x, \mathrm{grad}_{y^*})e^{\langle y, y^*\rangle} = P^\chi(x, y)e^{\langle y, y^*\rangle}\ . \end{cases}$$

The operator $Q^\chi(x, \mathrm{grad}_y)$ (resp. $P^\chi(x, \mathrm{grad}_{y^*})$) has order $d^*(\chi)$ (resp. $d(\chi)$).

For $s = (s_1, \cdots, s_n) \in \boldsymbol{C}^n$, set

$$P^s(x, y) = \exp\left(\sum_{i=1}^n s_i \log (P_i(x, y))\right)$$

and

$$Q^s(x, y^*) = \exp\left(\sum_{i=1}^n s_i \log (Q_i(x, y^*))\right)\ .$$

We consider P^s (resp. Q^s) as a function on the universal covering space of $V - S$ (resp. $V^* - S^*$).

LEMMA 3.1. (i) *If $\delta^*(\chi)_i \geqq 0$ for all i, there exists a polynomial $b_\chi(s)$ of degree $d^*(\chi)$ in $s = (s_1, \cdots, s_n)$ satisfying*

$$Q^\chi(x, \mathrm{grad}_y)P^s(x, y) = b_\chi(s)P^{s+\delta(\chi)}(x, y)\ .$$

(ii) *If $\delta(\chi)_i \geqq 0$ for all i, there exists a polynomial $b_\chi^*(s)$ of degree $d(\chi)$ in $s = (s_1, \cdots, s_n)$ satisfying*

$$P^\chi(x, \mathrm{grad}_{y^*})Q^s(x, y^*) = b^*_\chi(s)Q^{s+\delta^*(\chi)}(x, y^*)\ .$$

PROOF. We give a proof only for the first assertion. Denote by $F(x, y)$ the left hand side of the equality. It follows from the definition of $Q^\chi(x, \mathrm{grad}_y)$ that

$$Q^\chi(\rho_1(g)x, \mathrm{grad}_{\rho_2(g)y}) = \chi(g)Q^\chi(x, \mathrm{grad}_y)\ .$$

Let W be a simply connected neighbourhood of the identity element e of G. Define a function $\chi^s(g)$ on W by setting

$$\chi^s(g) = \exp\left(\sum_{i=1}^n s_i \log (\chi_i(g))\right)\ , \qquad \log (\chi_i(e)) = 0\ .$$

Then we have $F(\rho(g)(x, y)) = \chi^s(g)\chi(g)F(x, y)$ for all $g \in W$. By the prehomogeneity, $\rho(W)(x, y)$ contains an open neighbourhood of (x, y) for any $(x, y) \in V - S$. Hence the equality above implies that $P^{-s-\delta(\chi)}(x, y)F(x, y)$ is a constant which depends only upon s and χ. Denote it by $b_\chi(s)$. It is clear that $b_\chi(s)$ is a polynomial in s of degree not greater than $d^*(\chi)$.

Let $a_\chi(s)$ be the part of $b_\chi(s)$ homogeneous of degree $d^*(\chi)$. Then, by an elementary calculation, we have

$$a_\chi(s)P^\chi(x, y) = Q^\chi(x, \mathrm{grad}_y \log P^s) = Q^\chi(\phi_{P^s}(x, y)) .$$

Since F is a K-regular subspace, there exists an s_0 in $\boldsymbol{Z}^n$ such that $\phi_{P^{s_0}}$ is a biregular mapping of $V - S$ onto $V^* - S^*$. Then $a_\chi(s_0) \neq 0$. Thus the polynomial $b_\chi(s)$ is of degree $d^*(\chi)$.

LEMMA 3.2. *Let χ and ψ be characters in $X_\rho(G)_K$.*

(i) *If $\delta^*(\chi)_i \geqq 0$ and $\delta^*(\psi)_i \geqq 0$ for all i, then*

$$b_{\chi\psi}(s) = b_\chi(s)b_\psi(s + \delta(\chi)) .$$

(ii) *If $\delta(\chi)_i \geqq 0$ and $\delta(\psi)_i \geqq 0$ for all i, then*

$$b^*_{\chi\psi}(s) = b^*_\chi(s)b^*_\psi(s + \delta^*(\chi)) .$$

PROOF. It is easy to see that the operators $Q^\chi(x, \mathrm{grad}_y)$ and $Q^\psi(x, \mathrm{grad}_y)$ commute and $Q^{\chi\psi}(x, \mathrm{grad}_y) = Q^\psi(x, \mathrm{grad}_y)Q^\chi(x, \mathrm{grad}_y)$. Now the first assertion is an immediate consequence of the definition of $b_\chi(s)$. The second assertion is proved quite similarly.

By using the formulas in Lemma 3.2, we can define $b_\chi(s)$ and $b^*_\chi(s)$ for arbitrary character χ in $X_\rho(G)_K$. We call the polynomial $b_\chi(s)$ (resp. $b^*_\chi(s)$) *the (partial) b-function of* (G, ρ, V) (resp. (G, ρ^*, V^*)) *with respect to the K-regular subspace F* (resp. F^*) *corresponding to χ.*

In the case where $E = \{0\}$, $F = V$ and $\rho_2 = \rho$, the b-functions were introduced by Sato and precisely investigated in [11]. It is easy to see that our partial b-functions are the b-functions of (G_x°, ρ_2, F) in the sense of [11] and the results of Sato can be applied to our case without any essential change. The next lemma due to Sato plays an important role in § 5.

LEMMA 3.3 ([11, Theorem 2, Theorem 3, Corollary to Theorem 3]). *There exist a homomorphism $c: X_\rho(G)_K \to \boldsymbol{C}^\times$, non-zero linear forms $e_1, \cdots, e_m: \boldsymbol{C}^n \to \boldsymbol{C}$ and a Gamma factor*

$$\gamma(s) = \prod_{i=1}^m \left\{\prod_{j=1}^{\alpha_i} \Gamma(e_i(s) - p_{ij})\right\} \left\{\prod_{j=1}^{\beta_i} \Gamma(e_i(s) - q_{ij})\right\}^{-1} \quad (p_{ij}, q_{ij} \in \boldsymbol{C})$$

with the following properties:

(1) *All the coefficients of $e_1, \cdots, e_m$ are non-negative integers,*

(2) $b_\chi(s) = c(\chi)\gamma(s)/\gamma(s + \delta(\chi))$.

Notice that $b^*_\chi(s)$ has a similar expression in terms of the Gamma function.

For the general theory of b-functions, see [11] and [12]. The determination of b-functions for irreducible p.v.'s is treated by Kimura in [7].

4. Definition of zeta functions. Let (G, ρ, V) be a p.v. defined over $\boldsymbol{Q}$ and denote by S the singular set of (G, ρ, V). Let $P_1, \cdots, P_n$ be a complete system of $\boldsymbol{Q}$-irreducible relative invariants of (G, ρ, V). We denote by χ_i the character corresponding to P_i $(1 \leqq i \leqq n)$. For any $x \in V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$, denote by G_x° the connected component (with respect to the Zariski topology) of the identity component of the isotropy subgroup of G at x and put

$$(4\text{-}1) \qquad V'_{\boldsymbol{Q}} = \{x \in V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}};\ X(G_x^\circ)_{\boldsymbol{Q}} = \{1\}\} .$$

The set $V'_{\boldsymbol{Q}}$ is $\rho(G_{\boldsymbol{Q}})$-stable.

LEMMA 4.1. *If $V'_{\boldsymbol{Q}}$ is not empty, then* $\operatorname{rank} X_\rho(G)_{\boldsymbol{Q}} = \operatorname{rank} X(G)_{\boldsymbol{Q}}$.

PROOF. Take an $x \in V'_{\boldsymbol{Q}}$ and put $m = [G_x : G_x^\circ]$. Since $\chi|_{G_x^\circ} \equiv 1$ for any $\chi \in X(G)_{\boldsymbol{Q}}$, $\chi(g)^m = \chi(g^m) = 1$ for all $g \in G_x$. This implies that $\{\chi^m;\ \chi \in X(G)_{\boldsymbol{Q}}\} \subset X_\rho(G)_{\boldsymbol{Q}}$. Hence we have $\operatorname{rank} X_\rho(G)_{\boldsymbol{R}} = \operatorname{rank} X(G)_{\boldsymbol{Q}}$.

We always assume that $V'_{\boldsymbol{Q}}$ is not empty. Let Ω be a right invariant algebraic gauge form on G. Define a character Δ of G by the following formula: $\Omega(gx) = \Delta(g)\Omega(x)$. Then $\Delta \in X(G)_{\boldsymbol{Q}}$. By Lemma 4.1, there exists a natural number d such that $(\det \rho \cdot \Delta^{-1})^d \in X_\rho(G)_{\boldsymbol{Q}}$. Put $\delta = (\delta_1, \cdots, \delta_n) = d^{-1}\delta((\det \rho \cdot \Delta^{-1})^d)$:

$$(\det \rho(g)\Delta(g)^{-1})^d = \prod_{i=1}^n \chi_i(g)^{d\delta_i} .$$

Let $G_{\boldsymbol{R}}^+$ be a subgroup of the real Lie group $G_{\boldsymbol{R}}$ containing the connected component of the identity element. Then $V_{\boldsymbol{R}} - S_{\boldsymbol{R}}$ is decomposed into a finite number of $G_{\boldsymbol{R}}^+$-orbits (see the proof of Lemma 5.1). Let $V_{\boldsymbol{R}} - S_{\boldsymbol{R}} = V_1 \cup \cdots \cup V_\nu$ be the $G_{\boldsymbol{R}}^+$-orbit decompositon. Let $|\Delta|$ be the character of $G_{\boldsymbol{R}}^+$ defined by $|\Delta|(g) = |\Delta(g)|$. Normalize a $G_{\boldsymbol{R}}^+$-relative invariant measure $\omega(x)$ on $V_{\boldsymbol{R}} - S_{\boldsymbol{R}}$ with multiplier $|\Delta|$ by setting $\omega(x) = |P(x)|^{-\delta}dx$ where $|P(x)|^{-\delta}$ is an abbreviation for $|P_1(x)|^{-\delta_1} \cdots |P_n(x)|^{-\delta_n}$ and dx is a Euclidean measure on $V_{\boldsymbol{R}}$. Let dg be a right invariant measure on $G_{\boldsymbol{R}}^+$. Then

$$\int_{G_{\boldsymbol{R}}^+} F(h^{-1}g)dg = |\Delta|(h) \int_{G_{\boldsymbol{R}}^+} F(g)dg \quad (F \in L^1(G_{\boldsymbol{R}}^+; dg)) .$$

We fix a matrix expression of G and a basis of V compatible with the given $\boldsymbol{Q}$-structure of (G, ρ, V) and such that $\rho(G_{\boldsymbol{Z}})V_{\boldsymbol{Z}} \subset V_{\boldsymbol{Z}}$. Put

$$\Gamma = \{g \in G_{\boldsymbol{Z}} \cap G_{\boldsymbol{R}}^+;\ \chi_1(g) = \cdots = \chi_n(g) = 1\} .$$

For any $x \in V'_Q$, we set $G_x^+ = G_x \cap G_R^+$ and $\Gamma_x = \Gamma \cap G_x^+$. Then, by [3, Theorem 9-4], G_x^+ is unimodular and G_x^+/Γ_x has a finite invariant volume. We normalize a Haar measure $d\mu_x$ on G_x^+ such that

$$\int_{G_R^+} F(g)dg = \int_{G_R^+/G_x^+} \omega(\rho(g)x) \int_{G_x^+} F(gh)d\mu_x(h) \tag{4-2}$$

$(F \in L^1(G_R^+; dg),\ x \in V'_Q)$. Put $\mu(x) = \int_{G_x^+/\Gamma_x} d\mu_x$ $(x \in V'_Q)$.

Let L be a $\rho(\Gamma)$-invariant lattice in V_Q and set $L' = L \cap V'_Q$ and $L_i = L' \cap V_i$ $(1 \leqq i \leqq \nu)$. The sets $L', L_1, \cdots, L_\nu$ are also $\rho(\Gamma)$-invariant. Denote by $\Gamma\backslash L_i$ the set of all $\rho(\Gamma)$-orbits in L_i.

In the sequel, we use the symbols $|P(x)|^s$ and $|\chi(g)|^s$ as abbreviations for

$$\prod_{i=1}^n |P_i(x)|^{s_i} \quad \text{and} \quad \prod_{i=1}^n |\chi_i(g)|^{s_i}$$

respectively $(x \in V_R - S_R,\ g \in G_R^+,\ s \in C^n)$.

DEFINITION. The Dirichlet series

$$\xi_i(L; s) = \sum_{x \in \Gamma\backslash L_i} \mu(x)|P(x)|^{-s} \quad (s \in C^n,\ 1 \leqq i \leqq \nu)$$

are called the *zeta functions associated with* (G, ρ, V) (and L).

In the following, we assume that

(4-3) *the Dirichlet series $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ are absolutely convergent in a domain of the form $\{s \in C^n; \operatorname{Re} s_i > a_i\ (1 \leqq i \leqq n)\}$ for sufficiently large real numbers $a_1, \cdots, a_n$.*

For an $f \in \mathscr{S}(V_R)$, we consider the following integrals:

$$\Phi_i(f; s) = \int_{V_i} |P(x)|^s f(x)dx \quad (1 \leqq i \leqq \nu)$$

and

$$Z(f, L; s) = \int_{G_R^+/\Gamma} |\chi(g)|^s \sum_{x \in L'} f(\rho(g)x)dg\ .$$

When $\operatorname{Re} s_1 > 0, \cdots, \operatorname{Re} s_n > 0$, the integrals $\Phi_1(f; s), \cdots, \Phi_\nu(f; s)$ are absolutely convergent and represent holomorphic functions of s (cf. Lemma 5.2). The following lemma, which gives an integral representation of $\xi_1, \cdots, \xi_\nu$, is an immediate consequence of the assumption (4-3).

LEMMA 4.2. *Let $a_1, \cdots, a_n$ be as in* (4-3). *Then the integral $Z(f, L; s)$ $(f \in \mathscr{S}(V_R))$ is absolutely convergent in the domain*

$$B = \{s \in C^n; \operatorname{Re} s_i > \operatorname{Max}(a_i, \delta_i)\ (1 \leqq i \leqq n)\}$$

and the following identity holds:

$$Z(f, L; s) = \sum_{i=1}^{\nu} \xi_i(L; s)\Phi_i(f; s - \delta) \quad (s \in B) .$$

REMARK. It is a conjecture that the condition (4–3) always holds for $a_1 = \delta_1, \cdots, a_n = \delta_n$ (cf. [14, p. 154, Remark 1]). In [23], we shall establish the conjecture in a particular case.

