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ON LOCAL ISOMETRIC IMMERSIONS OF RIEMANNIAN SYMMETRIC SPACES

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Introduction. In the present paper we consider the problem concerning local isometric or conformal immersions of Riemannian symmetric spaces into the Euclidean spaces. The main results are announced in our recent note [2].

It is classically well known that any Riemannian manifold can be locally or globally isometrically immersed into the Euclidean spaces of sufficiently large dimension. For compact Riemannian symmetric spaces M it is known that many of them can be globally isometrically imbedded into the Euclidean spaces in codimension $\sim\dim M$ (Kobayashi [14]). On the other hand, Heitsch and Lawson [8] proved that the compact Lie groups $SO(2m+1)$ and $U(2m+1)$ with biinvariant Riemannian metrics cannot be globally conformally immersed into the Euclidean spaces in codimension $2m-1$ by calculating the Chern-Simons invariants. Later their method was extended by Donnelly [6], who proved that the Riemannian symmetric space $SU(2m+1)/SO(2m+1)$ cannot be globally conformally immersed in codimension $2m-1$. The purpose of this paper is to give a new estimate on the dimension of the Euclidean space into which Riemannian symmetric spaces $M=G/K$ can be locally isometrically or conformally immersed.

Let (M, g) be an n-dimensional Riemannian manifold and let R be the curvature tensor field of (M, g). Define a $\boldsymbol{Z}$-valued function r on M by setting

$$r(p)=(1/2)\max_{X,Y\in T_pM}\operatorname{rank} R(X, Y)\quad \text{for}\quad p\in M\,,$$

where $R(X, Y)$ is the curvature transformation of T_pM. If there exists an isometric immersion $\boldsymbol{f}$ of (M, g) into the m-dimensional Euclidean space $\boldsymbol{R}^m$, then $\boldsymbol{f}$ satisfies the so-called Gauss equation. Using this fact, we prove that in this case the function r defined above satisfies the inequality $r(p)\leqq m-n$ for each $p\in M$. In case $\boldsymbol{f}$ is a conformal immersion of (M, g) into $\boldsymbol{R}^m$, we prove, by considering the modified Gauss equation for conformal immersions, that the inequality $r(p)\leqq m-n+2$ holds for each $p\in M$ (Proposition 1.2).

Let $M = G/K$ be a Riemannian symmetric space. Then the function r takes a constant value and we denote it by $c(G/K)$. From the above result, it follows that G/K cannot be isometrically (resp. conformally) immersed into the Euclidean space in codimension $c(G/K) - 1$ (resp. $c(G/K) - 3$) even locally. The integer $c(G/K)$ can be expressed in the following Lie algebraic form. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$. We denote by $\rho: \mathfrak{k} \to \mathfrak{o}(\mathfrak{m})$ the infinitesimal linear isotropy representation of G/K at the origin. Then the integer $c(G/K)$ is given by $c(G/K) = (1/2) \max_{X,Y \in \mathfrak{m}} \operatorname{rank} \rho([X, Y])$ (see § 3). Our main aim is, applying the theory of Lie algebras, to determine the integers $c(G/K)$ for all Riemannian symmetric spaces. For this purpose we have only to determine the integers $c(G/K)$ in the case G/K is a simply connected irreducible Riemannian symmetric space of compact type (see Lemma 1.3).

Now our main result (Theorem 1.4) is stated as follows: Let G/K be a simply connected irreducible Riemannian symmetric space of compact type.

(1) If G/K is not isomorphic to any real Grassmann manifold, then $c(G/K) = (1/2)(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)$.

(2) If G/K is isomorphic to $SO(p+q)/SO(p) \times SO(q)$ $(p \geqq q \geqq 1)$, then

$$c(G/K) = \begin{cases} [pq/2]\,, & \text{if } \ q = \text{even or } \ 2q \geqq p \geqq q\,, \ \ q = \text{odd}\,, \\ p(q-1)/2 + q\,, & \text{if } \ p > 2q \ \text{ and } \ q = \text{odd}\,. \end{cases}$$

It is remarkable that the real Grassmann manifolds $SO(p+q)/SO(p) \times SO(q)$ with $p \geqq 2q + 2$ and $q = \text{odd}$ form an exceptional class among irreducible Riemannian symmetric spaces of compact type. By this theorem we know that most of the irreducible Riemannian symmetric spaces M cannot be isometrically or conformally immersed into the Euclidean spaces in codimension $\sim(1/2) \dim M$ even locally.

Unfortunately our estimates obtained above are not best possible in general. For example, it is known that the spaces of negative constant curvature M of dimension $n\ (\geqq 2)$ cannot be isometrically immersed into $\boldsymbol{R}^{2n-2}$ even locally (Ôtsuki [21]). On the other hand, from the above theorem we have $c(M) = 1$. Hence for $n \geqq 3$ our estimate on local isometric immersions is not best possible. For the Riemannian symmetric space $M = SO(5)$, the integer $c(M)$ is 4 and hence $SO(5)$ cannot be locally isometrically immersed into $\boldsymbol{R}^{13}$. But using a more delicate method, it can be proved that $SO(5)$ cannot be locally isometrically immersed into $\boldsymbol{R}^{15}$. (This result is best possible because $SO(5)$ is locally isomorphic to $Sp(2)$ and it is already known that $Sp(2)$ can be globally isometrically imbedded into $\boldsymbol{R}^{16}$ (Kobayashi [14]). For other examples, see Agaoka

[1].) However our estimates for the spaces $SO(2m+1)$, $U(2m+1)$ and $SU(2m+1)/SO(2m+1)$ are better than Heitsch, Lawson and Donnelly's. In fact for these spaces the integers $c(G/K)$ are quadratic polynomials of m and hence $c(G/K)-3 \gg 2m-1$ for large m.

We now explain the contents of this paper. In § 1 after showing the modified Gauss equation for conformal immersions, we prove Proposition 1.2 and state the main theorem (Theorem 1.4). In order to prove Theorem 1.4, we have to look for elements $X, Y \in \mathfrak{m}^c$ such that the map $\rho^c([X, Y]): \mathfrak{m}^c \to \mathfrak{m}^c$ takes a maximum rank, where $\mathfrak{m}^c$ (resp. ρ^c) is the complexification of $\mathfrak{m}$ (resp. ρ) (see § 3). For this purpose we prepare in § 2 several propositions concerning the root systems of compact irreducible Riemannian symmetric spaces. In particular a subset $\Gamma = \{\beta_1, \cdots, \beta_s\}$ ($s = \operatorname{rank} G/K - \operatorname{rank} G + \operatorname{rank} K$) of positive roots of $\mathfrak{g}^c$ satisfying certain conditions plays a fundamental role (Proposition 2.2). We prove this proposition in Appendix 1. Using the results in § 2, we prove Theorem 1.4. But we have to divide the proof into several cases according as the property of G/K. In § 3 we prove the theorem for "general" compact irreducible Riemannian symmetric spaces, which satisfy certain conditions on root systems. Many spaces are included in this case. The spaces which are not "general" are listed up in Proposition 3.4 and we have to prove the theorem individually. Sections 4~6 are devoted to the proof for these spaces. In Appendix 1 we give the proof of Proposition 2.2, using the classification of symmetric spaces. Finally in Appendix 2 we prove, as an application of the modified Gauss equation for conformal immersions, that an n-dimensional compact Riemannian manifold with non-positive sectional curvature cannot be globally conformally immersed into $\boldsymbol{R}^{2n-2}$. This is a generalization of the result of Moore in [20].

