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A RELATION BETWEEN THE TOTAL CURVATURE AND THE MEASURE OF RAYS, II

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1. Introduction. The total curvature of complete, noncompact, connected and oriented Riemannian 2-manifold M is defined to be an improper integral $\int_M Gdv$ of the Gaussian curvature G of M with respect to the Riemannian volume dv over M. It is well known that the total curvature of such an M is not a topological invariant but it depends on the choice of the Riemannian structure. The pioneering work of Cohn-Vossen on the total curvature states that if M is finitely connected and if M admits the total curvature, then $\int_M Gdv \leqq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M (see [2, Satz 6]). It is interesting to investigate a geometric influence of the total curvature on the Riemannian structure of M which defines it. The first attempt in this direction of the work was made by Maeda in [5], [6] and [7]. He investigated some relations between the measure of rays emanating from a fixed point and the total curvature of a complete Riemannian manifold homeomorphic to $\boldsymbol{R}^2$ whose Gaussian curvature is nonnegative everywhere.

From completeness and compactness of M it follows that through every point p on M there passes at least a ray $\gamma: [0, \infty) \to M$, where a ray is by definition a unit speed geodesic such that any subarc of it is a unique minimizing geodesic between the endpoints. Here all geodesics are parametrized by arc length unless otherwise mentioned. For a point p on M let T_pM and S_pM be the tangent space to M at p and the unit circle of T_pM centered at the origin. S_pM is endowed with a natural Lebesgue measure induced by the Riemannian structure of M. Let $A(p)$ be the set of all unit vectors tangent to rays emanating from p. $A(p)$ is closed because a limit geodesic of a sequence of rays is again a ray. Thus we are interested in the measure of the set $A(p)$. In a recent paper, Maeda has proved the following:

THEOREM (Maeda [7]). *If M is a complete Riemannian manifold homeomorphic to $\boldsymbol{R}^2$ and if the Gaussian curvature G of M is nonnegative everywhere, then*

$$\int_M G dv = 2\pi - \inf_{p \in M} \operatorname{meas}(A(p)) .$$

The purpose of the present note is to consider whether the above result is true for complete metrics on $\boldsymbol{R}^2$ on which G changes sign. We shall show that the equality in the above theorem does not hold in general where G changes sign. We shall furnish an example of a complete surface Σ homeomorphic to $\boldsymbol{R}^2$ embedded in the Euclidean 3-space E^3 on which the equality in the above theorem does not hold. In fact we have

$$2\pi - \int_\Sigma G^+ dv < \inf_{p \in \Sigma} \operatorname{meas}(A(p)) < 2\pi - \int_\Sigma G dv ,$$

where $G^+(x) := \operatorname{Max}\{G(x), 0\}$, $x \in \Sigma$. In general we have the following for a complete Riemannian manifold M homeomorphic to $\boldsymbol{R}^2$:

THEOREM 1. *Let M be a complete Riemannian manifold homeomorphic to $\boldsymbol{R}^2$. If M admits the total curvature, then*

$$2\pi - \int_M G^+ dv \leqq \inf_{p \in M} \operatorname{meas}(A(p)) \leqq 2\pi - \int_M G dv .$$

The infimum of meas $(A(p))$ is attained when G has compact support. However it is not certain whether the infimum is attained if the support of G is noncompact. But in the special case where $G > 0$ everywhere, we have the following as a direct consequence of the above Theorem 1:

THEOREM 2. *Let M be a complete Riemannian manifold whose Gaussian curvature is positive everywhere. If the infimum of* meas $(A(p))$ *is attained at some point on M, then the total curvature of M is equal to 2π.*

However the author does not know if the converse of Theorem 2 is true or not. Other geometric significance of the total curvature has been investigated by Innami [4] and Shiohama [9], [10]. Basic tools used in the proofs of our results will be given in §2 and the proofs are stated in §3. The example stated above will be furnished in §3.

2. **Preliminaries.** Let M be a connected, complete and noncompact Riemannian 2-manifold without boundary. The total curvature of M is defined as follows.

DEFINITION 1. M admits the total curvature if and only if for every monotone increasing sequence of compact domains $\{V_j\}$ of M such that $\bigcup_{j \geqq 1} V_j = M$, the sequence $\left\{\int_{V_j} G dv\right\}$ has a limit in $[-\infty, \infty]$.