5. Partial Fourier transforms of complex powers of relative invariants. 5.1. In this section, we keep the notation in § 2 and § 3 and we always assume the following two conditions (5–1) and (5–2):

(5–1) *(G, ρ_1, E) and (G, ρ_2, F) are defined over a subfield K of $\boldsymbol{R}$ and F is a K-regular subspace of the* p.v. $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, V_1 \oplus V_2)$.

(5–2) *The singular set S of (G, ρ, V) is a hypersurface.*

Here the subspaces E and F may be $\{0\}$ and V respectively. Lemma 2.4 (i) and (v) imply that these conditions are satisfied also by the p.v. $(G, \rho^*, V^*) = (G, \rho_1 \oplus \rho_2^*, E \oplus F^*)$ and F^*.

For simplicity, we further assume that

(5–3) *there exists a positive integer d such that* $(\det \rho_1)^d \in X_\rho(G)_K$.

Let G_R^+ be as in § 4.

LEMMA 5.1. *Under the assumption* (5-1), *the sets $V_R - S_R$ and $V_R^* - S_R^*$ decompose into the same finite number of G_R^+-orbits.*

PROOF. By (5–1) and Lemma 2.4, there exists a relative invariant $P(x, y)$ with real coefficients such that ϕ_P is a biregular mapping from $V - S$ onto $V^* - S^*$ defined over $\boldsymbol{R}$ (for the definition of ϕ_P, see (2–3)). Let χ be the rational character corresponding to P and let Q be a relative invariant of (G, ρ^*, V^*) corresponding to χ^{-1}. By Lemma 1.2 (ii), the character χ and χ^{-1} are defined over $\boldsymbol{R}$ and we may assume that Q has real coefficients. Then it follows from Lemma 2.4 (iv) that the inverse mapping of ϕ_P is ϕ_Q and is also defined over $\boldsymbol{R}$. Hence, by (2–4), the mapping ϕ_P gives a G_R^+-equivariant homeomorphism between $V_R - S_R$ and $V_R^* - S_R^*$. This implies that there exists a one to one correspondence between G_R^+-orbits in $V_R - S_R$ and those in $V_R^* - S_R^*$. Since G_R^+ contains the identity component of G_R, the number of G_R^+-orbits in $V_R - S_R$ is not greater than that of the topological components of $V_R - S_R$. By [3, Proposition 2.3], it is finite.

Let

$$V_R - S_R = V_1 \cup \cdots \cup V_\nu \quad \text{and} \quad V_R^* - S_R^* = V_1^* \cup \cdots \cup V_\nu^*$$

be their G_R^+-orbit decompositions. Let dx, dy and dy^* be Euclidean measures on E_R, F_R and F_R^* respectively. Put

$$\Phi_i(f; s) = \int_{V_i} |P(x, y)|^s f(x, y) dx dy$$

and

$$\Phi_i^*(f^*; s) = \int_{V_i^*} |Q(x, y^*)|^s f^*(x, y^*) dx dy^*$$

($1 \leqq i \leqq \nu$, $s \in \boldsymbol{C}^n$, $f \in \mathscr{S}(V_R)$, $f^* \in \mathscr{S}(V_R^*)$). The meromorphic properties of complex powers of polynomials were studied by Bernstein and Gelfand [1]. The following lemma is essentially due to them and is proved by the method indicated at the end of [1].

LEMMA 5.2. (i) *When* $\operatorname{Re} s_1 > 0, \cdots, \operatorname{Re} s_n > 0$, *the integrals* $\Phi_i(f; s)$ *and* $\Phi_i^*(f^*; s)$ *are absolutely convergent and represent holomorphic functions. Moreover they have analytic continuations to meromorphic functions of* s *in* $\boldsymbol{C}^n$.

(ii) *There exist* Γ-*factors* $\gamma_\rho(s)$ *and* $\gamma_{\rho^*}(s)$ *independent of* f *and* f^* *of the form*

$$\gamma_\rho(s) = \prod_{i=1}^m \Gamma(a_{i,1}s_1 + \cdots + a_{i,n}s_n + b_i) \quad (a_{ij}, b_i \in \boldsymbol{Q}) ,$$

$$\gamma_{\rho^*}(s) = \prod_{i=1}^m \Gamma(a_{i,1}^*s_1 + \cdots + a_{i,n}^*s_n + b_i^*) \quad (a_{ij}^*, b_i^* \in \boldsymbol{Q})$$

such that $\gamma_\rho(s)^{-1}\Phi_i(f; s)$ *and* $\gamma_{\rho^*}(s)^{-1}\Phi_i^*(f^*; s)$ *are entire functions.*

(iii) *The mappings*

$$\mathscr{S}(V_R) \ni f \mapsto \Phi_i(f; s) \in \boldsymbol{C} \quad \text{and} \quad \mathscr{S}(V_R^*) \ni f^* \mapsto \Phi_i^*(f^*; s) \in \boldsymbol{C}$$

are tempered distributions depending meromorphically on s. *Let* D_0 *be a bounded domain in* $\boldsymbol{R}^n$ *such that* $\Phi_i(f; s)$ *and* $\Phi_i^*(f^*; s)$ *are holomorphic functions in the tube domain* $D = D_0 + \sqrt{-1}\boldsymbol{R}^n$. *Then the orders of these tempered distributions are bounded for* $s \in D$.

By Lemma 2.5, we have $\det \rho_2(g)^2 \in X_\rho(G)_K$. Set

$$\lambda = (\lambda_1, \cdots, \lambda_n) = 2^{-1}\delta((\det \rho_2)^2) . \tag{5-4}$$

For any $f^* \in \mathscr{S}(V_R^*)$, we define the *partial Fourier transform* $\mathscr{F}f^*$ with respect to the K-regular subspace F^* by setting

$$\mathscr{F}f^*(x, y) = \int_{F_R^*} f^*(x, y^*) e[\langle y, y^* \rangle] dy^* \quad ((x, y) \in V_R) .$$

Let $\gamma(s)$ and $c(\chi)$ be as in Lemma 3.3. Put

$$c(s) = c(\chi_1)^{s_1} \cdots c(\chi_n)^{s_n}$$

and

$$d^*(s) = s_1 \cdot \deg_{y^*} Q_1 + \cdots + s_n \cdot \deg_{y^*} Q_n \, .$$

Then $c(\delta(\chi)) = c(\chi)$ and $d^*(\delta^*(\chi)) = d^*(\chi)$. Also put

$$\Phi(f; s) = {}^t(\Phi_1(f; s), \cdots, \Phi_\nu(f; s))$$

and

$$\Phi^*(f^*; s) = {}^t(\Phi_1^*(f^*; s), \cdots, \Phi_\nu^*(f^*; s)) \, .$$

Now we can state the first main theorem of the present paper.

THEOREM 1. *The functions* $\Phi_1(f; s), \cdots, \Phi_\nu(f; s)$ *and* $\Phi_1^*(f^*; s), \cdots, \Phi_\nu^*(f^*; s)$ *satisfy the following functional equation*:

$$\Phi(\mathscr{F} f^*; s) = c(-s)(-2\pi\sqrt{-1})^{d^*(s)}\gamma(s)A(s)\Phi^*(f^*; (s + \lambda)U) \tag{5-5}$$

where $A(s)$ *is a* $\nu \times \nu$ *matrix whose entries are polynomials in* $\exp(\pm\pi\sqrt{-1}s_1), \cdots, \exp(\pm\pi\sqrt{-1}s_n)$.

We are able to prove Theorem 1 by using the similar argument to that in [11] and [16] where the theorem is shown under the additional assumptions that $E = \{0\}$, $F = V$, $\rho = \rho_2$, $K = \boldsymbol{R}$, $X_\rho(G)_{\boldsymbol{R}} = X_\rho(G)_{\boldsymbol{C}}$ and G is a reductive algebraic group. For the sake of completeness, we shall give a proof.

PROOF OF THEOREM 1. As is easily seen, it is sufficient to prove the theorem for the case where the group $G_{\boldsymbol{R}}^+$ is the identity component of $G_{\boldsymbol{R}}$. Then the sign of any relative invariant does not change on a $G_{\boldsymbol{R}}^+$-orbit. For any i $(1 \leqq i \leqq \nu)$, set

$$\varepsilon(i) = (\varepsilon_1(i), \cdots, \varepsilon_n(i)) \, , \qquad \varepsilon_j(i) = \operatorname{sgn} P_j(x, y) \, , \quad (x, y) \in V_i$$

and

$$\varepsilon^*(i) = (\varepsilon_1^*(i), \cdots, \varepsilon_n^*(i)) \, , \qquad \varepsilon_j^*(i) = \operatorname{sgn} Q_j(x, y^*) \, , \quad (x, y^*) \in V_i^* \, .$$

Moreover we define $\varepsilon(i)^s$ and $\varepsilon^*(i)^s$ by the formulas

$$\varepsilon(i)^s = \prod_{j=1}^n \varepsilon_j(i)^{s_j} \quad \text{and} \quad \varepsilon^*(i)^s = \prod_{j=1}^n \varepsilon_j^*(i)^{s_j}$$

where $\varepsilon_j(i)^{s_j}$ (resp. $\varepsilon_j^*(i)^{s_j}$) $= \exp(2\pi\sqrt{-1}s_j)$ or $\exp(\pi\sqrt{-1}s_j)$ according as $\varepsilon_j(i)$ (resp. $\varepsilon_j^*(i)$) $= 1$ or -1.

LEMMA 5.3. (i) *For any* $\chi \in X_\rho(G)_K$ *such that* $\delta^*(\chi)_1, \cdots, \delta^*(\chi)_n \geqq 0$, *we obtain*

$$\Phi_i(Q^\chi(x, \operatorname{grad}_y)f; s) = (-1)^{d^*(\chi)}\varepsilon(i)^{\delta(\chi)}b_\chi(s)\Phi_i(f; s + \delta(\chi))$$

$(f \in \mathscr{S}(V_{\boldsymbol{R}}), \ 1 \leqq i \leqq \nu)$.

(ii) *For any* $\chi \in X_\rho(G)_K$ *such that* $\delta(\chi)_1, \cdots, \delta(\chi)_n \geqq 0$, *we obtain*

$$\Phi_i^*(P^\chi(x, \mathrm{grad}_{y^*})f^*; s) = (-1)^{d(\chi)}\varepsilon^*(i)^{\delta^*(\chi)} b_\chi^*(s)\Phi_i^*(f^*; s + \delta^*(\chi))$$

$(f^* \in \mathscr{S}(V_R^*),\ 1 \leqq i \leqq \nu)$.

PROOF. Integrating by parts, we can easily derive the formulas from Lemma 3.1.

For $f \in \mathscr{S}(V_R)$ and $f^* \in \mathscr{S}(V_R^*)$, put $f_g(x, y) = f(\rho(g)(x, y))$ and $f_g^*(x, y^*) = f^*(\rho^*(g)(x, y^*))$ $(g \in G_R^+)$. It is easy to check the following lemma.

LEMMA 5.4.
(i) $\Phi_i(f_g; s) = |\chi(g)|^{-s}|\det \rho(g)|^{-1}\Phi_i(f; s)$,
(ii) $\Phi_i^*(f_g^*; s) = |\chi^*(g)|^{-s}|\det \rho^*(g)|^{-1}\Phi_i^*(f^*; s)$ $(g \in G_R^+,\ 1 \leqq i \leqq \nu)$.

LEMMA 5.5. *The functions* $\Phi_1(f; s), \cdots, \Phi_\nu(f; s)$ *and* $\Phi_1^*(f^*; s), \cdots, \Phi_\nu^*(f^*; s)$ *satisfy the following functional equation*:

$$\Phi(\mathscr{F}f^*; s) = c(-s)(-2\pi\sqrt{-1})^{d^*(s)}\gamma(s)A(s)\Phi^*(f^*; (s + \lambda)U)$$

where $A(s)$ *is a* $\nu \times \nu$ *matrix whose* (i, j)*-entry* $A_{ij}(s)$ *is a product of* $\varepsilon^*(j)^{sU}\varepsilon(i)^{-s}$ *and an entire function* $t_{ij}(s)$ *of* s *with the period lattice* Z^n:

$$A_{ij}(s) = \varepsilon^*(j)^{sU}\varepsilon(i)^{-s}t_{ij}(s)\ .$$

PROOF. Consider the continuous linear forms T_s and T_s^* on $C_0^\infty(V_j^*)$ defined by $T_s(f^*) = \Phi_i(\mathscr{F}f^*; s)$ and $T_s^*(f^*) = \Phi_j^*(f^*; s)$. Since $\mathscr{F}(f_g^*)(x, y) = |\det \rho_2(g)|(\mathscr{F}f^*)_g(x, y)$, we have, by Lemma 5.4 (i),

$$T_s(f_g^*) = |\det \rho_1(g)|^{-1}|\chi(g)|^{-s}T_s(f^*) \quad (g \in G_R^+)\ .$$

On the other hand, it follows from (3-1) and Lemma 5.4 (ii) that

$$T_{(s+\lambda)U}^*(f_g^*) = |\det \rho_1(g)|^{-1}|\chi(g)|^{-s}T_{(s+\lambda)U}^*(f^*) \quad (g \in G_R^+)\ .$$

Therefore, by a theorem of Bruhat (see, e.g., [22, Theorem 5.2.1.4]), there exists a constant $h_{ij}(s)$ independent of f^* such that $T_s(f^*) = h_{ij}(s)T_{(s+\lambda)U}^*(f^*)$ for any $f^* \in C_0^\infty(V_j^*)$. The meromorphy of $h_{ij}(s)$ is an immediate consequence of Lemma 5.2 (i). This equality implies that

$$\text{(5-6)} \qquad \Phi_i(\mathscr{F}f^*; s) = \sum_{j=1}^{\nu} h_{ij}(s)\Phi_j^*(f^*; (s + \lambda)U)$$

for all $f^* \in C_0^\infty(V_R^* - S_R^*)$. If we define a tempered distribution T_s' on V_R^* by setting

$$T_s'(f^*) = \Phi_i(\mathscr{F}f^*; s) - \sum_{j=1}^{\nu} h_{ij}(s)\Phi_j^*(f^*; (s + \lambda)U)\ ,$$

then the support of T'_s is contained in the hypersurface S^*_R. For a $\chi \in X_\rho(G)_K$ with $\delta^*(\chi)_1, \cdots, \delta^*(\chi)_n \geqq 0$, let $Q^\chi(x, \mathrm{grad}_y)$ be the partial differential operator introduced in § 3. Then

$$\mathscr{F}(Q^\chi f^*)(x, y) = (2\pi\sqrt{-1})^{-d^*(\chi)} Q^\chi(x, \mathrm{grad}_y)\mathscr{F} f^*(x, y) .$$