Throughout this paper we always assume the differentiability of class C^∞.

1. The rank of the curvature transformations and the main theorem.

1.1. Let (M, g) be an n-dimensional Riemannian manifold. We denote by ∇ the covariant differentiation associated with the Levi-Civita connection of (M, g) and by R the curvature tensor field of (M, g). For a C^∞ function ϕ on M and tangent vectors $X_1, \cdots, X_k \in T_pM$ we define $\nabla_{X_1} \cdots \nabla_{X_k}\phi \in \boldsymbol{R}$ by $\nabla_{X_1} \cdots \nabla_{X_k}\phi = (\overbrace{\nabla \cdots \nabla}^{k}\phi)(X_1, \cdots, X_k)$ where $\overbrace{\nabla \cdots \nabla}^{k}\phi$ is the k-th covariant derivative of ϕ. Let $\boldsymbol{f} = (f^1, \cdots, f^m)$ be a mapping of M into the m-dimensional Euclidean space $\boldsymbol{R}^m$ and we set $\nabla_{X_1} \cdots \nabla_{X_k}\boldsymbol{f} = (\cdots, \nabla_{X_1} \cdots \nabla_{X_k} f^i, \cdots)$. Then we have:

$$\nabla_X\nabla_Y\boldsymbol{f} = \nabla_Y\nabla_X\boldsymbol{f}\,, \qquad \nabla_X\nabla_Y\nabla_Z\boldsymbol{f} = \nabla_X\nabla_Z\nabla_Y\boldsymbol{f}\,,$$
$$\nabla_X\nabla_Y\nabla_Z\boldsymbol{f} = \nabla_Y\nabla_X\nabla_Z\boldsymbol{f} - \nabla_{R(X,Y)Z}\boldsymbol{f}\,, \tag{1.1}$$

where X, Y, Z denote tangent vectors at $p\in M$.

Let $\langle\ ,\ \rangle$ be the standard inner product of $\boldsymbol{R}^m$. If $\boldsymbol{f}$ is an immersion of M into $\boldsymbol{R}^m$, then $\langle\nabla\boldsymbol{f}, \nabla\boldsymbol{f}\rangle$ gives a Riemannian metric on M. By definition an immersion is called *conformal* with respect to g if there exists a function ρ on M satisfying $\langle\nabla\boldsymbol{f}, \nabla\boldsymbol{f}\rangle = e^{2\rho}g$.

Let $\boldsymbol{f}$ be a conformal immersion of (M, g) into $\boldsymbol{R}^m$. We define an $\boldsymbol{R}^m$-valued symmetric 2-tensor field $\boldsymbol{\alpha}$ and a symmetric 2-tensor field β by setting

$$\boldsymbol{\alpha}(X, Y) = \nabla_X\nabla_Y\boldsymbol{f} - \{(\nabla_X\rho)\nabla_Y\boldsymbol{f} + (\nabla_Y\rho)\nabla_X\boldsymbol{f} - g(X, Y)\nabla_\xi\boldsymbol{f}\}\,;$$
$$\beta(X, Y) = e^{2\rho}\left\{\nabla_X\nabla_Y\rho - (\nabla_X\rho)(\nabla_Y\rho) + \frac{1}{2}g(\xi, \xi)g(X, Y)\right\},$$

where ξ stands for the dual of $\nabla\rho$, i.e., the vector field determined by the equality $g(\xi, X) = \nabla_X\rho$. Then we have the following

LEMMA 1.1 (cf. Gasqui [7]).

$$\langle\boldsymbol{\alpha}(X, Y), \nabla_Z\boldsymbol{f}\rangle = 0\,. \tag{1.2}$$

$$\begin{aligned}\langle\boldsymbol{\alpha}(X, Y), \boldsymbol{\alpha}(W, Z)\rangle &- \langle\boldsymbol{\alpha}(X, Z), \boldsymbol{\alpha}(W, Y)\rangle + \beta(X, Y)g(W, Z)\\ &+ g(X, Y)\beta(W, Z) - \beta(X, Z)g(W, Y) - g(X, Z)\beta(W, Y)\\ &= -e^{2\rho}g(R(X, W)Y, Z)\,.\end{aligned} \tag{1.3}$$

PROOF. Differentiating covariantly the equation $\langle\nabla\boldsymbol{f}, \nabla\boldsymbol{f}\rangle = e^{2\rho}g$, we have

$$\langle\nabla_X\nabla_Y\boldsymbol{f}, \nabla_Z\boldsymbol{f}\rangle + \langle\nabla_Y\boldsymbol{f}, \nabla_X\nabla_Z\boldsymbol{f}\rangle = 2(\nabla_X\rho)e^{2\rho}g(Y, Z)\,.$$

Then cyclic permutation of $\{X, Y, Z\}$ yields

$$\langle\nabla_X\nabla_Y\boldsymbol{f}, \nabla_Z\boldsymbol{f}\rangle = e^{2\rho}\{(\nabla_X\rho)g(Y, Z) + (\nabla_Y\rho)g(Z, X) - (\nabla_Z\rho)g(X, Y)\}$$

and (1.2) follows from this equation.

Next we differentiate the equality (1.2). Then we have

$$\langle\nabla_W\boldsymbol{\alpha}(X, Y), \nabla_Z\boldsymbol{f}\rangle + \langle\boldsymbol{\alpha}(X, Y), \nabla_W\nabla_Z\boldsymbol{f}\rangle = 0\,.$$

Interchanging X and W, we have

$$\langle\nabla_X\boldsymbol{\alpha}(W, Y), \nabla_Z\boldsymbol{f}\rangle + \langle\boldsymbol{\alpha}(W, Y), \nabla_X\nabla_Z\boldsymbol{f}\rangle = 0\,.$$