It turns out that the limit of $\left\{\int_{V_j} Gdv\right\}$ does not depend on the choice of $\{V_j\}$ if the total curvature of M exists.

REMARK. M admits the total curvature if and only if either $\int_M G^+dv < \infty$ or $\int_M G^-dv > -\infty$ is fulfilled, where $G^- = G - G^+$. If M does not admit the total curvature, then for any given real number A there exists a monotone increasing sequence $\{V_j\}$ of compact domains in M such that $\bigcup_{j\geq 1} V_j = M$ and that $\lim_{j\to\infty}\int_{V_j} Gdv = A$ (see [2, the footnote on p. 79]).

A geodesic $\gamma: R \to M$ is by definition a straight line if any subarc of γ is minimizing. The following theorem was proved by Cohn-Vossen which plays an essential role for the proofs of our results.

THEOREM A ([3, Satz 6]). *Let M be a complete Riemannian manifold homeomorphic to $\boldsymbol{R}^2$. If M admits the total curvature and if there exists a straight line $\gamma: R \to M$, then*

$$\int_M Gdv \leqq 0\ .$$

The proof of Theorem A is based upon the following Lemma B which Cohn-Vossen discussed in [3, § 5], and is valid in all dimensions.

LEMMA B. *Let N be an n-dimensional ($n \geqq 2$) complete noncompact Riemannian manifold and let $\gamma: [0, \infty) \to N$ be a ray and let x be a fixed point on N. For any positive ε, there exist a divergent sequence $\{t_j\}$ and minimizing geodesics $\sigma_j: [0, l_j] \to N$ such that $\sigma_j(0) = x$, $\sigma_j(l_j) = \gamma(t_j)$, $j = 1, \cdots$, and they satisfy $\measuredangle(\dot\sigma_j(l_j), \dot\gamma(t_j)) < \varepsilon$ for all $j = 1, \cdots$.*

PROOF OF LEMMA B. Set $f(t) := d(x, \gamma(t))$, where d is the distance function on N induced by the Riemannian metric. f is Lipschitz continuous with Lipschitz constant 1, and hence it is differentiable almost everywhere. f is differentiable at $t_0 > 0$ if and only if every minimizing geodesic joining x to $\gamma(t_0)$ makes a constant angle with γ at $\gamma(t_0)$. It then turns out that the constant is equal to $\cos^{-1}(f'(t_0))$. f is non-differentiable at $t_0 > 0$ if and only if there are two distinct minimizing geodesics joining x to $\gamma(t_0)$ such that their angles with γ at $\gamma(t_0)$ are not equal. Thus setting $\theta(t) := \cos^{-1}(f'(t))$ where it is defined, we have $t - f(t) = \int_0^t [1 - \cos\theta(u)]du - f(0)$. It follows from the triangle inequality that $t - f(t) \leqq d(x, \gamma(0))$ for all $t \geqq 0$. Therefore the integrand of the above equality is bounded above for all $t > 0$. Thus $\lim_{u\to\infty}\inf [1 - \cos\theta(u)] = 0$, and the proof is complete.

In the proof of Theorem A and later in §3, Lemma B is used on a closed unbounded domain D of M which is bounded by a geodesic polygon. The distance function $\hat{d}(x, y)$ on D is defined to be the infimum of lengths of all curves in D joining x and y. Every two points on D can be joined by a $\hat{d}$-minimizing segment in D whose length realizes the distance between the two points. The existence of such segments was already established Cohn-Vossen [2, §10, §11].

PROOF OF THEOREM A. Let $\{W_j\}$ be a monotone increasing sequence of compact domains satisfying $\bigcup_{j\geqq 1} W_j = M$ such that for each j, $W_j - \mathrm{int}(W_j)$ is a simply closed geodesic polygon. Let ε be an arbitrary small positive number and fix j. By Lemma B, we can choose large t_j and s_j in such a way that $\gamma(t_j)$ and $\gamma(-s_j)$ are joined by two segments T_j and S_j in $M - \mathrm{int}(W_j)$ satisfying the following properties: T_j and S_j are not homotopic to each other in $M - \mathrm{int}(W_j)$, they have the same minimal lengths among all curves in $M - \mathrm{int}(W_j)$ having the same endpoints and belonging to the same homotopy classes, and their angles at the endpoints are less than $\varepsilon/2$. The minimizing property of S_j and T_j in $M - \mathrm{int}(W_j)$ implies that if x is a non-differentiable point of S_j, then x belongs to ∂W_j and the angle of S_j at x is not smaller than π if it is measured with respect to $M - \mathrm{int}(W_j)$. Thus $S_j \cup T_j$ bounded a convex domain, say, D_j. The Gauss-Bonnet theorem implies that $\int_{D_j} G dv < \varepsilon$.