Hence Lemma 5.3 (i) yields that

$$(Q^\chi T'_s)(f^*) = (-2\pi\sqrt{-1})^{-d^*(\chi)} \varepsilon(i)^{\delta(\chi)} b_\chi(s) \Phi_i(\mathscr{F} f^*; s + \delta(\chi)) - \sum_{j=1}^{\nu} \varepsilon^*(j)^{\delta^*(\chi)} h_{ij}(s) \Phi^*_j(f^*; (s + \delta(\chi) + \lambda)U) .$$

Let $D = D_0 + \sqrt{-1}\boldsymbol{R}^n$ be as in Lemma 5.2 (iii). There exists a constant M such that the order of T'_s does not exceed M for all $s \in D$. If $\delta^*(\chi)_1, \cdots, \delta^*(\chi)_n \geqq M$, we have by [16, Lemma 1.3]

$$(Q^\chi T'_s)(f^*) = 0 \quad (s \in D) \tag{5-7}$$

for any $f^* \in \mathscr{S}(V^*_R)$. Comparing (5-6) with (5-7) for $f^* \in C_0^\infty(V^*_R - S^*_R)$, we obtain

$$h_{ij}(s + \delta(\chi)) = (-2\pi\sqrt{-1})^{d^*(\chi)} \varepsilon^*(j)^{\delta^*(\chi)} \varepsilon(i)^{-\delta(\chi)} b_\chi(s)^{-1} h_{ij}(s) . \tag{5-8}$$

The equalities (5-7) and (5-8) imply that the functional equation (5-6) is valid for any $f^* \in \mathscr{S}(V^*_R)$ and for any s in $D + \delta(\chi)$. By the principle of analytic continuation, we see that (5-6) holds for any s in $\boldsymbol{C}^n$. Making use of the cocycle property of $b_\chi(s)$ (cf. Lemma 3.2), we can easily check that (5-8) holds for any $\chi \in X_\rho(G)_K$. Hence the functions

$$t_{ij}(s) = c(s)(-2\pi\sqrt{-1})^{-d^*(s)} \gamma(s)^{-1} \varepsilon^*(j)^{-sU} \varepsilon(i)^s h_{ij}(s) \quad (1 \leqq i, j \leqq \nu)$$

are periodic functions with the period lattice $\boldsymbol{Z}^n = \delta(X_\rho(G)_K)$. We have by (5-6)

$$t_{ij}(s) = c(s)(-2\pi\sqrt{-1})^{-d^*(s)} \gamma(s)^{-1} \varepsilon^*(j)^{-sU} \varepsilon(i)^s \Phi_i(\mathscr{F} f^*; s) \Phi^*_j(f^*; (s + \lambda)U)^{-1}$$

for $f^* \in C_0^\infty(V^*_j)$. By Lemma 3.3 (ii), the function $\gamma(s)^{-1}$ is holomorphic if $\mathrm{Re}\, s_1, \cdots, \mathrm{Re}\, s_n$ are sufficiently large. Moreover, for a given s, we can choose an f^* such that $\Phi^*_j(f^*; (s + \lambda)U) \neq 0$. Therefore $t_{ij}(s)$ is holomorphic if $\mathrm{Re}\, s_1, \cdots, \mathrm{Re}\, s_n$ are sufficiently large. Since $t_{ij}(s)$ is periodic, this implies that $t_{ij}(s)$ is an entire function for any i, j.

The rest of this paragraph is devoted to the proof of the fact that $t_{ij}(s)$ is a polynomial of $\exp(\pm 2\pi\sqrt{-1}s_1), \cdots, \exp(\pm 2\pi\sqrt{-1}s_n)$.

Take bases of E_R and F^*_R and identify them with $\boldsymbol{R}^p$ and $\boldsymbol{R}^q$ respectively ($p = \dim E$, $q = \dim F^*$). Put

$$\| x, y^* \| = (x_1^2 + \cdots + x_p^2 + y_1^{*2} + \cdots + y_q^{*2})^{1/2}$$

for $(x, y^*) \in V_R^* = E_R \oplus F_R^*$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_q)$, set

$$|\alpha| = \alpha_1 + \cdots + \alpha_q \,, \qquad \frac{\partial^{|\alpha|}}{\partial y^{*\alpha}} = \frac{\partial^{\alpha_1}}{\partial y_1^{*\alpha_1}} \cdots \frac{\partial^{\alpha_q}}{\partial y_q^{*\alpha_q}}$$

as usual. We define a semi-norm $\nu_{M,N}$ on $\mathscr{S}(V_R^*)$ by

$$\nu_{M,N}(f^*) = \operatorname*{Sup}_{(x,y^*) \in V_R^*} \left\{(1 + \|x, y^*\|)^M \sum_{|\alpha| \leq N} \left|\frac{\partial^{|\alpha|}}{\partial y^{*\alpha}} f^*(x, y^*)\right|\right\} \qquad (M, N = 0, 1, 2, \cdots) \,.$$

Denote by $C^{M,N}(V_R^*)$ the subspace of $C^\infty(V_R^*)$ consisting of all functions f^* such that $\nu_{M,N}(f^*) < +\infty$.

The following lemma is easily proved.

LEMMA 5.6. *Let D_0 be a compact subset of $\boldsymbol{R}_+^n = \{(u_1, \cdots, u_n) \in \boldsymbol{R}^n;\ u_i > 0\ (1 \leqq i \leqq n)\}$ and put $D = D_0 + \sqrt{-1}\boldsymbol{R}^n$. Then there exist positive integers M, N, M^* and positive constants c, c^* such that*

$$|\Phi_i(\mathscr{F}f^*; s)| < c\nu_{M,N}(f^*) \,, \qquad |\Phi_i^*(f^*; s)| < c^*\nu_{M^*,0}(f^*) \qquad (1 \leqq i \leqq \nu,\ f^* \in \mathscr{S}(V_R^*),\ s \in D) \,.$$

As is already noticed in the proof of Lemma 5.5, we can find a constant β such that $\gamma(s)^{-1}$ is holomorphic in the domain

$$\Omega_\beta = \{s \in \boldsymbol{C}^n; \operatorname{Re} s_i > \beta\ (1 \leqq i \leqq n)\} \,.$$

Take two points $t = (t_1, \cdots, t_n)$ and $r = (r_1, \cdots, r_n)$ in $\boldsymbol{R}_+^n U^{-1}$ satisfying the conditions

$$\text{(5-9)} \qquad \begin{cases} \boldsymbol{R}_+^n U^{-1} \supset [t_1, t_1 + 1] \times \cdots \times [t_n, t_n + 1] \,, \\ t_i - \lambda_i - r_i > \beta \quad (1 \leqq i \leqq n) \,, \\ r_1, \cdots, r_n \in \boldsymbol{Z} \,. \end{cases}$$

Set $B = \{s \in \boldsymbol{C}^n; t_i \leqq \operatorname{Re} s_i + \lambda_i \leqq t_i + 1\ (1 \leqq i \leqq n)\}$.

LEMMA 5.7. *There exist positive integers M and N such that the functional equation in Lemma 5.5 holds for any $f^* \in C^{M,N}(V_R^*)$ and for any $s \in B$.*

PROOF. Let χ be a character in $X_\rho(G)_K$ such that $\delta(\chi) = r$. Since $\delta^*(\chi)_1, \cdots, \delta^*(\chi)_n \geqq 0$, it follows from Lemma 3.3 and Lemma 5.3 (i) that

$$\gamma(s)^{-1}\Phi_i(\mathscr{F}f^*; s) = (-1)^{d^*(\chi)}\varepsilon(i)^{\delta(\chi)}c(\chi^{-1})\gamma(s - \delta(\chi))^{-1} \times \Phi_i(Q^\chi(x, \operatorname{grad}_y)\mathscr{F}f^*; s - \delta(\chi)) \,.$$

By (5-9), $B - \delta(\chi)$ is contained in Ω_β. Hence, by Lemma 5.6, we have

$$\text{(5-10)} \qquad |\gamma(s)^{-1}\Phi_i(\mathscr{F}f^*; s)| < c_1|\gamma(s - \delta(\chi))|^{-1}\nu_{M,N}(f^*) \quad (s \in B)$$

for some constant c_1. Since $(B+\lambda)U$ is contained in $\boldsymbol{R}_+^n$, we may assume that

$$|\Phi_j^*(f^*; (s+\lambda)U)| < c_2\nu_{M,N}(f^*) \quad (s \in B)$$

for some constant c_2. For any $v > 0$, put $B(v) = \{s \in B; |\mathrm{Im}\, s| \leqq v\}$. The set $B(v) - \delta(\chi)$ is a compact subset of Ω_β. Therefore we obtain

$$|\gamma(s)^{-1}\Phi_i(\mathscr{F}f^*; s) - c(-s)(-2\pi\sqrt{-1})^{d^*(s)} \sum_{j=1}^{\nu} \varepsilon^*(j)^{sU}\varepsilon(i)^{-s}t_{ij}(s)\Phi_j^*(f^*; (s+\lambda)U)|$$
$$< c_3\nu_{M,N}(f^*) \quad (s \in B(v))$$

where c_3 is a constant depending only on v. For any $f^* \in C^{M,N}(V_R^*)$, there exists a sequence $\{f_j^*\}_{j=1}^\infty$ in $\mathscr{S}(V_R^*)$ such that $\nu_{M,N}(f_j^* - f^*) \to 0$ as $j \to \infty$. Hence the functional equation in Lemma 5.5 holds for any $f^* \in C^{M,N}(V_R^*)$ and for any $s \in B(v)$. Since v is an arbitrary positive number, we conclude that the functional equation holds for and $f^* \in C^{M,N}(V_R^*)$ and for any $s \in B$.

Now we construct functions contained in $C^{M,N}(V_R^*)$ explicitly. Let $\iota: \boldsymbol{R}_+^n \to G_R^+$ be an analytic homomorphism such that

$$\chi_i(\iota(u_1, \cdots, u_n)) = u_i \quad (1 \leqq i \leqq n)$$

and define a mapping $\iota_Q: V_j^* \to G_R^+$ by putting

$$\iota_Q(x, y^*) = \iota(|Q_1(x, y^*)|^{-1}, \cdots, |Q_n(x, y^*)|^{-1}) .$$

Set

$$K_j^* = \{(x, y^*) \in V_j^*; Q_i(x, y^*) = \varepsilon_i^*(j) \ (1 \leqq i \leqq n)\} .$$

We choose a differential form θ on V_j^* such that

$$dx_1 \wedge \cdots \wedge dx_p \wedge dy_1^* \wedge \cdots \wedge dy_q^* = dQ_1 \wedge \cdots \wedge dQ_n \wedge \theta .$$

Denote by $|\theta|$ the measure on K_j^* determined by θ. Take a ψ_j^* in $C_0^\infty(K_j^*)$ such that

$$\int_{K_j^*} \psi_j^*(x, y^*)|\theta| = 1 .$$

Let $q(u)$ be a function in $C^\infty(\boldsymbol{R})$ satisfying the following conditions:

(5–11) *All the derivatives of $q(u)$ are bounded functions on $\boldsymbol{R}$ and the support of $q(u)$ is contained in $[1, \infty)$.*

(5–12) *Put*

$$\hat{q}(z) = \int_0^\infty u^{z-1}q(u)du \quad (\mathrm{Re}\, z < 0) .$$

For every pair of positive numbers a_1, a_2 ($a_1 > a_2$), there exists a constant $c > 0$ such that

$$|\hat{q}(z)| \geqq c\cdot\exp(-|\mathrm{Im}\, z|^{1/2}) \quad (-a_1 < \mathrm{Re}\, z < -a_2) .$$

The existence of such a function $q(u)$ is guaranteed by [16, Lemma 1.4]. We define a function $f^*_{j,L}(x, y^*)$ by the formula

$$f^*_{j,L}(x, y^*) = \begin{cases} 0 & \text{if } (x, y^*) \notin V^*_j , \\ |Q_1 \cdots Q_n(x, y^*)|^{-L} \prod_{i=1}^{n} q(|Q_i(x, y^*)|)\psi^*_j(\rho^*(\iota_Q(x, y^*))(x, y^*)) & \\ & \text{if } (x, y^*) \in V^*_j . \end{cases}$$

It is obvious that the support of $f^*_{j,L}$ is contained in the set

$$\{(x, y^*) \in V^*_j ; |Q_i(x, y^*)| \geqq 1 \ (1 \leqq i \leqq n),\ \rho^*(\iota_Q(x, y^*))(x, y^*) \in \operatorname{Supp}(\psi^*_j)\} .$$

For given M, N, if L is sufficiently large, $f^*_{j,L} \in C^{M,N}(V^*_R)$. By the assumption (5-3), we can find a $\mu = (\mu_1, \cdots, \mu_n) \in Q^n$ such that $d \cdot \mu = \delta((\det \rho_1)^d)$.

LEMMA 5.8. *Let* $\tilde{\lambda}_k$ $(1 \leqq k \leqq n)$ *be the* k*-th component of* $(\mu - \lambda)U$. *When* $L > \operatorname{Re} s_1 + \tilde{\lambda}_1, \cdots, \operatorname{Re} s_n + \tilde{\lambda}_n$,

$$\Phi^*_i(f^*_{j,L}; s) = \begin{cases} \prod_{k=1}^{n} \hat{q}(s_k + \tilde{\lambda}_k - L) & (i = j) \\ 0 & (i \neq j) . \end{cases}$$

PROOF. It is clear that $\Phi^*_j(f^*_{j,L}; s) = 0$ for $i \neq j$. Since $d^{-1}\delta((\det \rho^*)^d) = \mu - \lambda$ for some positive integer d, we have

$$\begin{aligned} \Phi^*_j(f^*_{j,L}; s) &= \int_{V^*_j} \prod_{i=1}^{n} \{|Q_i(x, y^*)|^{s_i - L} q(|Q_i(x, y^*)|)\} \psi^*_j(\rho^*(\iota_Q(x, y^*))(x, y^*)) dx dy^* \\ &= \prod_{k=1}^{n} \int_0^\infty u^{s_k + \tilde{\lambda}_k - L - 1} q(u) du \int_{K^*_j} \psi^*_j(x, y^*)|\theta| = \prod_{k=1}^{n} \hat{q}(s_k + \tilde{\lambda}_k - L) . \end{aligned}$$

Let M, N and B be as in Lemma 5.7. Take an L such that $f^*_{j,L} \in C^{M,N}(V^*_R)$ and L is larger than the real parts of all the components of the vector $(s + \mu)U$ for $s \in B$. Then, by Lemma 5.7 and Lemma 5.8, we obtain

$$\begin{aligned} t_{ij}(s) &= c(s)(-2\pi\sqrt{-1})^{-d^*(s)} \gamma(s)^{-1} \varepsilon^*(j)^{-sU} \varepsilon(i)^s \\ &\quad \times \Phi_i(\mathscr{F} f^*_{j,L}; s) \prod_{k=1}^{n} \hat{q}\Big(\sum_{h=1}^{n} (s_h + \mu_h) u_{hk} - L\Big)^{-1} \quad (s \in B) \end{aligned}$$

where u_{hk} is the (h, k)-entry of U. It follows from (5-10) and (5-12) that

$$\begin{aligned} |t_{ij}(s)| &< c|c(s)(-2\pi\sqrt{-1})^{-d^*(s)} \varepsilon^*(j)^{-sU} \varepsilon(i)^s| \\ &\quad \times |\gamma(s - \delta(\chi))|^{-1} \prod_{k=1}^{n} \exp\Big(\Big|\sum_{h=1}^{n} \operatorname{Im} s_h u_{hk}\Big|^{1/2}\Big) \quad (s \in B) \end{aligned}$$

for some constant c. Hence the Stirling formula yields the following estimate:

$$|t_{ij}(s)| < c' \exp(a_1|\operatorname{Im} s_1| + \cdots + a_n|\operatorname{Im} s_n|) \quad (s \in B)$$

where $a_1, \cdots, a_n$ and c' are some positive constants. Since $t_{ij}(s)$ is a periodic function with the period lattice Z^n and B is a fundamental region of C^n for Z^n, this inequality holds for any $s \in C^n$. This implies that the function $t_{ij}(s)$ is a polynomial in $\exp(\pm 2\pi\sqrt{-1}s_1), \cdots, \exp(\pm 2\pi\sqrt{-1}s_n)$. Theorem 1 is now completely proved.