From these two equalities we obtain

$$\begin{aligned}\langle\boldsymbol{\alpha}(X, Y), \boldsymbol{\alpha}(W, Z)\rangle &- \langle\boldsymbol{\alpha}(X, Z), \boldsymbol{\alpha}(W, Y)\rangle\\ &= \langle\nabla_X\boldsymbol{\alpha}(W, Y) - \nabla_W\boldsymbol{\alpha}(X, Y), \nabla_Z\boldsymbol{f}\rangle\,.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\nabla_X\boldsymbol{\alpha}(W, Y) - \nabla_W\boldsymbol{\alpha}(X, Y) &= \nabla_X\nabla_W\nabla_Y\boldsymbol{f} - \nabla_W\nabla_X\nabla_Y\boldsymbol{f} - \{(\nabla_X\nabla_Y\rho)\nabla_W\boldsymbol{f} - (\nabla_W\nabla_Y\rho)\nabla_X\boldsymbol{f}\} \\ &\quad + \{g(W, Y)\nabla_{\nabla_X\xi}\boldsymbol{f} - g(X, Y)\nabla_{\nabla_W\xi}\boldsymbol{f}\} - \{(\nabla_W\rho)\nabla_X\nabla_Y\boldsymbol{f} - (\nabla_X\rho)\nabla_W\nabla_Y\boldsymbol{f}\} \\ &\quad + \{g(W, Y)\nabla_X\nabla_\xi\boldsymbol{f} - g(X, Y)\nabla_W\nabla_\xi\boldsymbol{f}\} .\end{aligned}$$

Then by use of (1.2) and the integrability condition (1.1) we have

$$\begin{aligned}\langle\nabla_X\boldsymbol{\alpha}(W, Y) - \nabla_W\boldsymbol{\alpha}(X, Y), \nabla_Z\boldsymbol{f}\rangle &= -e^{2\rho}g(R(X, W)Y, Z) - \beta(X, Y)g(W, Z) - g(X, Y)\beta(W, Z) \\ &\quad + \beta(X, Z)g(W, Y) + g(X, Z)\beta(W, Y) .\end{aligned}$$

This proves the equality (1.3). q.e.d.

We call (1.3) the *modified Gauss equation* for conformal immersions.

REMARK. If $\boldsymbol{f}$ is an isometric immersion, i.e., ρ is identically zero on M, then $\boldsymbol{\alpha}$ is nothing but the usual second fundamental form of $\boldsymbol{f}$. We also remark that in this case $\beta = 0$ and the equality (1.3) reduces to the usual Gauss equation.

1.2. Let r be the $\boldsymbol{Z}$-valued function on M defined in Introduction. Using Lemma 1.1, we prove the following proposition.

PROPOSITION 1.2. *If (M, g) can be isometrically immersed into $\boldsymbol{R}^m$, then for each point $p \in M$ the inequality $r(p) \leqq m - n$ holds. If (M, g) can be conformally immersed into $\boldsymbol{R}^m$, then $r(p) \leqq m - n + 2$ for each $p \in M$. In particular any open Riemannian submanifold of (M, g) containing p cannot be isometrically* (resp. *conformally*) *immersed into the Euclidean space in codimension $r(p) - 1$* (resp. $r(p) - 3$).

PROOF. Suppose that there exists a conformal immersion $\boldsymbol{f}: M \to \boldsymbol{R}^m$. Let $\boldsymbol{\alpha}$ and β be the symmetric tensor fields on M defined above and we denote by $T_p^\perp M$ the normal space to M at $p \in M$. For each $\xi \in T_p^\perp M$ we define a symmetric endomorphism A_ξ of T_pM by $g(A_\xi(X), Y) = \langle\boldsymbol{\alpha}(X, Y), \xi\rangle (X, Y \in T_pM)$ and let B be a symmetric endomorphism of T_pM defined by $g(B(X), Y) = \beta(X, Y)$. Then the modified Gauss equation (1.3) can be expressed in the form

$$\begin{aligned}(1.4)\qquad e^{2\rho}R(X, Y)Z = A_{\boldsymbol{\alpha}(Y,Z)}X - A_{\boldsymbol{\alpha}(X,Z)}Y - \beta(X, Z)\cdot Y \\ -g(X, Z)\cdot B(Y) + g(Y, Z)\cdot B(X) + \beta(Y, Z)\cdot X .\end{aligned}$$

Hence for all $X, Y \in T_pM$ we have

$$\begin{aligned}\operatorname{rank} R(X, Y) &\leqq \dim\{A_{\boldsymbol{\alpha}(X,Z)}Y \mid Z \in T_pM\} + \dim\{A_{\boldsymbol{\alpha}(Y,Z)}X \mid Z \in T_pM\} + 4 \\ &\leqq 2\dim T_p^\perp M + 4 ,\end{aligned}$$

which implies $r(p) \leqq \dim T_p^{\perp}M + 2 = m - n + 2$. In case f is an isometric immersion, we substitute $\rho = \beta = 0$ into the above equality (1.4). Then we have

$$\operatorname{rank} R(X, Y) \leqq 2 \dim T_p^{\perp}M$$

for all $X, Y \in T_pM$, implying $r(p) \leqq m - n$. q.e.d.

REMARK. The "isometric" part of this proposition is essentially equivalent to Theorem 1 in Matsumoto [17].

Now we consider the case where (M, g) is a Riemannian symmetric space G/K. In this case the function $r(p)$ is constant on G/K, as is stated in Introduction, and we denote this constant by $c(G/K) \in \boldsymbol{Z}$. Using the elementary facts on the curvature transformation of G/K at the origin (see [15]), we have

LEMMA 1.3. (1) *Let* $M = M_1 \times \cdots \times M_k$ *be a product of Riemannian symmetric spaces. Then* $c(M) = \sum_{i=1}^{k} c(M_i)$.

(2) *Let* M *be a Riemannian symmetric space of compact type and let* M^* *be its non-compact dual space. Then* $c(M^*) = c(M)$.

We determine the number $c(G/K)$ for each simply connected irreducible Riemannian symmetric space G/K of compact type. Then by Lemma 1.3 and the fact $c(\boldsymbol{R}^n) = 0$, we know the value $c(G/K)$ for all Riemannian symmetric spaces G/K. (We remark that the integer $c(G/K)$ is determined by the infinitesimal property of G/K.)

The rest of this paper is devoted to the proof of the following main theorem.

THEOREM 1.4. *Let* $M = G/K$ *be a simply connected irreducible Riemannian symmetric space of compact type. If* G/K *is not isomorphic to any real Grassmann manifold, then*

$$c(G/K) = (1/2)\cdot(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)\ .$$

For real Grassmann manifolds $G/K = SO(p+q)/SO(p) \times SO(q)$ $(p \geqq q \geqq 1)$,

$$c(G/K) = \begin{cases} [pq/2]\ , & \textit{if} \quad q = \text{even} \quad \textit{or} \quad 2q \geqq p \geqq q\ , \quad q = \text{odd}\ , \\ p(q-1)/2 + q\ , & \textit{if} \quad p > 2q \quad \textit{and} \quad q = \text{odd}\ , \end{cases}$$

where [ ] *is the Gauss symbol.*

We remark that the equality $c(G/K) = (1/2)(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)$ holds except for the case $G/K = SO(p+q)/SO(p) \times SO(q)$ with $p \geqq 2q + 2$ and $q = \text{odd}$, which includes the standard sphere S^n $(n \geqq 4)$.

2. Riemannian symmetric spaces. In order to prove Theorem 1.4, we prepare in this section several propositions concerning irreducible Riemannian symmetric spaces of compact type.