Now by choosing a subsequence $\{D_k\}$ of $\{D_j\}$ if necessary, we may assume that $\{D_k\}$ is a monotone increasing sequence with $\bigcup_k D_k = M$. Then $\lim_{k\to\infty} \int_{D_k} G dv = \int_M G dv \leqq \varepsilon$, and the proof is complete since ε is arbitrary.

From now on, let M be a complete Riemannian manifold homeomorphic to $\boldsymbol{R}^2$. Let q be an arbitrary fixed point on M. Since $A(q)$ is closed in $S_q(M)$, $S_q(M) - A(q)$ consists of a disjoint union of open subarcs of $S_q(M)$. Set $\bigcup_{\lambda\in\Lambda} F_\lambda := S_q(M) - A(q)$, where Λ is an index set, and each F_λ is an open subarc of $S_q(M)$. For each $\lambda \in \Lambda$ let $D_\lambda(q)$ be a unique component of the set $D(q) := M - \{\exp_q tu;\, u \in A(q),\, t \geqq 0\}$ which contains $\{\exp_q tv;\, v \in F_\Lambda,\, 0 < t < \text{the convexity radius at } q\}$. Clearly $\bar{D}_\lambda(q) - D_\lambda(q)$ consists of rays, say, $\sigma, \tau: [0, \infty) \to M$ with $\sigma(0) = \tau(0) = q$. The following fact was first proved by Maeda [5] under the assumption that $G \geqq 0$, and later the assumption $G \geqq 0$ was removed by the author. Here a proof simpler than that in [8] will be stated.

LEMMA C (compare [8, Lemma 4]). *With the same notations as above, we have*:

(1) *For any* $\varepsilon \in (0, \sphericalangle(\dot{\sigma}(0), \dot{\tau}(0)/2)$ *there exists an* $R = R(\varepsilon)$ *such that if* U *is the unique unbounded component of the set* $D_\lambda(q) \cap \{x \in M; d(q, x) > R\}$, *then for any* $x \in U$ *and for any minimizing geodesic* $\alpha: [0, l] \to M$ *with* $\alpha(0) = q$, $\alpha(l) = x$, *either* $\sphericalangle(\dot{\sigma}(0), \dot{\alpha}(0)) < \varepsilon$ *or else* $\sphericalangle(\dot{\tau}(0), \dot{\alpha}(0)) < \varepsilon$ *holds.*

(2) *For any fixed* $t > R$ *if* $c_t: [0, 1] \to U$ *is a curve whose image bounds a unique unbounded component of* $\{x \in U; d(q, x) > t\}$, *then there are a point* x *on* $c_t([0, 1])$ *and two distinct minimizing geodesics* $\beta, \gamma: [0, t] \to M$ *such that their images are in* D_λ *and that they satisfy*

$$\sphericalangle(\dot{\beta}(0), \dot{\sigma}(0)) < \varepsilon , \qquad \sphericalangle(\dot{\gamma}(0), \dot{\tau}(0)) < \varepsilon .$$

PROOF OF LEMMA C. First of all it follows from the construction of the boundary of $\bar{D}_\lambda$ that every minimizing geodesic joining q to any point in D_λ does not intersect its boundary and has its image in $\bar{D}_\lambda$. Therefore (1) is a direct consequence of the fact that there is no ray emanating from q whose initial tangent vector belongs to F_λ