REMARK 1. The assumption (5-2) can be replaced by the following assumption:

For a generic point x of (G, ρ_1, E), the singular set S_x of (G_x°, ρ_2, F) is a hypersurface.

REMARK 2. The condition (5-3) is assumed for the sake of simplicity. We are able to avoid it. In the application to functional equations of zeta functions in the next section, this condition is satisfied.

REMARK 3. Let H be a subgroup of $X_\rho(G)_R$ containing the character $\det \rho_2(g)^2$. Put $m = \operatorname{rank} H$. Let $P_1, \cdots, P_m$ (resp. $Q_1, \cdots, Q_m$) be relative invariants with real coefficients of (G, ρ, V) (resp. (G, ρ^*, V^*)) such that the characters corresponding to $P_1, \cdots, P_m$ (resp. $Q_1, \cdots, Q_m$) generate the group H. Then if we modify the definitions of $U, \lambda, \Phi_i(f; s)$ and $\Phi_i^*(f^*; s)$, an analogue of Theorem 1 remains valid.

REMARK 4. An algorithm to calculate $A(s)$ explicitly is obtained for a fairly wide class of p.v.'s by the method of micro local calculus (see [10] and [19]).

5.2. For a later application, we shall prove a lemma which enables us to reduce the calculation of partial Fourier transforms to the special case where $E = \{0\}$, $F = V$ and $\rho = \rho_2$.

Put $r = n - \operatorname{rank} X_{\rho_1}(G)_K$. As is observed in § 2, we may assume that $P_{r+1}, \cdots, P_n$ (resp. $Q_{r+1}, \cdots, Q_n$) are independent of the second component $y \in F$ (resp. $y^* \in F^*$) and

$$P_i(x, y) = P_i(x) = Q_i(x) = Q_i(x, y^*) \quad (r + 1 \leqq i \leqq n) . \tag{5-13}$$

In this case, the matrix U is of the form

$$U = \begin{pmatrix} U_0 & U_1 \\ 0 & E_{n-r} \end{pmatrix} \tag{5-14}$$

where $U_0 \in GL(r)_Z$, $U_1 \in M(r, n-r; Z)$ and E_{n-r} is the identity square matrix of size $n - r$. By Lemma 2.4 (iii), the group $X_{\rho|\rho_2}(G_x^\circ)_K$ coincides with $X_{\rho^*|\rho_2^*}(G_x^\circ)_K$ for any K-rational generic point x of (G, ρ_1, E). It follows from Corollary to Lemma 2.2 that $\chi_1|_{G_x^\circ}, \cdots, \chi_r|_{G_x^\circ}$ and $\chi_1^*|_{G_x^\circ}, \cdots, \chi_r^*|_{G_x^\circ}$ form two systems of generators of this group. The matrix U_0 gives the relation between these two systems of generators, namely,

$$\chi_i|_{G_x^\circ} = \prod_{j=1}^{r} (\chi_j^*|_{G_x^\circ})^{u_{ij}} \quad (U_0 = (u_{ij})_{1\leq i,j\leq r}) .$$

By (5-4), we have

$$\det \rho_2(g)^2|_{G_x^\circ} = \prod_{i=1}^{r} (\chi_i(g)|_{G_x^\circ})^{2\lambda_i} .$$

Put $\lambda^0 = (\lambda_1, \cdots, \lambda_r, 0, \cdots, 0) \in (2^{-1}\boldsymbol{Z})^n$. Let S^1 be the singular set of (G, ρ_1, E). Consider the projection mappings

$$p: V_R - S_R \to E_R - S_R^1 \quad \text{and} \quad p^*: V_R^* - S_R^* \to E_R - S_R^1 .$$

These mappings are G_R^+-equivariant and surjective. Let $E_R - S_R^1 = \omega_1 \cup \cdots \cup \omega_t$ be the G_R^+-orbit decomposition. For an $x \in \omega_i$, put $G_x^+ = G_x \cap G_R^+$. For simplicity, we assume that

$$G_x^+ \subset (G_x^\circ)_R . \tag{5-15}$$

There exists a one to one correspondence between G_R^+-orbits in $p^{-1}(\omega_i)$ (resp. $p^{*-1}(\omega_i)$) and G_x^+-orbits in $F_R - S_{x,R}$ (resp. $F_R^* - S_{x,R}^*$). Hence by Lemma 5.1, the number k_i of G_R^+-orbits in $p^{-1}(\omega_i)$ is equal to that of G_R^+-orbits in $p^{*-1}(\omega_i)$ $(1 \leqq i \leqq t)$. We have $k_1 + \cdots + k_t = \nu$. We may assume that

$$p^{-1}(\omega_i) = V_{k_1+\cdots+k_{i-1}+1} \cup \cdots \cup V_{k_1+\cdots+k_{i-1}+k_i}$$

and

$$p^{*-1}(\omega_i) = V^*_{k_1+\cdots+k_{i-1}+1} \cup \cdots \cup V^*_{k_1+\cdots+k_{i-1}+k_i} \quad (1 \leqq i \leqq t) .$$

For an $x \in \omega_i$, set

$$F(x)_j = \{y \in F_R;\, (x, y) \in V_{k_1+\cdots+k_{i-1}+j}\}$$

and

$$F(x)_j^* = \{y \in F_R^*;\, (x, y^*) \in V^*_{k_1+\cdots+k_{i-1}+j}\} \quad (1 \leqq j \leqq k_i) .$$

Then the G_x^+-orbit decompositions of $F_R - S_{x,R}$ and $F_R^* - S_{x,R}^*$ are given by

$$F_R - S_{x,R} = F(x)_1 \cup \cdots \cup F(x)_{k_i} \quad \text{and} \quad F_R^* - S_{x,R}^* = F(x)_1^* \cup \cdots \cup F(x)_{k_i}^* .$$

We set

$$\begin{aligned}\Phi_i(x, f; s) &= \int_{F(x)_i} |P(x, y)|^s f(y)dy \\ &= \prod_{j=r+1}^{n} |P_j(x)|^{s_j} \int_{F(x)_i} \prod_{j=1}^{r} |P_j(x, y)|^{s_j} f(y)dy\end{aligned}$$

and

$$\Phi_i^*(x, f^*; s) = \int_{F(x)_i^*} |Q(x, y^*)|^s f^*(y^*) dy^*$$
$$= \prod_{j=r+1}^{n} |P_j(x)|^{s_j} \int_{F(x)_i^*} \prod_{j=1}^{r} |Q_j(x, y^*)|^{s_j} f^*(y^*) dy^*$$

($f \in \mathscr{S}(F_R)$, $f^* \in \mathscr{S}(F_R^*)$, $s \in C^n$).

The Fourier transform $\hat{f}^*$ of $f^* \in \mathscr{S}(F_R^*)$ is defined to be

$$\hat{f}^*(y) = \int_{F_R^*} f^*(y^*) e[\langle y, y^* \rangle] dy^* .$$

Set

$$\Phi(x, \hat{f}^*; s) = {}^t(\Phi_1(x, \hat{f}^*; s), \cdots, \Phi_{k_i}(x, \hat{f}^*; s)) ,$$
$$\Phi^*(x, f^*; s) = {}^t(\Phi_1^*(x, f^*; s), \cdots, \Phi_{k_i}^*(x, f^*; s))$$

and

$$\tilde{U}_0 = \begin{pmatrix} U_0 & 0 \\ 0 & E_{n-r} \end{pmatrix} .$$

Since the condition (5-1) implies that (G_x°, ρ_2, F) is regular over K, by Theorem 1 and Remark 3, there exists a k_i by k_i matrix $A(x; s)$ of meromorphic functions of s, which is independent of f^*, such that

$$\Phi(x, \hat{f}^*; s) = A(x; s) \Phi^*(x, f^*; (s + \lambda^0) \tilde{U}_0) . \tag{5-16}$$

Note that the matrix $A(x; s)$ depends only on $s_1, \cdots, s_r$.

LEMMA 5.9. *Let $x^{(1)}, \cdots, x^{(t)}$ be points in $\omega_1, \cdots, \omega_t$ respectively. If $|P_{r+1}(x^{(i)})| = \cdots = |P_n(x^{(i)})| = 1$ for every $i = 1, \cdots, t$, then*

$$\begin{pmatrix} A(x^{(1)}; s) & & \\ & \ddots & \\ & & A(x^{(t)}; s) \end{pmatrix} = c(-s)(-2\pi\sqrt{-1})^{d*(s)} \gamma(s) A(s) .$$

PROOF. For $f \in \mathscr{S}(F_R)$ and $f^* \in \mathscr{S}(F_R^*)$, we put $f_g(y) = f(\rho_2(g)y)$ and $f_g^*(y^*) = f^*(\rho_2^*(g)y^*)$ $(g \in G_R^+)$. Then,

$$\Phi(\rho_1(g)x, f; s) = |\chi(g)|^{s+\lambda} \Phi(x, f_g; s)$$

and

$$\Phi^*(\rho_1(g)x, f^*; s) = |\chi^*(g)|^{s-\lambda U} \Phi^*(x, f_g^*; s)$$

for any $x \in E_R - S_R^1$ and any $g \in G_R^+$. The following identity is an immediate consequence of these formulas: $A(\rho_1(g)x; s) = |\chi(g)|^z A(x; s)$ where $z = s + \lambda - (s + \lambda^0) \tilde{U}_0 U^{-1}$. Hence, for any $x \in \omega_i$, we have

$$A(x; s) = |P(x)|^z A(x^{(i)}; s) . \tag{5-17}$$

Since $z_1 = \cdots = z_r = 0$, we write $|P(x)|^z$ here for $|P(x, y)|^z$. Suppose that $\operatorname{Re} s_1, \cdots, \operatorname{Re} s_n > 0$ and $f^* \in C_0^\infty(V_R^* - S_R^*)$. Then

$$\Phi(\mathscr{F}f^*; s) = {}^t\left(\int_{\omega_1} \Phi(x, (\mathscr{F}f^*)_x; s)dx, \cdots, \int_{\omega_t} \Phi(x, (\mathscr{F}f^*)_x; s)dx\right) \tag{5-18}$$

and

$$\Phi^*(f^*; s) = {}^t\left(\int_{\omega_1} \Phi^*(x, f_x^*; s)dx, \cdots, \int_{\omega_t} \Phi^*(x, f_x^*; s)dx\right) \tag{5-19}$$

where $(\mathscr{F}f^*)_x$ and f_x^* stand for the functions on F_R and F_R^* defined by $(\mathscr{F}f^*)_x(y) = \mathscr{F}f^*(x, y)$ and $f_x^*(y^*) = f^*(x, y^*)$ respectively. We get $(\mathscr{F}f^*)_x = \hat{f}_x^*$. Denote by $A'(s)$ the left hand side of the equality in the lemma. Then the identity

$$\Phi(\mathscr{F}f^*; s) = A'(s)\Phi^*(f^*; z + (s + \lambda^0)\tilde{U}_0) \quad (f^* \in C_0^\infty(V_R^* - S_R^*))$$

follows from (5-16), (5-17), (5-18) and (5-19). Since $z + (s + \lambda^0)\tilde{U}_0 = (s + \lambda)U$, we have $A'(s) = c(-s)(-2\pi\sqrt{-1})^{d^*(s)}\gamma(s)A(s)$.

6. Functional equations of zeta functions. Throughout this section, in addition to the conditions (5-1) for $K = \boldsymbol{Q}$ and (5-2), we assume that

(6-1) *for any $z = (x, y) \in V_Q - S_Q$, the group $X(G_z^0)_Q$ is trivial, namely, $V_Q' = V_Q - S_Q$ (for the definition of V_Q', see* (4-1)).

By Lemma 2.4 (iii), this assumption is also satisfied by the p.v. $(G, \rho^*, V^*) = (G, \rho_1 \oplus \rho_2^*, E \oplus F^*)$. The condition (5-3) follows immediately from (6-1) (cf. Lemma 4.1).

As in the previous section, let G_R^+ be a subgroup of G_R containing the connected component of the identity element and let

$$V_R - S_R = V_1 \cup \cdots \cup V_\nu \quad \text{and} \quad V_R^* - S_R^* = V_1^* \cup \cdots \cup V_\nu^*$$

be the G_R^+-orbit decompositions.

We fix a matrix expression of G and bases of E and F compatible with the given $\boldsymbol{Q}$-structures of (G, ρ_1, E) and (G, ρ_2, F) such that $\rho_1(G_Z)E_Z \subset E_Z$ and $\rho_2(G_Z)F_Z \subset F_Z$. We define a $\boldsymbol{Q}$-structure on (G, ρ_2^*, F^*) by taking the basis dual to that of F. Let M and N be $\rho_1(G_Z)$- and $\rho_2(G_Z)$-stable lattices in E_Q and F_Q respectively. Denote by N^* the lattice dual to N:

$$N^* = \{y^* \in F_Q^*; \langle y, y^* \rangle \in \boldsymbol{Z} \text{ for all } y \in N\}.$$

It is obvious that N^* is $\rho_2^*(G_Z)$-stable. Put $L = M \oplus N$ and $L^* = M \oplus N^*$. Then L and L^* are $\rho(G_Z)$- and $\rho^*(G_Z)$-stable lattices in V_Q and V_Q^* respectively.