2.1. Let G/K be an irreducible Riemannian symmetric space of compact type and let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). We denote by θ the involutive automorphism of G associated with G/K. We also denote by the same letter θ the involutive automorphism of $\mathfrak{g}$ induced by θ. Let us define an inner product $(\ ,\ )$ of $\mathfrak{g}$ by setting $(X, Y) = -B(X, Y)$, $X, Y \in \mathfrak{g}$, where B stands for the Killing form of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ obtained by θ. Then $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{m}$ and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. We put $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$. Then we have $\mathfrak{t} = \mathfrak{a} + \mathfrak{b}$ (orthogonal direct sum). In particular we have $\theta\mathfrak{t} = \mathfrak{t}$. In the following discussions we fix a θ-order "$<$" in $\mathfrak{t}$, i.e., a linear order in $\mathfrak{t}$ satisfying: If $H > 0$, $H \notin \mathfrak{b}$, then $\theta H < 0$. We denote by $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$. We extend θ and $\mathrm{Ad}(g)$ $(g \in G)$ to complex linear isomorphisms of $\mathfrak{g}^c$ by complex linearity and denote them by the same letters.

Let $\alpha \in \mathfrak{t}$. We define a subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}^c$ by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^c \mid [H, X] = \sqrt{-1}(\alpha, H)X \text{ for all } H \in \mathfrak{t}\} .$$

An element $\alpha \in \mathfrak{t}$ is called a *root* of $\mathfrak{g}^c$ (with respect to $\mathfrak{t}^c$) if $\mathfrak{g}_\alpha \neq \{0\}$. Let Δ (resp. Δ^+) denote the set of non-zero roots (resp. positive roots) of $\mathfrak{g}^c$. Clearly we have $\theta\Delta = \Delta$ and $\theta\mathfrak{g}_\alpha = \mathfrak{g}_{\theta\alpha}$ for each $\alpha \in \Delta$.

Let τ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. Then it can be easily verified that $\tau\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$. The following is easy to prove.

PROPOSITION 2.1. *There exists a set of vectors* $\{Z_\alpha \in \mathfrak{g}_\alpha \mid \alpha \in \Delta\}$ *satisfying*

(1) $\tau Z_\alpha = Z_{-\alpha}$, $\theta Z_\alpha = Z_{\theta\alpha}$;

(2) $[Z_\alpha, Z_{-\alpha}] = 2\sqrt{-1}/(\alpha, \alpha)\cdot\alpha$.

We fix once for all such a set of vectors $\{Z_\alpha \in \mathfrak{g}_\alpha \mid \alpha \in \Delta\}$ stated in Proposition 2.1.

Let us define a non-negative integer $s(G/K)$ by setting $s(G/K) = \operatorname{rank} G/K - \operatorname{rank} G + \operatorname{rank} K$. Then the following proposition plays an important role in the proof of Theorem 1.4. The proof will be given in Appendix 1.

PROPOSITION 2.2. *Assume that* $s = s(G/K) > 0$. *Then there exists a subset* $\Gamma = \{\beta_1, \cdots, \beta_s\}$ *of* Δ^+ *satisfying*:

(1) $\theta\beta_i = -\beta_i$, *i.e.*, $\beta_i \in \Delta^+ \cap \mathfrak{a}$;

(2) $\beta_i \pm \beta_j \notin \Delta \cup \{0\}$ $(i \neq j)$.

REMARK. (1) The set $\Gamma = \{\beta_1, \cdots, \beta_s\}$ having the above properties is not uniquely determined. In Appendix 1 we explicitly construct $\Gamma = \{\beta_1, \cdots, \beta_s\}$ for each G/K, using the classification of Riemannian symmetric spaces.

(2) The spaces satisfying $s(G/K) = 0$, i.e., the spaces of split rank, are exhausted by the following; the compact simple Lie groups, ***AII*** $SU(2n)/Sp(n)$, ***DII*** $SO(2n+2)/SO(2n+1)$ and ***EIV*** E_6/F_4. These spaces are exceptional in our point of view and we have to prove Theorem 1.4 for these spaces individually (§ 4 and § 6).

Let $\Gamma = \{\beta_1, \cdots, \beta_s\}$ be a subset of Δ^+ with the properties stated in Proposition 2.2. Then since the length of the β_i-series containing β_j $(i \neq j)$ is 1, we have $(\beta_i, \beta_j) = 0$ for $i \neq j$. For each $\beta_i \in \Gamma$ we denote by X_{β_i} and Y_{β_i} the real part and the imaginary part of Z_{β_i}, respectively. Then:

$$X_{\beta_i} = (1/2)(Z_{\beta_i} + \tau Z_{\beta_i}) = (1/2)(Z_{\beta_i} + Z_{-\beta_i}) = (1/2)(Z_{\beta_i} + \theta Z_{\beta_i}) \in \mathfrak{k} \; ;$$
$$Y_{\beta_i} = (1/2\sqrt{-1})(Z_{\beta_i} - \tau Z_{\beta_i}) = (1/2\sqrt{-1})(Z_{\beta_i} - Z_{-\beta_i})$$
$$= (1/2\sqrt{-1})(Z_{\beta_i} - \theta Z_{\beta_i}) \in \mathfrak{m} \, .$$

We set $\mathfrak{a}_0 = \sum_{i=1}^{s} \boldsymbol{R}\beta_i$, $\mathfrak{b}_0 = \sum_{i=1}^{s} \boldsymbol{R}X_{\beta_i}$. Then we have $\mathfrak{a}_0 \subset \mathfrak{a}$, $\mathfrak{b}_0 \subset \mathfrak{k}$ and it is easily observed that $\dim \mathfrak{a}_0 = \dim \mathfrak{b}_0 = s$. Let $\mathfrak{a}_1$ be the orthogonal complement of $\mathfrak{a}_0$ in $\mathfrak{a}$. Then we have $\dim \mathfrak{a}_1 = \dim \mathfrak{a} - \dim \mathfrak{a}_0 = \operatorname{rank} G/K - s = \operatorname{rank} G - \operatorname{rank} K$. With the above notations, we prove

PROPOSITION 2.3. (1) *$\mathfrak{b}_0$ is orthogonal to $\mathfrak{b}$ and $\mathfrak{b}_1 = \mathfrak{b}_0 + \mathfrak{b}$ (orthogonal direct sum) is a Cartan subalgebra of $\mathfrak{k}$.*

(2) *$\mathfrak{t}_1 = \mathfrak{a}_1 + \mathfrak{b}_1$ (orthogonal direct sum) is a Cartan subalgebra of $\mathfrak{g}$.*