To prove (2) note that there is an open set around $\sigma(t)$ in which every point can be joined to q by a unique minimizing geodesic which makes an angle with σ at q less than ε. Note also that for every $u \in [0, 1]$ every minimizing geodesic joining q to $c_t(u)$ does not meet c_t ([0, 1]) except at $c_t(u)$. Let $I_\sigma := \{u \in [0, 1]$; every minimizing geodesic joining q to $c_t(u)$ makes an angle with σ at q less than $\varepsilon\}$. Then it follows from what is noted above that $I_\sigma \neq \emptyset$ and that if $u \in I_\sigma$, then $u' \in I_\sigma$ holds for all $u' \in [0, u)$. Similarly we define $I_\tau := \{u \in [0, 1]$; every minimizing geodesic joining q to $c_t(u)$ makes an angle with τ at q less than $\varepsilon\}$. Then we conclude that I_τ is a nonempty subinterval of [0, 1] containing 1. If there is a $u_0 \in [0, 1] - I_\sigma \cup I_\tau$, then the point $x = c_t(u_0)$ has the desired property. Suppose that $[0, 1] = I_\sigma \cup I_\tau$. Then it follows from $I_\sigma \cap I_\tau = \emptyset$ that one of the two intervals is open and the other is closed. Without loss of generality we may assume that I_σ is open. Set $u_0 = \sup I_\sigma$. Then a minimizing geodesic $\beta: [0, t] \to M$ is obtained as a limit of minimizing geodesics $\beta_j: [0, t] \to M$ with $\beta_j(0) = q$, $\beta_j(t) = c_t(u_j)$ such that $u_j \in I_\sigma$ and $\lim u_j = u_0$. On the other hand, it follows from $u_0 \in I_\tau$ that there is a minimizing geodesic γ joining q to $c_t(u_0)$ with the desired property. Thus the proof is complete.

The following result was established in the previous work of the author and the proof is omitted here.

THEOREM D ([8, Theorem 2]). *Let* M *be a complete Riemannian manifold homeomorphic to* $\boldsymbol{R}^2$. *If* M *admits the total curvature, then for every point* p *on* M,

$$\text{meas}\,(A(p)) \geqq 2\pi - \int_M G^+ dv\ .$$

3. The proof of Theorem 1. By means of Theorem D we only need to prove that

$$(*)\qquad \inf_{p \in M} \text{meas}\,(A(p)) \leqq 2\pi - \int_M G dv\ .$$

If the total curvature of M is nonpositive, then $(*)$ is obvious. Taking Theorem A into account, we may therefore assume that *M admits no straight line.* If the left hand side of $(*)$ is zero, then $(*)$ is nothing but the Cohn-Vossen theorem. Therefore we may assume that *through every point p on M there pass at least two* (indeed more than two) *distinct rays.*

Let $\{V_j\}$ be a monotone increasing sequence of compact sets such that $\bigcup_{j \geqq 1} V_j = M$. Let ε be an arbitrary fixed positive number, and fix j. The proof is divided into three steps as follows.

STEP 1. Since M has no straight line by assumption, there is an $R_j > 0$ such that for each point x on $M - B_{R_j}(V_j)$ all rays emanating from x do not intersect V_j. Indeed, we otherwise would have a divergent sequence of points $\{x_k\}$ and rays $\{\sigma_k\}$, each σ_k emanating from $x_k = \sigma_k(0)$ and passing through a point on V_j. Then the compactness of V_j would make it possible to choose a subsequence of $\{\sigma_k\}$ which converges to a straight line, a contradiction. Let q be a point on $M - B_{R_j}(V_j)$ and set $S_q(M) - A(q) = \bigcup_{\lambda \in \Lambda} F_\lambda$ as before. Then there exists a $\lambda \in \Lambda$ such that F_λ is a proper subarc of $S_q(M)$ and that $D_\lambda(q)$ contains V_j. Let $\sigma, \tau \colon [0, \infty) \to M$ be distinct rays with $\sigma(0) = \tau(0) = q$ which bound $D_\lambda(q)$. The existence of distinct rays emanating from q is guaranteed by the assumption that the left hand side of $(*)$ is positive.