Set

$$\Gamma = \{g \in G_Z \cap G_R^+; \chi_i(g) = 1 \ (1 \leqq i \leqq n)\} .$$

By (6–1), applying the argument in § 4 to (G, ρ, V) (resp. (G, ρ^*, V^*)), we can define zeta functions $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ (resp. $\xi_1^*(L^*; s), \cdots, \xi_\nu^*(L^*; s)$).

We further assume that

(6–2) *the Dirichlet series* $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ (*resp.* $\xi_1^*(L^*; s), \cdots, \xi_\nu^*(L^*; s)$) *are absolutely convergent for* $\mathrm{Re}\, s_1 > a_1, \cdots, \mathrm{Re}\, s_n > a_n$ (*resp.* $\mathrm{Re}\, s_1 > a_1^*, \cdots, \mathrm{Re}\, s_n > a_n^*$) *for some positive real numbers* $a_1, \cdots, a_n$ (*resp.* $a_1^*, \cdots, a_n^*$).

As in § 4, let δ and δ^* be the elements in Q^n such that

$$(\det \rho(g)\Delta(g)^{-1})^d = \chi_1(g)^{d\delta_1} \cdots \chi_n(g)^{d\delta_n}$$

and

$$(\det \rho^*(g)\Delta(g)^{-1})^{d^*} = \chi_1^*(g)^{d^*\delta_1^*} \cdots \chi_n^*(g)^{d^*\delta_n^*}$$

for some integers d and d^*. Then we have $\delta^* = (\delta - 2\lambda)U$. Put

$$B = \{s \in C^n; \mathrm{Re}\, s_i > \mathrm{Max}\,(a_i, \delta_i) \ (1 \leqq i \leqq n)\}$$

and

$$B^* = \{s \in C^n; \mathrm{Re}\, s_i > \mathrm{Max}\,(a_i^*, \delta_i^*) \ (1 \leqq i \leqq n)\} .$$

By Lemma 3.2, we have the following integral representations:

$$(6\text{–}3) \qquad Z(f, L; s) = \int_{G_R^+/\Gamma} |\chi(g)|^s \sum_{z \in L - S_Q} f(\rho(g)z)dg$$
$$= \sum_{i=1}^{\nu} \xi_i(L; s)\Phi_i(f; s - \delta) \quad (f \in \mathscr{S}(V_R),\ s \in B)$$

and

$$(6\text{–}4) \qquad Z^*(f^*, L^*; s) = \int_{G_R^+/\Gamma} |\chi^*(g)|^s \sum_{z^* \in L^* - S_Q^*} f^*(\rho^*(g)z^*)dg$$
$$= \sum_{i=1}^{\nu} \xi_i^*(L^*; s)\Phi_i^*(f^*; s - \delta^*) \quad (f^* \in \mathscr{S}(V_R^*),\ s \in B^*) .$$

Denote by D (resp. D^*) the convex hull of $(B^*U^{-1} + \lambda) \cup B$ (resp. $(B - \lambda)U \cup B^*$) in C^n. Notice that $(D - \lambda)U = D^*$.

LEMMA 6.1. *Let f^* be a function in $\mathscr{S}(V_R^*)$ such that f^* and $\mathscr{F}f^*$ vanish on the singular sets S^* and S respectively. Then $Z(\mathscr{F}f^*, L; s)$ and $Z^*(f^*, L^*; s)$ have analytic continuations to holomorphic functions of s in D and D^* respectively. Moreover they satisfy the functional equation*

$$Z^*(f^*, L^*; (s-\lambda)U) = v(N^*)^{-1}Z(\mathscr{F}f^*, L; s) \quad (s \in D)$$

where $v(N^*) = \int_{F_R^*/N^*} dy^*$.

PROOF. Take points $b \in Z^n \cap B$ and $b^* \in Z^n \cap (B^*U^{-1} + \lambda)$. Put $\beta = (\beta_1, \cdots, \beta_n) = b - b^*$ and $\chi^\beta = \chi_1^{\beta_1} \cdots \chi_n^{\beta_n}$. We define four domains D_+, D_-, D_+^* and D_-^* as follows:

$$D_+ = \{s \in C^n; s + t\beta \in B \text{ for some } t \geqq 0\} ,$$
$$D_- = \{s \in C^n; s - t\beta \in B \text{ for some } t \geqq 0\} ,$$
$$D_+^* = \{s \in C^n; s - t\beta U \in B^* \text{ for some } t \geqq 0\} ,$$
$$D_-^* = \{s \in C^n; s + t\beta U \in B^* \text{ for some } t \geqq 0\} .$$

Set

$$Z_+(f, L; s) = \int_{|\chi^\beta(g)|\geqq 1} |\chi(g)|^s \sum_{z \in L-S} f(\rho(g)z)dg ,$$
$$Z_-(f, L; s) = \int_{|\chi^\beta(g)|\leqq 1} |\chi(g)|^s \sum_{z \in L-S} f(\rho(g)z)dg ,$$
$$Z_+^*(f^*, L^*; s) = \int_{|\chi^\beta(g)|\leqq 1} |\chi^*(g)|^s \sum_{z^* \in L^*-S^*} f^*(\rho^*(g)z^*)dg$$

and

$$Z_-^*(f^*, L^*; s) = \int_{|\chi^\beta(g)|\geqq 1} |\chi^*(g)|^s \sum_{z^* \in L^*-S^*} f^*(\rho^*(g)z^*)dg$$
$$(f \in \mathscr{S}(V_R),\ f^* \in \mathscr{S}(V_R^*)).$$

Since $Z(f, L; s)$ (resp. $Z^*(f^*, L^*; s)$) is absolutely convergent in B (resp. B^*), $Z_\pm(f, L; s)$ (resp. $Z_\pm^*(f^*, L^*; s)$) is absolutely convergent in $D_\pm$ (resp. $D_\pm^*$) and we have

$$Z(f, L; s) = Z_+(f, L; s) + Z_-(f, L; s) \quad (s \in B)$$

and

$$Z^*(f^*, L^*; s) = Z_+^*(f^*, L^*; s) + Z_-^*(f^*, L^*; s) \quad (s \in B^*) .$$

We are assuming that f^* and $\mathscr{F}f^*$ vanish on S^* and S respectively. Hence the Poisson summation formula yields the following equality:

$$|\chi(g)|^\lambda \sum_{z \in L-S} \mathscr{F}f^*(\rho(g)z) = v(N^*) \sum_{z^* \in L^*-S^*} f^*(\rho^*(g)z^*) \quad (g \in G_R^+) .$$

By this formula, we obtain at least formally

$$Z_-^*(f^*, L^*; (s-\lambda)U) = v(N^*)^{-1}Z_+(\mathscr{F}f^*, L; s) \tag{6-5}$$

and

$$Z_+^*(f^*, L^*; (s-\lambda)U) = v(N^*)^{-1}Z_-(\mathscr{F}f^*, L; s) . \tag{6-6}$$

The right (resp. left) hand side of the equality (6-5) is absolutely convergent in D_+ (resp. $D^*_-U^{-1}+\lambda$). By the choice of β, the segment joining b and b^* is contained in D_+ and the set $D_+\cap(D^*_-U^{-1}+\lambda)$ is a non-empty connected open set containing the neighbourhood of b^*. Hence the functions $Z_+(\mathscr{F}f^*, L; s)$ and $Z^*_-(f^*, L^*; (s-\lambda)U)$ are continued holomorphically in $D_+\cup(D^*_-U^{-1}+\lambda)$ and the equality (6-5) actually holds in this domain. The same argument shows that the functions $Z_-(\mathscr{F}f^*, L; s)$ and $Z^*_+(f^*, L^*; (s-\lambda)U)$ are continued holomorphically in $D_-\cup(D^*_+U^{-1}+\lambda)$ and the equality (6-6) also holds. Thus we get the functional equation

$$(6\text{-}7)\quad \begin{aligned} Z^*(f^*, L^*; (s-\lambda)U) &= Z^*_+(f^*, L^*; (s-\lambda)U) + v(N^*)^{-1}Z_+(\mathscr{F}f^*, L; s)\\ &= v(N^*)^{-1}Z(\mathscr{F}f^*, L; s)\end{aligned}$$

and both sides of the equation are holomorphic functions of s in $\{D_+\cup(D^*_-U^{-1}+\lambda)\}\cap\{D_-\cup(D^*_+U^{-1}+\lambda)\}$. This domain contains the union of B, $B^*U^{-1}+\lambda$ and the segment joining b and b^*. Hence, by [5, Theorem 2.5.10], the identity (6-7) holds for $s\in D$.

We shall construct rapidly decreasing functions with the property mentioned in the lemma above by the method indicated in [14, p. 169, Additional remark 2]. We may assume that (5-13) holds for $P_1, \cdots, P_n$ and $Q_1, \cdots, Q_n$. Put $\chi_F=\chi_1\cdots\chi_r$ and $\chi_{F^*}=\chi^*_1\cdots\chi^*_r$.

LEMMA 6.2. (i) *For an* $f^*_i\in C_0^\infty(V^*_i)$, *put* $f^*=P^{\chi_F}(x, \mathrm{grad}_{y^*})f^*_i$. *Then the function* $\mathscr{F}f^*$ *vanishes in* S_R.

(ii) *For an* $f_i\in C_0^\infty(V_i)$, *put* $f^*=\mathscr{F}^{-1}(Q^{\chi_{F^*}}(x, \mathrm{grad}_y)f_i)$ *where* $\mathscr{F}^{-1}$ *stands for the inverse transformation of the partial Fourier transform* $\mathscr{F}$. *Then the function* f^* *vanishes in* S^*_R.

PROOF. (i) Integrating by parts, we have

$$\mathscr{F}f^*(x, y) = (-2\pi\sqrt{-1})^d P^{\chi_F}(x, y)\mathscr{F}f^*_i(x, y)$$

where $d=\deg_y P_1+\cdots+\deg_y P_r$. The assumption (5-13) implies that $S_R=S^1_R\times F_R\cup\{(x, y)\in V_R; P_1(x, y)\cdots P_r(x, y)=0\}$. Since $\mathscr{F}f^*_i(x, y)=0$ for any $(x, y)\in S^1_R\times F_R$, $\mathscr{F}f^*$ vanishes in S_R.

(ii) We omit the similar proof to that of (i).

Put $b_F(s)=b_{\chi_{F^*}}(s)$ and $b_{F^*}(s)=b^*_{\chi_F}(s)$. We call $b_F(s)$ (resp. $b_{F^*}(s)$) *the partial b-function of* (G, ρ, V) (resp. (G, ρ^*, V^*)) *with respect to the* *Q-regular subspace* F (resp. F^*).

THEOREM 2. (i) *The Dirichlet series* $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ (*resp.* $\xi^*_1(L^*; s), \cdots, \xi^*_\nu(L^*; s)$) *have analytic continuations to meromorphic functions of* s *in* D (*resp.* D^*).

(ii) *The functions* $b_F(s-\delta)\xi_1(L; s), \cdots, b_F(s-\delta)\xi_\nu(L; s)$ (*resp.* $b_{F^*}(s-\delta^*)\xi_1^*(L^*; s), \cdots, b_{F^*}(s-\delta^*)\xi_\nu^*(L^*; s)$) *are holomorphic in* D (*resp.* D^*).

(iii) *Put*

$$\xi(L; s) = {}^t(\xi_1(L; s), \cdots, \xi_\nu(L; s))$$

and

$$\xi^*(L^*; s) = {}^t(\xi_1^*(L^*; s), \cdots, \xi_\nu^*(L^*; s)) .$$

Then the following functional equation holds for s *in* D:

$$(6\text{-}8) \qquad v(N^*)\xi^*(L^*; (s-\lambda)U) = c(\delta - s)(-2\pi\sqrt{-1})^{d^*(s-\delta)}\gamma(s-\delta)\, {}^tA(s-\delta)\xi(L; s) .$$

PROOF. Let f_i^* and f^* be as in Lemma 6.2 (i). By (6-3), (6-4) and Lemma 6.1, we have

$$(6\text{-}9) \qquad \xi_i^*(L^*; (s-\lambda)U)\Phi_i^*(f^*; (s-\lambda)U-\delta^*) = v(N^*)^{-1}\sum_{j=1}^{\nu}\xi_j(L; s)\Phi_j(\mathscr{F}f^*; s-\delta)$$

where both sides of the equality are holomorphic functions of s in D. We are able to take an f_i^* such that the support of f_i^* is contained in a connected component of V_i^* and $\Phi_i^*(f_i^*; (s-\lambda)U - \delta^* + \delta^*(\chi_F)) \neq 0$. Then the sign of any relative invariant does not change in the support of f_i^*. Lemma 5.3 (ii) implies that

$$\Phi_i^*(f^*; (s-\lambda)U-\delta^*) = (-1)^{d(\chi_F)}\varepsilon_i^{*\delta^*(\chi_F)} b_{F^*}((s-\lambda)U-\delta^*)\Phi_i^*(f_i^*; (s-\lambda)U - \delta^* + \delta^*(\chi_F)) .$$

Hence the function $b_{F^*}((s-\lambda)U-\delta^*)\xi_i^*(L^*; (s-\lambda)U)$ is holomorphic in D. Since $(D-\lambda)U = D^*$, this proves the assertions (i) and (ii) for ξ_i^*. The similar argument applied to f_i and f^* given in Lemma 6.2 (ii) shows that $\xi_1, \cdots, \xi_\nu$ have the analytic properties asserted in (i) and (ii). Since $(\delta - 2\lambda)U = \delta^*$, the functional equation (6-8) is an immediate consequence of (6-9) and Theorem 1.

COROLLARY 1. *Let* (G, ρ, V) *is a* p.v, *with a reductive algebraic group* G *satisfying the conditions* (5-1), (6-1) *and* (6-2) *for* $E = \{0\}$, $F = V$ *and* $K = \boldsymbol{Q}$. *Then the zeta functions* $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ *have analytic continuations to meromorphic functions of* s *in* $\boldsymbol{C}^n$. *Moreover the functions* $b_V(s-\delta)\xi_1(L; s), \cdots, b_V(s-\delta)\xi_\nu(L; s)$ *are entire functions of* s.

PROOF. When G is reductive, the condition (5-2) is derived from the condition (5-1) for $E = \{0\}$ and $F = V$ (see [13, § 4, Remark 26]). Hence we are able to apply Theorem 2 to (G, ρ, V) and our task is only to show that the convex hull D of $(B^* U^{-1} + \lambda) \cup B$ coincides with $\boldsymbol{C}^n$. By [13, § 4, Proposition 24], we may assume that $U = -E_n$. This implies that $D = \boldsymbol{C}^n$.