PROOF. (1) Since $(\mathfrak{t}^c, \mathfrak{g}_\alpha) = 0$ for $\alpha \in \Delta$, it follows that $(\mathfrak{b}_0, \mathfrak{b}) = 0$. Consequently we have $\dim \mathfrak{b}_1 = \dim \mathfrak{b}_0 + \dim \mathfrak{b} = s + (\operatorname{rank} G - \operatorname{rank} G/K) = \operatorname{rank} K$. Hence in order to show that $\mathfrak{b}_1$ is a Cartan subalgebra of $\mathfrak{k}$, we have only to show that $\mathfrak{b}_1$ is abelian. This can be verified by the following equalities:

$$[H, X_{\beta_i}] = (1/2)[H, Z_{\beta_i} + Z_{-\beta_i}] = (\sqrt{-1}/2)(\beta_i, H)(Z_{\beta_i} - Z_{-\beta_i}) = 0 \, ,$$
$$[X_{\beta_i}, X_{\beta_j}] = (1/4)[Z_{\beta_i} + Z_{-\beta_i}, Z_{\beta_j} + Z_{-\beta_j}] = 0 \, ,$$

where $H \in \mathfrak{b}$, $\beta_i, \beta_j \in \Gamma$. (Note that $\beta_i \in \Delta \cap \mathfrak{a}$ and $\beta_i \pm \beta_j \notin \Delta$ for $i \neq j$.) Hence we have $[\mathfrak{b}_1, \mathfrak{b}_1] = \{0\}$, proving that $\mathfrak{b}_1$ is a Cartan subalgebra of $\mathfrak{k}$.

(2) Clearly we have $[\mathfrak{a}_1, \mathfrak{a}_1] = [\mathfrak{a}_1, \mathfrak{b}] = [\mathfrak{b}_1, \mathfrak{b}_1] = \{0\}$. We also obtain $[\mathfrak{a}_1, \mathfrak{b}_0] = \{0\}$, because $[H, X_{\beta_i}] = (\sqrt{-1}(\beta_i, H)/2)(Z_{\beta_i} - Z_{-\beta_i}) = 0$, where $H \in \mathfrak{a}_1$ and $\beta_i \in \Gamma$. Therefore we have $[\mathfrak{t}_1, \mathfrak{t}_1] = \{0\}$. Since $(\mathfrak{a}_1, \mathfrak{b}_1) = 0$, it follows that $\dim \mathfrak{t}_1 = \dim \mathfrak{a}_1 + \dim \mathfrak{b}_1 = \operatorname{rank} G - \operatorname{rank} K + \operatorname{rank} K = \operatorname{rank} G$.

This proves that $\mathfrak{t}_1$ is a Cartan subalgebra of $\mathfrak{g}$. q.e.d.

REMARK. Let $\Gamma' = \{\beta'_1, \cdots, \beta'_t\}$ be a subset of Δ^+ satisfying the properties (1) and (2) in Proposition 2.2. Then by Proposition 2.3 (1) it can be easily proved that $t \leqq s(G/K)$, i.e., $s(G/K)$ is the largest integer possessing the property in Proposition 2.2.

2.2. We now set $W = \sum_{i=1}^{s} Y_{\beta_i}$ $(\in \mathfrak{m})$ and $g = \exp(-\pi/2 \cdot W)$ $(\in G)$. Then we have

PROPOSITION 2.4. *Let* $H \in \mathfrak{t}$. *Then*:

$$\mathrm{Ad}(g) \cdot H = H + \sum_{i=1}^{s} (\beta_i, H) X_{\beta_i} - \sum_{i=1}^{s} (\beta_i, H)/(\beta_i, \beta_i) \cdot \beta_i .$$

Consequently $\mathrm{Ad}(g) \cdot \mathfrak{a}_0 = \mathfrak{b}_0$, $\mathrm{Ad}(g)|_{\mathfrak{a}_1 + \mathfrak{b}} = id$ *and* $\mathrm{Ad}(g) \cdot \mathfrak{t} = \mathfrak{t}_1$.

Before proceeding to the proof we show

LEMMA 2.5. *For each* $H \in \mathfrak{t}$, $\beta_i \in \Gamma$, *it holds*

$$\mathrm{Ad}(\exp tY_{\beta_i}) \cdot H = H - (\sin t)(\beta_i, H) X_{\beta_i} + (\cos t - 1) \cdot (\beta_i, H)/(\beta_i, \beta_i) \cdot \beta_i \quad (t \in \boldsymbol{R}) .$$

PROOF. By Proposition 2.1, we obtain the following equalities:

$$[Y_{\beta_i}, H] = -(\beta_i, H) X_{\beta_i} \quad H \in \mathfrak{t} ;$$
$$[Y_{\beta_i}, X_{\beta_i}] = 1/(\beta_i, \beta_i) \cdot \beta_i .$$

Hence by induction on n, we can easily prove

$$(\mathrm{ad}\, Y_{\beta_i})^{2n+1} \cdot H = (-1)^{n+1} (\beta_i, H) X_{\beta_i} ,$$
$$(\mathrm{ad}\, Y_{\beta_i})^{2n+2} \cdot H = (-1)^{n+1} (\beta_i, H)/(\beta_i, \beta_i) \cdot \beta_i .$$

Therefore we have

$$\begin{aligned} \mathrm{Ad}(\exp tY_{\beta_i}) \cdot H &= H + \sum_{n=1}^{\infty} t^n/n! \cdot (\mathrm{ad}\, Y_{\beta_i})^n \cdot H \\ &= H + (\beta_i, H) \cdot \sum_{n=0}^{\infty} (-1)^{n+1} t^{2n+1}/(2n+1)! \cdot X_{\beta_i} \\ &\quad + ((\beta_i, H)/(\beta_i, \beta_i)) \cdot \sum_{n=0}^{\infty} (-1)^{n+1} t^{2n+2}/(2n+2)! \cdot \beta_i . \end{aligned}$$

Since $\sum_{n=0}^{\infty} (-1)^n t^{2n+1}/(2n+1)! = \sin t$ and $\sum_{n=0}^{\infty} (-1)^{n+1} t^{2n+2}/(2n+2)! = \cos t - 1$, we obtain the desired equality. q.e.d.

PROOF OF PROPOSITION 2.4. We first remark the following equalities: $[Y_{\beta_i}, \beta_j] = [Y_{\beta_i}, X_{\beta_j}] = [Y_{\beta_i}, Y_{\beta_j}] = 0$ for $i \neq j$. Hence we have $\mathrm{Ad}(\exp tW) = \mathrm{Ad}(\exp tY_{\beta_1}) \cdots \mathrm{Ad}(\exp tY_{\beta_s})$ and $\mathrm{Ad}(\exp tY_{\beta_i}) \cdot \beta_j = \beta_j$, $\mathrm{Ad}(\exp tY_{\beta_i}) \cdot X_{\beta_j} = X_{\beta_j}$ for $i \neq j$. Thus by Lemma 2.5, we obtain

$$\mathrm{Ad}(\exp tW)\cdot H = H - (\sin t)\cdot\sum_{i=1}^{s}(\beta_i, H)X_{\beta_i}$$
$$+ (\cos t - 1)\cdot\sum_{i=1}^{s}(\beta_i, H)/(\beta_i, \beta_i)\cdot\beta_i \quad (t\in \boldsymbol{R}) .$$

Putting $t = -\pi/2$ into the above equality, we have the desired equality.