STEP 2. Since V_j is contained in $D_\lambda(q)$, there is a positive number η such that $\eta = \inf\{\measuredangle(\dot\sigma(0), \dot\alpha(0)), \measuredangle(\dot\tau(0), \dot\alpha(0));\ \alpha$ is a minimizing geodesic joining q to every point on $V_j\}$. It follows from Lemma C, (2) that there exist a point p and two minimizing geodesics $a, b \colon [0, l] \to D_\lambda(q)$ such that $a(0) = b(0) = q$, $a(l) = b(l) = p$ and $\measuredangle(\dot a(0), \dot\sigma(0)) < \eta/2$, $\measuredangle(\dot b(0), \dot\tau(0)) < \eta/2$. Thus the subdomain of $D_\lambda(q)$ which is bounded by $a([0, l])$ and $b([0, l])$ contains V_j in its interior.

Consider $\bar D_\lambda(q)$ to be a complete Riemannian manifold with nonempty boundary. The distance function $\hat d$ is naturally defined on $\bar D_\lambda(q)$ by the metric on M restricted to $\bar D_\lambda(q)$. Then every two points can be joined by a $\hat d$-minimizing segment which may have a nondifferentiable point in its interior. It follows from Lemma B that there are large numbers s_j and t_j and $\hat d$-minimizing segments a_j and b_j joining p to $\sigma(s_j)$ and $\tau(t_j)$

respectively such that the angles between a_j and σ at $\sigma(s_j)$ and between b_j and τ at $\tau(t_j)$ are less than $\varepsilon/2$. It should be checked that these angles are all positive. This is observed as follows. It suffices for the proof of $\measuredangle(\dot{a}_j, \dot{\sigma})_{|\sigma(s_j)} > 0$ to show that a_j is a geodesic in M. Suppose a_j has a nondifferentiable point in its interior. Then the break point coincides with q. (This might happen when $\measuredangle(\dot{\sigma}(0), \dot{\tau}(0))$ is close to 2π.) Since a_j have $\hat{d}$-minimizing property and since a and b are minimizing, the subarc of a_j between p and q has the same length as a and b. Thus $L(a_j) = \hat{d}(p, q) = L(a) + L(\sigma|[0, s_j])$. But since $\measuredangle(\dot{a}(0), \dot{\sigma}(0)) < \eta/2$ we have $L(a) + L(\sigma|[0, s_j]) > (s_j - r) + d(\sigma(r), a(r)) + (L(a) - r)$ for a small $r > 0$, and this value is realized by a broken geodesic in $D_\lambda(q)$ joining $\sigma(s_j)$, $\sigma(r)$, $a(r)$ and $p = a(l)$ in an obvious way. This contradicts the $\hat{d}$-minimizing property of a_j, proving the positivity of the angle.

STEP 3. Let D_j be the compact subdomain of $\bar{D}_\lambda(q)$ which is bounded by a_j, b_j, $\sigma([0, s_j])$ and $\tau([0, t_j])$. The above argument ensures that D_j contains the compact domain bounded by a geodesic biangle a and b, and hence $D_j \supset V_j$. By choosing a subsequence $\{D_k\}$ of $\{D_j\}$ if necessary, we may assume that $\{D_k\}$ is monotone increasing and $\bigcup_k D_k = M$. For each k the Gauss-Bonnet theorem applies to yield

$$\int_{D_k} G dv < \text{meas}(\bar{F}_\lambda) + \varepsilon \leqq 2\pi - \text{meas}(A(q)) + \varepsilon$$
$$\leqq 2\pi - \inf_{q \in M} \text{meas}(A(p)) + \varepsilon .$$

This completes the proof of Theorem 1 since ε is an arbitrary positive number.

PROOF OF THEOREM 2. Let q be a point of M at which the infimum of the function $x \mapsto \text{meas}(A(x))$ is attained. Let $\bigcup_{\lambda \in \Lambda} F_\lambda = S_q(M) - A(q)$ and for each $\lambda \in \Lambda$ let $D_\lambda(q)$ be defined as in §2. It follows from Lemma C, (2) that $D_\lambda(q)$ for each $\lambda \in \Lambda$ is covered by a monotone increasing sequence of compact subdomains in $D_\lambda(q)$ each of which is bounded by a geodesic biangle. Because of $G > 0$ the total curvature of $D_\lambda(q)$ exists and

$$\int_{D_\lambda(q)} G dv \geqq \text{meas}(\bar{F}_\lambda) .$$

Moreover we have

$$\int_M G dv \geqq \sum_{\lambda \in \Lambda} \int_{D_\lambda(q)} G dv \geqq \sum_{\lambda \in \Lambda} \text{meas}(\bar{F}_\lambda) = 2\pi - \text{meas}(A(q)) ,$$

where the first inequality is ensured by the assumption $G > 0$. Therefore all equalities hold by means of the Maeda theorem. The first equality implies that $M - \bigcup_{\lambda \in \Lambda} D_\lambda(q)$ has measure zero, in other words, $A(q)$ has no interior in $S_q(M)$. Hence meas $(A(q)) = 0$ implies the conclusion.