Corollary 1 is generalized as follows:

COROLLARY 2. *Let* (G, ρ, V) *be a regular* p.v. *defined over* $\boldsymbol{Q}$ *with a reductive algebraic group* G. *Assume that* (G, ρ, V) *is decomposed into a direct sum* $(G, \rho_1 \oplus \rho_2, W_1 \oplus W_2)$ *over* $\boldsymbol{Q}$. *Further assume that the conditions* (5–1), (6–1) *and* (6–2) *hold for* $E = W_1$, $F = W_2$, $K = \boldsymbol{Q}$ *and for* $E = W_2$, $F = W_1$, $K = \boldsymbol{Q}$. *Then if* L *is decomposed into a direct sum of a* $\rho_1(\Gamma)$*-invariant lattice* L_1 *in* $W_{1\boldsymbol{Q}}$ *and a* $\rho_2(\Gamma)$*-invariant lattice* L_2 *in* $W_{2\boldsymbol{Q}}$, *the zeta functions* $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ *multiplied by* $b_{W_1}(s - \delta) \times b_{W_2}(s - \delta)$ *are entire functions of* s.

PROOF. Since (G, ρ, V) is regular and G is reductive, the condition (5–2) is automatically satisfied. The matrix U and the vector λ are defined for each of two $\boldsymbol{Q}$-regular subspaces W_1 and W_2. We denote them by U_1, U_2, $\lambda^{(1)}$ and $\lambda^{(2)}$. For sufficiently large positive numbers $a_1, \cdots, a_n$, put

$$B = \{s \in \boldsymbol{C}^n; \operatorname{Re} s_1 > a_1, \cdots, \operatorname{Re} s_n > a_n\} .$$

Let D_i $(i = 1, 2)$ be the convex hull of $(BU_i^{-1} + \lambda^{(i)}) \cup B$. Then, by Theorem 2 (ii), the functions $b_{W_i}(s - \delta)\xi_j(L; s)$ $(i = 1, 2,\ 1 \leqq j \leqq \nu)$ are holomorphic in D_i. Hence $b_{W_1}(s - \delta) b_{W_2}(s - \delta)\xi_j(L; s)$ $(1 \leqq j \leqq \nu)$ are holomorphic functions of s in the convex hull of

$$B \cup \{BU_1^{-1} + \lambda^{(1)}\} \cup \{BU_2^{-1} + \lambda^{(2)}\} .$$

Since G is reductive and $\rho_1^* \oplus \rho_2$ is the contragredient representation of $\rho_1 \oplus \rho_2^*$, by [13, § 4, Proposition 24], we may assume that $U_1 = -U_2$. Therefore the convex hull of the set above is equal to $\boldsymbol{C}^n$.

REMARK 1. When G is not reductive, the author does not know whether the zeta functions have analytic continuations to meromorphic functions of s in $\boldsymbol{C}^n$.

REMARK 2. If (G, ρ, V) does not satisfy the condition (6–1), namely, $V'_{\boldsymbol{Q}}$ is a proper subset of $V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$, the study of zeta functions becomes extremely difficult. An example of zeta functions of this kind is the Siegel zeta function of a ternary zero form (cf. [17], [18] and [25]).

REMARK 3. Theorem 2 was previously proved for some special cases.

If G is reductive and S is an absolutely irreducible hypersurface, Theorem 2 was already established by Sato and Shintani in [14].

The Eisenstein series of the group $SL(n)$ can be viewed as an example of zeta functions associated with p.v.'s. Arithmetic approaches to the Eisenstein series given in Langlands [8], Maass [9], Selberg [15] and Terras [21] are well-understood from our point of view (cf. [24]).

In [20], Suzuki showed Theorem 2 for certain zeta functions in two variables related to quadratic forms (cf. § 7 Remark 2 to Example B).

In [17, Chapter 1], Shintani studied certain Dirichlet series in two variables which we shall reexamine in the next section.

7. Examples. In this section we frequently use the symbols introduced in the previous sections without any special reference.

7.1. Example (A). Let $G = SL(2) \times GL(1)^3$, $V^{(1)} = V^{(2)} = V^{(3)} = M(2, 1)$ and $V = V^{(1)} \oplus V^{(2)} \oplus V^{(3)}$. We define a representation ρ of G on V by setting

$$\rho(g)v = \rho(h, t_1, t_2, t_3)(x, y, z) = (hxt_1^{-1}, hyt_2^{-1}, hzt_3^{-1}) .$$

Put

$$\begin{aligned} P_1(v) &= P_1(x, y, z) = \det(y, z) , \\ P_2(v) &= P_2(x, y, z) = \det(x, z) , \\ P_3(v) &= P_3(x, y, z) = \det(x, y) \end{aligned}$$

and

$$S = \bigcup_{i=1}^{3} \{v \in V;\ P_i(v) = 0\} . \tag{7-1}$$

It is easy to check that the triple (G, ρ, V) is a p.v. with the singular set S. Hence the condition (5-2) is satisfied by (G, ρ, V). The polynomials P_1, P_2 and P_3 are irreducible relative invariants which correspond to the characters

$$\begin{cases} \chi_1(g) = (t_2 t_3)^{-1} , \\ \chi_2(g) = (t_1 t_3)^{-1} , \\ \chi_3(g) = (t_1 t_2)^{-1} , \end{cases} \tag{7-2}$$

respectively. There exists a natural $\boldsymbol{Q}$-structure on (G, ρ, V):

$$\begin{aligned} G_{\boldsymbol{Q}} &= SL(2)_{\boldsymbol{Q}} \times \boldsymbol{Q}^\times \times \boldsymbol{Q}^\times \times \boldsymbol{Q}^\times , \\ V_{\boldsymbol{Q}} &= M(2, 1; \boldsymbol{Q}) \oplus M(2, 1; \boldsymbol{Q}) \oplus M(2, 1; \boldsymbol{Q}) , \\ G_{\boldsymbol{Z}} &= SL(2)_{\boldsymbol{Z}} \times \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\} , \\ V_{\boldsymbol{Z}} &= M(2, 1; \boldsymbol{Z}) \oplus M(2, 1; \boldsymbol{Z}) \oplus M(2, 1; \boldsymbol{Z}) . \end{aligned}$$

We put $G_{\boldsymbol{R}}^+ = G_{\boldsymbol{R}} = SL(2)_{\boldsymbol{R}} \times \boldsymbol{R}^\times \times \boldsymbol{R}^\times \times \boldsymbol{R}^\times$. Then the set $V_{\boldsymbol{R}} - S_{\boldsymbol{R}}$ is the union of two $G_{\boldsymbol{R}}^+$-orbits V_+ and V_-:

$$V_+ = \{v \in V_{\boldsymbol{R}};\ P_1(v)P_2(v)P_3(v) > 0\} , \qquad V_- = \{v \in V_{\boldsymbol{R}};\ P_1(v)P_2(v)P_3(v) < 0\} .$$

The group Γ is given by

$$\Gamma = \{(h, t_1, t_2, t_3) \in G_Z;\ t_1 = t_2 = t_3 = \pm 1\} .$$

The isotropy subgroup G_v at a generic point $v \in V - S$ is independent of the choice of v and coincides with $\{\pm(E_2, 1, 1, 1)\}$. This implies the condition (6-1): $V'_Q = V_Q - S_Q$. For any $v \in V'_Q$, $G_v^+ = \Gamma_v = \{\pm(E_2, 1, 1, 1)\}$. The character $\det \rho$ is in $X_\rho(G)_Q$ and $\delta = \delta(\det \rho) = (1, 1, 1)$. Let $dv = dxdydz$ be the standard Euclidean measure on V_R. We can normalize a Haar measure dg on G_R^+ such that

$$\int_{G_R^+} F(\rho(g)v_0)dg = 2\int_{V_\pm} F(v)|P(v)|^{-\delta}dv \quad (v_0 \in V_\pm,\ F \in L^1(V_\pm, |P(v)|^{-\delta}dv)) .$$

The normalization of dg is independent of v_0. Then, for the Haar measure $d\mu_v$ on G_v^+ normalized by (4-2), we have

$$\int_{G_v^+} d\mu_v = 2 \quad \text{and} \quad \mu(v) = \int_{G_v^+/\Gamma_v} d\mu_v = 1 \quad (v \in V'_Q) .$$

Let L be a Γ-invariant lattice in V_Q and set $L_\pm = L \cap V_\pm$. The zeta functions associated with (G, ρ, V) are defined by the formula

$$\xi_\pm(L; s) = \sum_{v \in \Gamma \backslash L_\pm} |P(v)|^{-s} \quad (s \in C^3) . \tag{7-3}$$

For the lattice $L = V_Z$, these Dirichlet series are easily calculated and we get

$$\xi_\pm(s) = \xi_\pm(V_Z; s) = \zeta(s_1)\zeta(s_2)\zeta(s_3)\zeta(s_1 + s_2 + s_3 - 1) . \tag{7-4}$$

This implies that the series $\xi_\pm(L; s)$ are absolutely convergent for $\operatorname{Re} s_1, \operatorname{Re} s_2, \operatorname{Re} s_3 > 1$. Since $\delta = (1, 1, 1)$,

$$B = \{s \in C^3;\ \operatorname{Re} s_1, \operatorname{Re} s_2, \operatorname{Re} s_3 > 1\} .$$

The p.v. (G, ρ, V) has the following seven Q-regular subspaces:

$$\begin{cases} V , \\ V^{(2)} \oplus V^{(3)},\ V^{(1)} \oplus V^{(3)},\ V^{(1)} \oplus V^{(2)} , \\ V^{(1)},\ V^{(2)},\ V^{(3)} . \end{cases}$$

Since (G, ρ, V) has obvious symmetry for the permutations of the indices 1, 2, 3, we shall calculate the explicit forms of the functional equations obtained by the partial Fourier transforms with respect to V, $E = V^{(1)}$ and $F = V^{(2)} \oplus V^{(3)}$.

The notions such as ρ^*, U, λ are defined for each Q-regular subspace. In order to indicate the dependence on Q-regular subspace, we use the subscripts E, F and V. For example, the symbols ρ_V^*, ρ_E^* and ρ_F^* stand for the representations of G (partially) contragredient to ρ with respect to V, E and F respectively.

Put $J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and identify the vector spaces V^*, E^* and F^* with V, E and F via the non-degenerate bilinear forms

$$\langle (x, y, z), (x^*, y^*, z^*)\rangle = {}^t x J x^* + {}^t y J y^* + {}^t z J z^* , \qquad \langle x, x^*\rangle = {}^t x J x^*$$

and

$$\langle (y, z), (y^*, z^*)\rangle = {}^t y J y^* + {}^t z J z^* ,$$

respectively. Then the representations ρ_V^*, ρ_E^*, ρ_F^* are realized on V as follows:

$$\begin{aligned}\rho_V^*(g)v &= (hxt_1, hyt_2, hzt_3) ,\\ \rho_E^*(g)v &= (hxt_1, hyt_2^{-1}, hzt_3^{-1}) ,\\ \rho_F^*(g)v &= (hxt_1^{-1}, hyt_2, hzt_3) .\end{aligned}$$

The singular sets of the p.v.'s (G, ρ_X^*, V) $(X=E, F, V)$ coincide with S and the polynomials P_1, P_2, P_3 are also irreducible relative invariants of (G, ρ_X^*, V). Since $\rho|_\Gamma = \rho_E^*|_\Gamma = \rho_F^*|_\Gamma = \rho_V^*|_\Gamma$, the zeta functions associated with (G, ρ_X^*, V) $(X=E, F, V)$ are also given by the formula (7-3). Therefore the conditions (5-1) for $K=\boldsymbol{Q}$, (5-2), (6-1) and (6-2) are satisfied in the present three cases.

By an easy calculation, we get

$$U_V=\begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix}, \qquad U_E=\begin{pmatrix}1&0&0\\1&0&-1\\1&-1&0\end{pmatrix}, \qquad U_F=\begin{pmatrix}-1&0&0\\-1&0&1\\-1&1&0\end{pmatrix}.$$

Moreover

$$\lambda_V=(1, 1, 1) , \qquad \lambda_E=(-1, 1, 1) , \qquad \lambda_F=(2, 1, 1) .$$

For an $f\in\mathscr{S}(V_{\boldsymbol{R}})$, set

$$\Phi_\pm(f; s)=\int_{V_\pm} |P(v)|^s f(v)dv .$$

Let $\mathscr{F}_V f$, $\mathscr{F}_E f$ and $\mathscr{F}_F f$ be the (partial) Fourier transforms of f with respect to V, E and F respectively.

The explicit forms of the functional equations in Theorem 1 are given by the following lemma.

LEMMA 7.1.

(i)

$$\begin{pmatrix}\Phi_+(\mathscr{F}_V f;\ s_1, s_2, s_3)\\ \Phi_-(\mathscr{F}_V f;\ s_1, s_2, s_3)\end{pmatrix}$$
$$= 2(2\pi)^{-2(s_1+s_2+s_3)-5}\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_3+1)\Gamma(s_1+s_2+s_3+2)$$
$$\times\begin{pmatrix} \sin(s_1+s_2)\pi+\sin(s_2+s_3)\pi+\sin(s_3+s_1)\pi & -\sin(s_1+s_2+s_3)\pi-\sin\pi s_1-\sin\pi s_2-\sin\pi s_3\\ -\sin(s_1+s_2+s_3)\pi-\sin\pi s_1-\sin\pi s_2-\sin\pi s_3 & \sin(s_1+s_2)\pi+\sin(s_2+s_3)\pi+\sin(s_3+s_1)\pi\end{pmatrix}$$
$$\times\begin{pmatrix}\Phi_+(f;\ -1-s_1, -1-s_2, -1-s_3)\\ \Phi_-(f;\ -1-s_1, -1-s_2, -1-s_3)\end{pmatrix}.$$

(ii)

$$\begin{pmatrix}\Phi_+(\mathscr{F}_E f;\ s_1, s_2, s_3)\\ \Phi_-(\mathscr{F}_E f;\ s_1, s_2, s_3)\end{pmatrix} = 2(2\pi)^{-(s_2+s_3)-2}\Gamma(s_2+1)\Gamma(s_3+1)$$
$$\times\begin{pmatrix}-\cos((s_2+s_3)\pi/2) & \cos((s_2-s_3)\pi/2)\\ \cos((s_2-s_3)\pi/2) & -\cos((s_2+s_3)\pi/2)\end{pmatrix}$$
$$\times\begin{pmatrix}\Phi_+(f;\ s_1+s_2+s_3+1, -s_3-1, -s_2-1)\\ \Phi_-(f;\ s_1+s_2+s_3+1, -s_3-1, -s_2-1)\end{pmatrix}.$$

(iii)

$$\begin{pmatrix}\Phi_+(\mathscr{F}_F f;\ s_1, s_2, s_3)\\ \Phi_-(\mathscr{F}_F f;\ s_1, s_2, s_3)\end{pmatrix} = 2(2\pi)^{-(2s_1+s_2+s_3)-3}\Gamma(s_1+1)\Gamma(s_1+s_2+s_3+2)$$
$$\times\begin{pmatrix}-\sin((s_2+s_3)\pi/2) & \sin((2s_1+s_2+s_3)\pi/2)\\ \sin((2s_1+s_2+s_3)\pi/2) & -\sin((s_2+s_3)\pi/2)\end{pmatrix}$$
$$\times\begin{pmatrix}\Phi_+(f;\ -s_1-s_2-s_3-2, s_3, s_2)\\ \Phi_-(f;\ -s_1-s_2-s_3-2, s_3, s_2)\end{pmatrix}.$$

By Lemma 5.9, we are able to reduce the lemma to the following well-known formula (cf. [4, p. 360]):

$$\begin{pmatrix}\int |y|_+^s e^{2\pi i x y}dy\\ \int |y|_-^s e^{2\pi i x y}dy\end{pmatrix} = \sqrt{-1}(2\pi)^{-s-1}\Gamma(s+1)\begin{pmatrix} e^{\pi s\sqrt{-1}/2} & -e^{-\pi s\sqrt{-1}/2}\\ -e^{-\pi s\sqrt{-1}/2} & e^{\pi s\sqrt{-1}/2}\end{pmatrix}\cdot\begin{pmatrix}|x|_+^{-1-s}\\ |x|_-^{-1-s}\end{pmatrix}.$$

We omit further detail of the proof of Lemma 7.1.