We now prove the latter part of the proposition. In view of the formulas obtained above, we can easily check that $\mathrm{Ad}(g)|_{\mathfrak{a}_1+\mathfrak{b}} = id$. Also we have $\mathrm{Ad}(g)\cdot\beta_i = (\beta_i, \beta_i)X_{\beta_i}$ for $\beta_i\in\Gamma$, because $(\beta_i, \beta_j) = 0$ for $i\neq j$. Hence $\mathrm{Ad}(g)\cdot\mathfrak{a}_0 = \mathfrak{b}_0$. Consequently we have $\mathrm{Ad}(g)\cdot\mathfrak{t} = \mathfrak{t}_1$ and the proof is completed. q.e.d.

Let $\alpha\in\Delta$. We set $\tilde{\alpha} = \mathrm{Ad}(g)\cdot\alpha$ $(\in\mathfrak{t}_1)$ and $\tilde{Z}_\alpha = \mathrm{Ad}(g)\cdot Z_\alpha$ $(\in\mathfrak{g}^c)$. Then it is easily seen that $\tilde{\alpha}$ is a non-zero root of $\mathfrak{g}^c$ with respect to $\mathfrak{t}_1^c$ and that $\boldsymbol{C}\tilde{Z}_\alpha$ is the root space of $\mathfrak{g}^c$ corresponding to $\tilde{\alpha}$. (We remark that $\theta\mathfrak{t}_1 = \mathfrak{t}_1$.) We put $\Delta_\sharp = \Delta\cap(\mathfrak{a}_0+\mathfrak{b})$. Then we have $\theta\tilde{\alpha} = \tilde{\alpha}$ if and only if $\alpha\in\Delta_\sharp$.

Let $\alpha\in\Delta_\sharp$. Since $\theta\tilde{\alpha} = \tilde{\alpha}$, it follows that $\theta\tilde{Z}_\alpha\in\boldsymbol{C}\tilde{Z}_\alpha$. Hence there exists a complex number ε_α such that $\theta\tilde{Z}_\alpha = \varepsilon_\alpha\tilde{Z}_\alpha$. Since $\theta^2 = id$, we have $\varepsilon_\alpha = \pm 1$.

LEMMA 2.6. (1) *Let* $\alpha\in\Delta_\sharp$. *Then* $\varepsilon_\alpha = \varepsilon_{-\alpha}$.
(2) *Let* $\alpha, \beta\in\Delta_\sharp$. *Then if* $\alpha+\beta\in\Delta_\sharp$, $\varepsilon_{\alpha+\beta} = \varepsilon_\alpha\cdot\varepsilon_\beta$.

PROOF. (1) Applying θ on both sides of the equality $[\tilde{Z}_\alpha, \tilde{Z}_{-\alpha}] = 2\sqrt{-1}/(\alpha, \alpha)\cdot\tilde{\alpha}$, we obtain $\varepsilon_\alpha\cdot\varepsilon_{-\alpha} = 1$. This proves (1). The assertion (2) is obvious. q.e.d.

Now we set $\Delta_\sharp(+) = \{\alpha\in\Delta_\sharp \mid \varepsilon_\alpha = 1\}$ and $\Delta_\sharp(-) = \{\alpha\in\Delta_\sharp \mid \varepsilon_\alpha = -1\}$. Then we have

$$\mathfrak{k}^c = \mathfrak{b}_1^c + \sum_{\alpha\in\Delta_\sharp(+)}\boldsymbol{C}\tilde{Z}_\alpha + \sum_{\alpha\in\Delta^+\backslash\Delta_\sharp}\boldsymbol{C}(\tilde{Z}_\alpha + \theta\tilde{Z}_\alpha) \quad (\text{direct sum}) ,$$
$$\mathfrak{m}^c = \mathfrak{a}_1^c + \sum_{\alpha\in\Delta_\sharp(-)}\boldsymbol{C}\tilde{Z}_\alpha + \sum_{\alpha\in\Delta^+\backslash\Delta_\sharp}\boldsymbol{C}(\tilde{Z}_\alpha - \theta\tilde{Z}_\alpha) \quad (\text{direct sum}) .$$

Let $H\in\mathfrak{a}_0^c+\mathfrak{b}^c$. We define two subsets $\kappa_1(H)$ and $\kappa_2(H)$ of Δ by setting $\kappa_1(H) = \{\alpha\in\Delta_\sharp(-) \mid (\alpha, H) = 0\}$ and $\kappa_2(H) = \{\alpha\in\Delta^+\backslash\Delta_\sharp \mid (\alpha, H) = 0\}$.

LEMMA 2.7. *For* $H\in\mathfrak{a}_0^c+\mathfrak{b}^c$, *set* $\tilde{H} = \mathrm{Ad}(g)\cdot H\in\mathfrak{b}_1^c$. *Then the following equality holds*:

$$\dim_{\boldsymbol{C}}\mathrm{Ker}(\mathrm{ad}\,\tilde{H}|_{\mathfrak{m}^c}) = \mathrm{rank}\,G - \mathrm{rank}\,K + \sharp\kappa_1(H) + \sharp\kappa_2(H) ,$$

where $\sharp\kappa_i(H)$ *denotes the cardinality of* $\kappa_i(H)$.

PROOF. We first note that $\mathrm{ad}\,\tilde{H}\cdot\mathfrak{a}_1^c = \{0\}$, $\mathrm{ad}\,\tilde{H}\cdot\tilde{Z}_\alpha = \sqrt{-1}(\alpha, H)\tilde{Z}_\alpha$ and $\mathrm{ad}\,\tilde{H}\cdot\theta\tilde{Z}_\alpha = \theta(\mathrm{ad}\,\theta\tilde{H}\cdot\tilde{Z}_\alpha) = \theta(\mathrm{ad}\,\tilde{H}\cdot\tilde{Z}_\alpha) = \sqrt{-1}(\alpha, H)\theta\tilde{Z}_\alpha$ for $\alpha\in\Delta$.

Hence we know that each factor that appears in the decomposition of $\mathfrak{m}^c$ obtained above is invariant under $(\mathrm{ad}\,\tilde{H}|_{\mathfrak{m}^c})$. Therefore we have:

$$\mathrm{Ker}\,(\mathrm{ad}\,\tilde{H}|_{\mathfrak{m}^c}) = \mathfrak{a}_1^c + \sum_{\alpha\in\kappa_1(H)} C\tilde{Z}_\alpha + \sum_{\alpha\in\kappa_2(H)} C(\tilde{Z}_\alpha - \theta\tilde{Z}_\alpha)\,.$$

Consequently we have $\dim_C \mathrm{Ker}\,(\mathrm{ad}\,\tilde{H}|_{\mathfrak{m}^c}) = \dim_C \mathfrak{a}_1^c + \#\kappa_1(H) + \#\kappa_2(H) = \mathrm{rank}\,G - \mathrm{rank}\,K + \#\kappa_1(H) + \#\kappa_2(H)$. q.e.d.