Finally we furnish an example of a surface Σ in E^3, on which both inequalities in Theorem 1 hold. The Gaussian curvature G of Σ has compact support and Σ has two "hills" on a plane. The construction is carried out as follows. For positive numbers $a < b$ set $f(t) := h(b-t)/\{h(b-t) + h(t-a)\}$, where $h: R \to R$ is defined by

$$h(t) := \begin{cases} \exp(-1/t^2) & \text{for } t \geqq 0 \\ 0 & \text{for } t < 0 . \end{cases}$$

Computations show that $f(t) = 0$ for $t \geqq b$, $0 < f(t) < 1$ for $a < t < b$ and $f(t) = 1$ for $a \geqq t$, and $0 \geqq f'(t) \geqq f'((a+b)/2) = -8(b-a)^{-3}$. For an arbitrary fixed point $p = (p_1, p_2)$ with $\|p\| > 2b$, let Σ be defined as the graph of $x_3 = x_3(x_1, x_2)$;

$$x_3 := \begin{cases} f(\|x\|) & \text{for } \|x - p\| > b \\ f(\|x - p\|) & \text{for } \|x - p\| \leqq b , \end{cases}$$

where $x = (x_1, x_2) \in \boldsymbol{R}^2$. It is elementary to verify that

$$\int_\Sigma G dv = 0 \quad \text{and} \quad \int_\Sigma G^+ dv = 4\pi(1 - \sin\{\tan^{-1}(b-a)^3/8\}) .$$

Therefore we can choose a and b in such a way that

$$\pi < \int_\Sigma G dv < 2\pi$$

is satisfied. We can also choose p sufficiently far from the origin so that meas $(A(x)) \geqq \pi$ holds for any point x on Σ. This is possible because every compact set on E^2 is contained in a cone of arbitrary small angle at the vertex is taken to be sufficiently far from the compact set. Therefore we have

$$2\pi - \int_\Sigma G^+ dv < \pi < \inf_{x \in \Sigma} \text{meas}\,(A(x)) < \pi = 2\pi - \int_\Sigma G dv .$$

It should be noted that the Cohn-Vossen theorem was extended to a finitely connected noncompact G-surface on which angular measure is defined. The total excess with respect to the angular measure plays the same role as the total curvature. For details see [1, §43 and §44].

References

[1] H. Busemann, The Geometry of Geodesics, Academic Press, New York, 1955.

[2] S. Cohn-Vossen, Kürzeste Wege und Totalkrümmung auf Flächen, Compositio Math. 2 (1935), 63–133.

[3] S. COHN-VOSSEN, Totalkrümmung und geodätische Linien auf einfach zusammenhängenden offenen volständigen Flächenstücken, Recueil Math. Moscow 43 (1936), 139–163.

[4] N. INNAMI, Differentiability of Busemann functions and total excess, Math. Z. 180 (1982), 235–247.

[5] M. MAEDA, On the existence of rays, Sci. Rep. Yokohama National University 26 (1979), 1–4.

[6] M. MAEDA, Remarks on the distribution of rays, to appear in Sci. Rep. Yokohama National University.

[7] M. MAEDA, A geometric significance of total curvature on complete open surfaces, to appear in Geometry of Geodesics and Related Topics, Advanced Studies in Pure Mathematics 3, Kinokuniya, Tokyo and North-Holland Publ. Amsterdam.

[8] K. SHIGA, On a relation between the total curvature and the measure of rays, Tsukuba J. Math. 6 (1982), 41–50.

[9] K. SHIOHAMA, Busemann functions and total curvature, Inventions Math. 53 (1979), 281–297.

[10] K. SHIOHAMA, A role of total curvature on complete noncompact Riemannian 3-manifolds, to appear in Illinois J. Math.

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