The polynomials $b_V(s)$, $b_E(s)$ and $b_F(s)$ are easily computed by (5-8) and the lemma above:

LEMMA 7.2.

$$b_V(s) = -s_1s_2s_3(s_1+s_2+s_3+1)(s_1+s_2+s_3)(s_1+s_2+s_3-1),$$
$$b_E(s) = s_2s_3, \qquad b_F(s) = -s_1(s_1-1)(s_1-2)(s_1+s_2+s_3+1).$$

Let $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ be $SL(2)_Z$-invariant lattices in $M(2,1;\boldsymbol{Q})$ and put

$$L^{(i)*} = \{x^* \in V_Q^{(i)};\ {}^t xJx^* \in \boldsymbol{Z} \text{ for all } x \in L^{(i)}\}\ .$$

We set

$$L = L^{(1)} \oplus L^{(2)} \oplus L^{(3)}\ , \qquad L_V^* = L^{(1)*} \oplus L^{(2)*} \oplus L^{(3)*}\ ,$$
$$L_E^* = L^{(1)*} \oplus L^{(2)} \oplus L^{(3)}\ , \qquad \text{and} \qquad L_F^* = L^{(1)} \oplus L^{(2)*} \oplus L^{(3)*}\ .$$

THEOREM 3. (i) *The Dirichlet series* $\xi_\pm(L;s)$, $\xi_\pm(L_V^*;s)$, $\xi_\pm(L_E^*;s)$, $\xi_\pm(L_F^*;s)$ *have analytic continuations to meromorphic functions of* s *in* $\boldsymbol{C}^3$.

(ii) *These functions multiplied by* $(s_1-1)(s_2-1)(s_3-1)(s_1+s_2+s_3-2)$ *are entire functions.*

(iii) *They satisfy the following functional equations*:

$$v(L_V^*)\begin{pmatrix}\xi_+(L_V^*;\,1-s_1,1-s_2,1-s_3)\\ \xi_-(L_V^*;\,1-s_1,1-s_2,1-s_3)\end{pmatrix}$$
$$= 2(2\pi)^{-2(s_1+s_2+s_3)+1}\Gamma(s_1)\Gamma(s_2)\Gamma(s_3)\Gamma(s_1+s_2+s_3-1)$$
$$\times\begin{pmatrix}\sin(s_1+s_2)\pi+\sin(s_2+s_3)\pi+\sin(s_3+s_1)\pi & \sin(s_1+s_2+s_3)\pi+\sin\pi s_1+\sin\pi s_2+\sin\pi s_3\\ \sin(s_1+s_2+s_3)\pi+\sin\pi s_1+\sin\pi s_2+\sin\pi s_3 & \sin(s_1+s_2)\pi+\sin(s_2+s_3)\pi+\sin(s_3+s_1)\pi\end{pmatrix}$$
$$\times\begin{pmatrix}\xi_+(L;\,s_1,s_2,s_3)\\ \xi_-(L;\,s_1,s_2,s_3)\end{pmatrix},$$

$$v(L^{(1)*})\begin{pmatrix}\xi_+(L_E^*;\,s_1+s_2+s_3-1,1-s_3,1-s_2)\\ \xi_-(L_E^*;\,s_1+s_2+s_3-1,1-s_3,1-s_2)\end{pmatrix}$$
$$= 2(2\pi)^{-s_2-s_3}\Gamma(s_2)\Gamma(s_3)$$
$$\times\begin{pmatrix}\cos((s_2+s_3)\pi/2) & \cos((s_2-s_3)\pi/2)\\ \cos((s_2-s_3)\pi/2) & \cos((s_2+s_3)\pi/2)\end{pmatrix}\cdot\begin{pmatrix}\xi_+(L;\,s_1,s_2,s_3)\\ \xi_-(L;\,s_1,s_2,s_3)\end{pmatrix},$$

$$v(L^{(2)*}\oplus L^{(3)*})\begin{pmatrix}\xi_+(L_F^*;\,2-s_1-s_2-s_3,s_3,s_2)\\ \xi_-(L_F^*;\,2-s_1-s_2-s_3,s_3,s_2)\end{pmatrix}$$
$$= 2(2\pi)^{-(2s_1+s_2+s_3)+1}\Gamma(s_1)\Gamma(s_1+s_2+s_3-1)$$
$$\times\begin{pmatrix}\sin((s_2+s_3)\pi/2) & \sin((2s_1+s_2+s_3)\pi/2)\\ \sin((2s_1+s_2+s_3)\pi/2 & \sin((s_2+s_3)\pi/2)\end{pmatrix}\cdot\begin{pmatrix}\xi_+(L;\,s_1,s_2,s_3)\\ \xi_-(L;\,s_1,s_2,s_3)\end{pmatrix}.$$

PROOF. Since G is reductive, the first assertion is a special case of Corollary 1 to Theorem 2. The third assertion follows from Theorem 2 (iii) and Lemma 7.1. By Corollary 1 and Corollary 2 to Theorem 2, the functions $b_V(s-1)\xi_\pm(L;s)$ and $b_E(s-1)b_F(s-1)\xi_\pm(L;s)$ are entire functions. Here $s - 1 = (s_1 - 1, s_2 - 1, s_3 - 1)$. Hence Lemma 7.2 implies the second assertion.

REMARK. For $L = V_Z$, the results in Theorem 3 are consistent with the functional equation of the Riemann zeta function.

7.2. Example (B). Let $A(m, n)$ be the number of distinct solutions of the congruence $x^2 \equiv n \pmod{m}$. We define four Dirichlet series $\xi_i(s_1, s_2)$ and $\xi_i^*(s_1, s_2)$ $(i = 1, 2)$ by the following formulas:

$$\xi_i(s_1, s_2) = 2^{-1} \sum_{m,n=1}^{\infty} A(4m, (-1)^{i-1}n)m^{-s_1}n^{-s_2} ,$$

$$\xi_i^*(s_1, s_2) = \sum_{m,n=1}^{\infty} A(m, (-1)^{i-1}n)m^{-s_1}(4n)^{-s_2} .$$

These Dirichlet series were closely investigated by Shintani in [17, Chapter 1]. In particular he proved that:

THEOREM 4 (Shintani). (i) *The Dirichlet series* $\xi_i(s_1, s_2)$ *and* $\xi_i^*(s_1, s_2)$ $(i = 1, 2)$ *multiplied by*

$$\Gamma((s_1 + 1)/2)^{-1}s_1(2s_1 - 1)\zeta(2s_1)(s_2 - 1)(s_1 - 1)^2(2s_1 + 2s_2 - 3)$$

have analytic continuations to entire functions in C^2.

(ii) *They satisfy the following functional equations*:

$$\begin{pmatrix}\xi_1(s_1, 3/2 - s_1 - s_2)\\ \xi_2(s_1, 3/2 - s_1 - s_2)\end{pmatrix}$$
$$= 2^{-1}\pi^{1/2}(2/\pi)^{s_1+2s_2}\Gamma(s_2)\Gamma(s_1 + s_2 - 1/2)$$
$$\times \begin{pmatrix}\sin((s_1 + 2s_2)\pi/2) & \sin(\pi s_1/2)\\ \cos(\pi s_1/2) & \cos((s_1 + 2s_2)\pi/2)\end{pmatrix} \cdot \begin{pmatrix}\xi_1^*(s_1, s_2)\\ \xi_2^*(s_1, s_2)\end{pmatrix} .$$

(iii) *The functions*

$$(2\pi)^{-s_1}(\sin(\pi s_1/2))^{-1}\Gamma(s_1)\zeta(2s_1)\xi_1(s_1, s_2) ,$$
$$(2\pi)^{-s_1}\Gamma(s_1)\zeta(2s_1)\xi_2(s_1, s_2) ,$$
$$(2\pi)^{-s_1}(\sin(\pi s_1/2))^{-1}\Gamma(s_1)\zeta(2s_1)\xi_1^*(s_1, s_2) ,$$
$$(2\pi)^{-s_1}\Gamma(s_1)\zeta(2s_1)\xi_2^*(s_1, s_2)$$

are all invariant under the substitution $(s_1, s_2) \to (1 - s_1, s_1 + s_2 - 1/2)$.

We shall give a proof of the theorem above as an application of the results in § 6.

Let E be the vector space of 2 by 2 symmetric matrices and set $F = C^2$. In the following we consider an element y of F as a column vector $y = {}^t(y_1, y_2)$. Put $G = GL(2) \times GL(1)$ and $V = E \oplus F$. Define a representation ρ of G on V by setting

$$\rho(g, t)(x, y) = (g x {}^t g, t \cdot {}^t g^{-1} y) .$$

The triple (G, ρ, V) is a p.v. with irreducible relative invariants

$$P_1(x, y) = {}^t y x y , \qquad P_2(x, y) = P_2(x) = \det x$$

and the singular set S is given by

$$S = \{(x, y); P_1(x, y) = 0\} \cup \{(x, y); P_2(x) = 0\} .$$

The characters χ_1 and χ_2 defined by

$$\chi_1(g, t) = t^2 \quad \text{and} \quad \chi_2(g, t) = \det g^2$$

correspond to P_1 and P_2 respectively.

We consider the standard $\boldsymbol{Q}$-structure on (G, ρ, V):

$$G_{\boldsymbol{Q}} = GL(2)_{\boldsymbol{Q}} \times GL(1)_{\boldsymbol{Q}} , \qquad V_{\boldsymbol{Q}} = \{E \cap M(2; \boldsymbol{Q})\} \oplus \boldsymbol{Q}^2 ,$$
$$G_{\boldsymbol{Z}} = GL(2)_{\boldsymbol{Z}} \times \{\pm 1\} , \qquad V_{\boldsymbol{Z}} = \{E \cap M(2; \boldsymbol{Z})\} \oplus \boldsymbol{Z}^2 .$$

Then the p.v. (G, ρ, V) has the three $\boldsymbol{Q}$-regular subspaces E, F and V. Identify E and F with their dual vector spaces via the bilinear forms

$$\langle x, x^* \rangle = \operatorname{tr}(x J x^* {}^t J) \quad (x, x^* \in E) \quad \text{and} \quad \langle y, y^* \rangle = {}^t y J y \quad (y, y^* \in F)$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The representations ρ_E^*, ρ_F^* and ρ_V^* (partially) contragredient to ρ are given by

$$\rho_E^*(g, t)(x, y) = (\det g^{-2} \cdot g x {}^t g, t \cdot {}^t g^{-1} y) ,$$
$$\rho_F^*(g, t)(x, y) = (g x {}^t g, \det g \cdot t^{-1} \cdot {}^t g^{-1} y)$$

and

$$\rho_V^*(g, t)(x, y) = (\det g^{-2} \cdot g x {}^t g, \det g \cdot t^{-1} \cdot {}^t g^{-1} y) ,$$

respectively. Here the subscripts E, F and V have the same meaning as in Example (A). These formulas show that the triple (G, ρ_E^*, V), (G, ρ_F^*, V) and (G, ρ_V^*, V) are p.v.'s with the same relative invariants and the same singular set as those of (G, ρ, V).

The matrix U and the vectors λ, δ are easily calculated and we get

$$U_E = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} , \qquad U_F = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} , \qquad U_V = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} ,$$
$$\lambda_E = (0, 3/2) , \qquad \lambda_F = (1, -1/2) , \qquad \lambda_V = (1, 1) ,$$
$$\delta = \delta_E^* = \delta_F^* = \delta_V^* = (1, 1) .$$

Let $G_{\boldsymbol{R}}^+$ be the connected component of the identity element of $G_{\boldsymbol{R}}$. The $G_{\boldsymbol{R}}^+$-orbit decomposition of $V_{\boldsymbol{R}} - S_{\boldsymbol{R}}$ does not depend on the representations ρ, ρ_E^*, ρ_F^* and ρ_V^*, and is given by

$$V_R - S_R = V_1^+ \cup V_1^- \cup V_2^+ \cup V_2^-$$

where

$$V_i^\pm = \left\{(x, y) \in V_R ; \begin{array}{l} \operatorname{sgn} P_1(x, y) = \pm 1 \\ \operatorname{sgn} P_2(x, y) = (-1)^i \end{array}\right\} \quad (i = 1, 2) .$$

The group Γ coincides with $SL(2)_Z \times \{1\}$.

For any $(x, y) \in V_R - S_R$ and for any representation of ρ, ρ_E^*, ρ_F^*, and ρ_V^*, the group $G_{x,y}^+ = \Gamma_{x,y}$ are trivial. Let $dx = dx_{11}dx_{12}dx_{22}$ for $x = (x_{ij})_{i,j=1,2}$ and $dy = dy_1 dy_2$ for $y = {}^t(y_1, y_2)$. Then we can normalize a Haar measure $dgd^\times t$ on G_R^+ such that

$$\begin{aligned} &\int_{V_i^\pm} f(x, y) |P_1(x, y)|^{-1} |P_2(x)|^{-1} dxdy \\ &= \int_{G_R^+} f(\rho(g, t)(x_0, y_0)) dgd^\times t \\ &= \int_{G_R^+} f(\rho_X^*(g, t)(x_0, y_0)) dgd^\times t \\ &\qquad (X = E, F, V, \ (x_0, y_0) \in V_i^\pm, \ f \in L^1(V_i^\pm, |P_1P_2|^{-1}dxdy)) . \end{aligned}$$

The normalization is independent of the choice of (x_0, y_0).