2.3. Finally we study the sets $\Delta_\#(+)$ and $\Delta_\#(-)$ more closely. We show

PROPOSITION 2.8. (1) $\pm\Gamma = \{\pm\beta_1, \cdots, \pm\beta_s\} \subset \Delta_\#(-)$.

(2) *Assume that* $\alpha\in\Delta\cap\mathfrak{b}$. *Then* $\alpha\in\Delta_\#(-)$ *if and only if* $\alpha\pm\beta_i\in\Delta$ *for some* $\beta_i\in\Gamma$.

We first prove

LEMMA 2.9. (1) *Assume that* $\alpha\in\Delta_\#$ *and that there exists* $\beta_i\in\Gamma$ *such that* $2(\alpha,\beta_i)/(\beta_i,\beta_i) = 1$ *and* $\alpha+\beta_i\notin\Delta$. *Then*:

$$\mathrm{Ad}\,(\exp tY_{\beta_i})\cdot Z_\alpha = \cos(t/2)\cdot Z_\alpha + 2\sin(t/2)[Y_{\beta_i}, Z_\alpha]\,;$$
$$\mathrm{Ad}\,(\exp tY_{\beta_i})\cdot Z_{-\alpha} = \cos(t/2)\cdot Z_{-\alpha} + 2\sin(t/2)[Y_{\beta_i}, Z_{-\alpha}] \quad (t\in \boldsymbol{R})\,.$$

(2) *Assume that* $\alpha\in\Delta\cap\mathfrak{b}$ *and that there exists* $\beta_i\in\Gamma$ *such that* $\alpha\pm\beta_i\in\Delta$. *Then*:

$$\mathrm{Ad}\,(\exp tY_{\beta_i})\cdot Z_\alpha = \cos t\cdot Z_\alpha + \sin t\cdot[Y_{\beta_i}, Z_\alpha]\,, \quad (t\in\boldsymbol{R})\,.$$

PROOF. (1) Under the assumptions we know that $\alpha-\beta_i\in\Delta\cup\{0\}$ and $\alpha-2\beta_i\notin\Delta\cup\{0\}$. Hence we have

$$\begin{aligned}(\mathrm{ad}\,Y_{\beta_i})^2\cdot Z_\alpha &= (1/4)[Z_{\beta_i}, [Z_{-\beta_i}, Z_\alpha]] = (1/4)[[Z_{\beta_i}, Z_{-\beta_i}], Z_\alpha] \\ &= (-1/4)2(\alpha,\beta_i)/(\beta_i,\beta_i)\cdot Z_\alpha = -(1/4)Z_\alpha\,.\end{aligned}$$

Therefore by induction on n ($\in\boldsymbol{Z}$, $\geqq 0$), we can easily prove that $(\mathrm{ad}\,Y_{\beta_i})^{2n}\cdot Z_\alpha = (-1/4)^n\cdot Z_\alpha$ and $(\mathrm{ad}\,Y_{\beta_i})^{2n+1}\cdot Z_\alpha = (-1/4)^n\cdot[Y_{\beta_i}, Z_\alpha]$. Consequently

$$\begin{aligned}\mathrm{Ad}\,(\exp tY_{\beta_i})\cdot Z_\alpha &= \Big(\sum_{n=0}^{\infty}(-1)^n(t/2)^{2n}/(2n)!\Big)\cdot Z_\alpha \\ &\quad + 2\Big(\sum_{n=0}^{\infty}(-1)^n(t/2)^{2n+1}/(2n+1)!\Big)\cdot[Y_{\beta_i}, Z_\alpha] \\ &= \cos(t/2)\cdot Z_\alpha + 2\sin(t/2)\cdot[Y_{\beta_i}, Z_\alpha]\,.\end{aligned}$$

This proves the first equality. In the same manner the second equality can also be proved.

(2) We first note that since $\alpha\in\mathfrak{b}$ and $\beta_i\in\Gamma\subset\mathfrak{a}_0$, we have $(\alpha,\beta_i)=0$. Hence we know that $\alpha\pm2\beta_i\notin\Delta\cup\{0\}$. In fact if $\alpha+2\beta_i\in\Delta\cup\{0\}$ (resp.

$\alpha - 2\beta_i \in \Delta \cup \{0\}$), then it follows that $\alpha - 2\beta_i \in \Delta \cup \{0\}$ (resp. $\alpha + 2\beta_i \in \Delta \cup \{0\}$), because $(\alpha, \beta_i) = 0$. Consequently the β_i-series of α contains at least five roots. But this contradicts the fact that for any $\gamma, \delta \in \Delta$, the length of the δ-series of γ is at most four (see Bourbaki [4]). Thus $\alpha \pm 2\beta_i \notin \Delta \cup \{0\}$. Hence we obtain the following:

$$[Z_{-\beta_i}, [Z_{\beta_i}, Z_\alpha]] = [Z_{\beta_i}, [Z_{-\beta_i}, Z_\alpha]] = -2Z_\alpha ,$$

(see Helgason [9] Chap. III). Therefore

$$(\mathrm{ad}\, Y_{\beta_i})^2 \cdot Z_\alpha = (1/4)\{[Z_{-\beta_i}, [Z_{\beta_i}, Z_\alpha]] + [Z_{\beta_i}, [Z_{-\beta_i}, Z_\alpha]]\} = -Z_\alpha .$$

Hence by induction on n, we can easily show that $(\mathrm{ad}\, Y_{\beta_i})^{2n} \cdot Z_\alpha = (-1)^n Z_\alpha$ and $(\mathrm{ad}\, Y_{\beta_i})^{2n+1} \cdot Z_\alpha = (-1)^n [Y_{\beta_i}, Z_\alpha]$. Consequently we have

$$\begin{aligned} \mathrm{Ad}\,(\exp t Y_{\beta_i}) \cdot Z_\alpha &= \Big(\sum_{n=0}^{\infty} (-1)^n t^{2n}/(2n)!\Big) \cdot Z_\alpha \\ &\quad + \Big(\sum_{n=0}^{\infty} (-1)^n t^{2n+1}/(2n+1)!\Big) \cdot [Y_{\beta_i}, Z_\alpha] \\ &= \cos t \cdot Z_\alpha + \sin t \cdot [Y_{\beta_i}, Z_\alpha] . \end{aligned}$$

Thus the proof of the lemma is completed. q.e.d.

LEMMA 2.10. *Let* $\alpha \in \Delta_\sharp$. *Then* $\mathrm{Ad}\,(g^{-2}) \cdot \theta Z_\alpha = \varepsilon_\alpha \cdot Z_\alpha$.