Notice that $\rho|_\Gamma = \rho_E^*|_\Gamma = \rho_F^*|_\Gamma = \rho_V^*|_\Gamma$. Hence, for any Γ-invariant lattice L in V_Q, the zeta functions associated with (G, ρ, V), (G, ρ_E^*, V), (G, ρ_F^*, V) and (G, ρ_V^*, V) are given by

$$\xi_i^\pm(L; s_1, s_2) = \sum_{(x,y) \in \Gamma \backslash L \cap V_i^\pm} |P_1(x, y)|^{-s_1} |P_2(x)|^{-s_2} .$$

Since the mapping $(x, y) \mapsto (-x, y)$ induces a one to one correspondence between $\Gamma \backslash L \cap V_i^+$ and $\Gamma \backslash L \cap V_i^-$, we obtain

$$\xi_i^+(L; s_1, s_2) = \xi_i^-(L; s_1, s_2) \quad (i = 1, 2) .$$

From now on, we simply write $\xi_i(L; s_1, s_2)$ for $\xi_i^\pm(L; s_1, s_2)$.

Let $E_Z = E \cap M(2; Z)$ and $F_Z = Z^2$. Then

$$E_Z^* = \left\{\begin{pmatrix} x_1^* & x_2^*/2 \\ x_2^*/2 & x_3^* \end{pmatrix}; x_i^* \in Z\right\}$$

is the lattice dual to E_Z. It is easy to check that

$$\xi_i(E_Z \oplus F_Z; s_1, s_2) = 2^{2s_2}\zeta(2s_1)\xi_i^*(s_1, s_2) , \tag{7-5}$$

$$\xi_i(E_Z^* \oplus F_Z; s_1, s_2) = 2^{2s_2}\zeta(2s_1)\xi_i(s_1, s_2) \quad (i = 1, 2) . \tag{7-6}$$

This shows that the Dirichlet series $\xi_i(L; s_1, s_2)$ are absolutely convergent for $\mathrm{Re}\, s_1$, $\mathrm{Re}\, s_2 > 1$. Thus we can see that the conditions (5-1) for $K = \boldsymbol{Q}$, (5-2), (6-1) and (6-2) are satisfied by the $\boldsymbol{Q}$-regular subspaces E, F and V.

For an $f \in \mathscr{S}(V_R)$, put

$$\Phi_i(f; s) = \int_{V_i^+ \cup V_i^-} |P_1(x, y)|^{s_1} |P_2(x)|^{s_2} f(x, y) dx dy \quad (i = 1, 2) .$$

Denote by $\mathscr{F}_E f$, $\mathscr{F}_F f$, $\mathscr{F}_V f$ the (partial) Fourier transforms of $f \in \mathscr{S}(V_R)$ with respect to E, F and V respectively.

LEMMA 7.3. *The functions $\Phi_i(f; s)$ satisfy the following functional equations*:

$$\text{(7-7)} \quad \begin{pmatrix} \Phi_1(\mathscr{F}_E f; s) \\ \Phi_2(\mathscr{F}_E f; s) \end{pmatrix} = \Gamma(s_2 + 1)\Gamma(s_1 + s_2 + 3/2) 2^{-s_1 - 2s_2 - 2} \pi^{-5/2 - s_1 - 2s_2} \times \begin{pmatrix} -\cos((s_1 + 2s_2)\pi/2) & -\sin(\pi s_1/2) \\ \cos(\pi s_1/2) & \sin((s_1 + 2s_2)\pi/2) \end{pmatrix} \times \begin{pmatrix} \Phi_1(f; s_1, -3/2 - s_1 - s_2) \\ \Phi_2(f; s_1, -3/2 - s_1 - s_2) \end{pmatrix} ,$$

$$\text{(7-8)} \quad \begin{pmatrix} \Phi_1(\mathscr{F}_F f; s) \\ \Phi_2(\mathscr{F}_F f; s) \end{pmatrix} = \pi^{-2(s_1+1)} \Gamma(s_1 + 1)^2 \begin{pmatrix} 2\sin^2(\pi s_1/2) & 0 \\ 0 & -\sin(\pi s_1) \end{pmatrix} \times \begin{pmatrix} \Phi_1(f; -s_1 - 1, s_1 + s_2 + 1/2) \\ \Phi_2(f; -s_1 - 1, s_1 + s_2 + 1/2) \end{pmatrix} ,$$

$$\text{(7-9)} \quad \begin{pmatrix} \Phi_1(\mathscr{F}_V f; s) \\ \Phi_2(\mathscr{F}_V f; s) \end{pmatrix} = \Gamma(s_1 + 1)^2 \Gamma(s_2 + 1) \Gamma(s_1 + s_2 + 3/2) 2^{-s_1 - 2s_2 - 2} \pi^{-9/2 - 3s_1 - 2s_2} \times \begin{pmatrix} -2\sin^2(\pi s_1/2)\cos((s_1 + 2s_2)\pi/2) & \sin(\pi s_1)\sin(\pi s_1/2) \\ \sin(\pi s_1)\sin(\pi s_1/2) & -\sin((s_1 + 2s_2)\pi/2)\sin(\pi s_1) \end{pmatrix} \times \begin{pmatrix} \Phi_1(f; -1 - s_1, -1 - s_2) \\ \Phi_2(f; -1 - s_1, -1 - s_2) \end{pmatrix} .$$

PROOF. The functional equation (7-7) follows easily from Lemma 5.9 and [17, Chapter 1, Lemma 1 (i)]. By Lemma 5.9, the functional equation (7-8) is reduced to the formulas for the Fourier transforms of $|x^2 \pm y^2|^s$ (cf. [4, Chapter III 2.6]). Combining (7-7) with (7-8), we obtain the last functional equation (7-9).

The partial b-functions with respect to the regular subspaces E, F and V are easily computed.

LEMMA 7.4. *We have, up to non-zero constant multiples,*

$$b_E(s) = s_2(s_2 - 1)(s_1 + s_2 + 1/2)\,, \qquad b_F(s) = s_1^2\,,$$
$$b_V(s) = s_1^2 s_2(s_1 + s_2 + 1/2)(s_1 + s_2 - 1/2)\,.$$

Let M and N be Γ-invariant lattices in E_Q and F_Q respectively. We denote by M^* (resp. N^*) the lattice dual to M (resp. N). Set

$$L = M \oplus N\,, \qquad L_E^* = M^* \oplus N\,, \qquad L_F^* = M \oplus N^*\,, \qquad L_V^* = M^* \oplus N^*\,.$$

THEOREM 5. (i) *The Dirichlet series* $\xi_i(L; s)$, $\xi_i(L_E^*; s)$, $\xi_i(L_F^*; s)$ *and* $\xi_i(L_V^*; s)$ $(i = 1, 2)$ *multipled by* $(s_1 - 1)^2(s_2 - 1)(s_1 + s_2 - 3/2)$ *have analytic continuations to entire functions in* C^2.

(ii) *They satisfy the following functional equations*:

$$\begin{pmatrix} \xi_1(L_E^*; s_1, 3/2 - s_1 - s_2) \\ \xi_2(L_E^*; s_1, 3/2 - s_1 - s_2) \end{pmatrix}$$
$$= v(M^*)^{-1} 2^{1-s_1-2s_2} \pi^{1/2-s_1-2s_2} \Gamma(s_2)\Gamma(s_1 + s_2 - 1/2)$$
$$\times \begin{pmatrix} \sin((s_1 + 2s_2)\pi/2) & \sin(\pi s_1/2) \\ \cos(\pi s_1/2) & \cos((s_1 + 2s_2)\pi/2) \end{pmatrix} \cdot \begin{pmatrix} \xi_1(L; s_1, s_2) \\ \xi_2(L; s_1, s_2) \end{pmatrix},$$

$$\begin{pmatrix} \xi_1(L_F^*; 1 - s_1, s_1 + s_2 - 1/2) \\ \xi_2(L_F^*; 1 - s_1, s_1 + s_2 - 1/2) \end{pmatrix}$$
$$= v(N^*)^{-1} \pi^{-2s_1} \Gamma(s_1)^2 \begin{pmatrix} 2\cos^2(\pi s_1/2) & 0 \\ 0 & \sin(\pi s_1) \end{pmatrix} \cdot \begin{pmatrix} \xi_1(L; s_1, s_2) \\ \xi_2(L; s_1, s_2) \end{pmatrix},$$

$$\begin{pmatrix} \xi_1(L_V^*; 1 - s_1, 1 - s_2) \\ \xi_2(L_V^*; 1 - s_1, 1 - s_2) \end{pmatrix}$$
$$= v(L_V^*)^{-1} 2^{1-s_1-2s_2} \pi^{-3s_1-2s_2+1/2} \Gamma(s_1)^2 \Gamma(s_2) \Gamma(s_1 + s_2 - 1/2)$$
$$\times \begin{pmatrix} 2\cos^2(\pi s_1/2)\sin(\pi(s_1 + 2s_2)/2) & \sin(\pi s_1)\cos(\pi s_1/2) \\ \sin(\pi s_1)\cos(\pi s_1/2) & \sin(\pi s_1)\cos(\pi(s_1 + 2s_2)/2) \end{pmatrix}$$
$$\times \begin{pmatrix} \xi_1(L; s_1, s_2) \\ \xi_2(L; s_1, s_2) \end{pmatrix}.$$

PROOF. As in the proof of Theorem 3, Corollaries 1, 2 to Theorem 2 and Lemma 7.4 imply the first assertion. The functional equations are immediate consequences of Theorem 2 (iii) and Lemma 7.3.

Now the theorem of Shintani is easily derived from (7-5) (7-6) and Theorem 5. Moreover Shintani's result on singularities of ξ_i and ξ_i^* is improved in our Theorem 5 (i).

REMARK 1. Shintani's method for proving Theorem 4 (ii) is essentially the same as ours. For the functional equations in Theorem 4 (iii), he reduced them to the functional equation of the Legendre function (see the proof of [17, Lemma 1 (ii)]). We note that the functional equation of the Legendre function is derived from (7-8) which is the base of the functional equations of $\xi_i(L; s)$ with respect to the $\boldsymbol{Q}$-regular subspace F.

REMARK 2. Suzuki gave a generalization of this example to the vector space of n by n symmetric matrices for $n \geqq 4$ in [20]. Another generalization will be seen in [24, § 4]. More precise investigation of zeta functions associated with the p.v. treated here will be made in [25, § 2].

REFERENCES

[1] I. N. BERNSTEIN AND S. I. GELFAND, Meromorphic property of the functions P^λ, Functional Anal. Appl. 3 (1969), 68-69.

[2] A. BOREL, Linear algebraic groups, Benjamin, New York, 1969.

[3] A. BOREL AND HARISH-CHANDRA, Arithmetic subgroups of algebraic groups, Ann. of Math. 75 (1962), 485-535.

[4] I. M. GELFAND AND G. E. SHILOV, Generalized functions, vol. 1, Academic Press, New York, 1964.

[5] L. HÖRMANDER, An introduction to complex analysis in several variables, Van Nostrand, Princeton, N.J., 1966.

[6] J. IGUSA, On certain representations of semi-simple algebraic groups, Lecture Notes in Math. 185, Springer-Verlag, Berlin, Heidelberg, New York, 1971, 76-86.

[7] T. KIMURA, The b-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, Nagoya Math. J. 85 (1982), 1-80.

[8] R. P. LANGLANDS, Dirichlet series associated with quadratic forms, Lecture Notes in Math. 544, Springer-Verlag, Berlin, Heidelberg, New York, 1976, 236-268.

[9] H. MAASS, Siegel's modular forms and Dirichlet series, Lecture Notes in Math. 216, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[10] M. MURO, Micro local calculus and Fourier transforms, Publ. RIMS, Kyoto Univ. 12 Suppl. (1977), 301-307.

[11] M. SATO, Theory of prehomogeneous vector spaces (notes by T. Shintani in Japanese), Sugaku no Ayumi 15 (1970), 85-157.

[12] M. SATO, M. KASHIWARA, T. KIMURA AND T. OSHIMA, Micro local analysis of prehomogeneous vector spaces, Invent. Math. 62 (1980), 117-179.

[13] M. SATO AND T. KIMURA, A classification of irreducible prehomogeneous vector spaces and their invariants, Nagoya Math. J. 65 (1977), 1-155.

[14] M. SATO AND T. SHINTANI, On zeta functions associated with prehomogenous vector spaces, Ann. of Math. 100 (1974), 131-170.

[15] A. SELBERG, A new type of zeta functions connected with quadratic forms, Report of the Institute in the Theory of Numbers, Univ. of Colorado, Boulder, Colo. 1959, 207-210.

[16] T. SHINTANI, On Dirichlet series whose coefficients are class numbers of integral binary cubic forms, J. Math. Soc. Japan 24 (1972), 132-188.

[17] T. SHINTANI, On zeta functions associated with the vector space of quadratic forms, J. Fac. Sci. Univ. Tokyo 22 (1975), 25-65.

[18] C. L. SIEGEL, Über die Zetafunktionen indefiniter quadratischer Formen, Math. Z. 43 (1938), 682-708.

[19] T. SUZUKI, Fourier tranforms of relative invariants of prehomogeneous vector spaces (in Japanese), Master's Thesis, Nagoya Univ., 1975.

[20] T. SUZUKI, On zeta functions associated with quadratic forms of variable coefficients, Nagoya Math. J. 73 (1979), 117-147.

[21] A. TERRAS, A generalization of Epstein's zeta function, Nagoya Math. J. 42 (1971), 173-188.

[22] G. WARNER, Harmonic analysis on semi-simple Lie groups I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[23] F. SATO, Zeta functions in several variables associated with prehomogeneous vector spaces II: A convergence criterion, to appear in Tôhoku Math. J.

[24] F. SATO, Zeta functions in several variables associated with prehomogeneous vector spaces III: Eisenstein series for indefinite quadratic forms, to appear in Ann. of Math.

[25] F. SATO, On zeta functions of ternary zero forms, J. Fac. Sci. Univ. Tokyo. 28 (1982), 585-604.

[26] F. SATO, Zeta functions in several variables associated with prehomogeneous vector spaces I: Functional equations, Proc. Japan Acad. 57 (1981), 74-79.

DEPARTMENT OF MATHEMATICS
RIKKYO UNIVERSITY
NISHI-IKEBUKURO, TOKYO 171
JAPAN