PROOF. We note that since $W = \sum_{i=1}^{s} Y_{\beta_i} \in \mathfrak{m}$, it follows that $\theta W = -W$. Hence $\theta(g) = \theta(\exp(-\pi/2 \cdot W)) = \exp(-\pi/2 \cdot \theta W) = \exp(\pi/2 \cdot W) = g^{-1}$. Thus $\theta \tilde{Z}_\alpha = \theta(\mathrm{Ad}\,(g) \cdot Z_\alpha) = \mathrm{Ad}\,(\theta(g)) \cdot \theta Z_\alpha = \mathrm{Ad}\,(g^{-1}) \cdot \theta Z_\alpha = \mathrm{Ad}\,(g) \cdot \mathrm{Ad}\,(g^{-2}) \cdot \theta Z_\alpha$. On the other hand since $\theta \tilde{Z}_\alpha = \varepsilon_\alpha \cdot \tilde{Z}_\alpha = \varepsilon_\alpha\, \mathrm{Ad}\,(g) \cdot Z_\alpha$, we obtain $\mathrm{Ad}\,(g^{-2}) \cdot \theta Z_\alpha = \varepsilon_\alpha Z_\alpha$. q.e.d.

PROOF OF PROPOSITION 2.8. (1) By Lemma 2.10 we obtain $\mathrm{Ad}\,(g^{-2}) \cdot (X_{\beta_i} - \sqrt{-1}\, Y_{\beta_i}) = \varepsilon_{\beta_i}(X_{\beta_i} + \sqrt{-1}\, Y_{\beta_i})$. Comparing the imaginary parts of both sides, we have $\mathrm{Ad}\,(g^{-2}) \cdot Y_{\beta_i} = -\varepsilon_{\beta_i} Y_{\beta_i}$. On the other hand since $[Y_{\beta_i}, Y_{\beta_j}] = 0$ for $\beta_i, \beta_j \in \Gamma$, it follows that $\mathrm{Ad}\,(g^{-2}) \cdot Y_{\beta_i} = Y_{\beta_i}$. This proves $\varepsilon_{\beta_i} = -1$. Hence we have $\pm\beta_i \in \Delta_\sharp(-)$ (see Lemma 2.6).

(2) We first assume that $\alpha \pm \beta_i \in \Delta$ for some $\beta_i \in \Gamma$. Then for any $\beta_j \in \Gamma$ $(j \neq i)$ it holds that $\alpha \pm \beta_j \notin \Delta$. In fact if $\alpha + \beta_j \in \Delta$, then $(\alpha + \beta_i) - (\alpha + \beta_j) = \beta_i - \beta_j \in \Delta \cup \{0\}$, because $(\alpha + \beta_i, \alpha + \beta_j) = (\alpha, \alpha) > 0$. This is a contradiction. $\alpha - \beta_j \notin \Delta$ can be analogously proved. Hence we have $\mathrm{Ad}\,(\exp t Y_{\beta_j}) \cdot Z_\alpha = Z_\alpha$ for any $\beta_j \in \Gamma$ $(j \neq i)$. Therefore $\mathrm{Ad}\,(g^{-2}) \cdot \theta Z_\alpha = \mathrm{Ad}\,(\exp \pi W) \cdot Z_{\theta\alpha} = \mathrm{Ad}\,(\exp \pi Y_{\beta_1}) \cdots \mathrm{Ad}\,(\exp \pi Y_{\beta_s}) \cdot Z_\alpha = \mathrm{Ad}\,(\exp \pi Y_{\beta_i}) \cdot Z_\alpha = -Z_\alpha$ (see Lemma 2.9 (2)). Then by Lemma 2.10 we have $\varepsilon_\alpha = -1$, i.e., $\alpha \in \Delta_\sharp(-)$. Conversely we suppose that $\alpha \pm \beta_i \notin \Delta$ for any $\beta_i \in \Gamma$. (We remark that $\alpha + \beta_i \notin \Delta$ if and only if $\alpha - \beta_i \notin \Delta$ because $(\alpha, \beta_i) = 0$.) Then it is clear that $\mathrm{Ad}\,(\exp t Y_{\beta_i}) \cdot Z_\alpha = Z_\alpha$ for any

$\beta_i \in \Gamma$ and hence $\mathrm{Ad}(g^{-2})\cdot\theta Z_\alpha = Z_\alpha$. This shows that $\varepsilon_\alpha = 1$, implying $\alpha \in \Delta_\sharp(+)$. q.e.d.

REMARK. (1) If $s(G/K) = 0$, then $\Delta_\sharp = \Delta \cap \mathfrak{b}$ because $\mathfrak{a}_0 = 0$. Consequently by Proposition 2.8 (2) we have $\Delta_\sharp(-) = \varnothing$.

(2) As is easily observed that Proposition 2.1 also holds even if we replace Z_{β_i} and $Z_{-\beta_i}$ by $-Z_{\beta_i}$ and $-Z_{-\beta_i}$, respectively for some $\beta_i \in \Gamma$. Such a modification brings about a change in the sign of Y_{β_i} and hence alters the sets $\Delta_\sharp(+)$ and $\Delta_\sharp(-)$. (See § 5 and § 6. We remark that the union $\Delta_\sharp = \Delta_\sharp(+) \cup \Delta_\sharp(-)$ is unchanged by this modification.) However it should be noted that under such a change in the sign of Y_{β_i} Proposition 2.8 remains to be true. The proof of this fact is left to the reader.

3. Proof of Theorem 1.4. (General case). In this section we determine the integers $c(G/K)$ for many G/K that are "general" in our sense.

Let G/K be a Riemannian symmetric space of compact type and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$. In a usual way $\mathfrak{m}$ can be identified with the tangent space at the origin $o \in G/K$. We denote by $\rho: \mathfrak{k} \to \mathfrak{o}(\mathfrak{m})$ the linear isotropy representation of G/K at $o \in G/K$. Then as is well known, the curvature transformation $R(X, Y)$ ($X, Y \in \mathfrak{m}$) at the origin $o \in G/K$ is given by $R(X, Y) = -\rho([X, Y]): \mathfrak{m} \to \mathfrak{m}$ (see Kobayashi and Nomizu [15]). Hence we have

$$c(G/K) = (1/2) \max_{X,Y \in \mathfrak{m}} \mathrm{rank}\, \rho([X, Y]) .$$

The following lemma is easy to verify.

LEMMA 3.1. *Let $\rho^c: \mathfrak{k}^c \to \mathfrak{o}(\mathfrak{m}^c)$ be the complexification of the linear isotropy representation ρ. Then*

$$c(G/K) = (1/2) \max_{X,Y \in \mathfrak{m}^c} \mathrm{rank}\, \rho^c([X, Y]) .$$

By this lemma we may consider the problem in the complex category.

Now we define a non-negative integer $c_0(G/K)$ by $c_0(G/K) = (1/2)(\dim G/K - \mathrm{rank}\, G + \mathrm{rank}\, K)$. We first prove

LEMMA 3.2. $c(G/K) \leqq c_0(G/K)$.

PROOF. Let X, Y be arbitrary elements of $\mathfrak{m}$. Since $[X, Y] \in \mathfrak{k}$ and since $\mathfrak{b}_1$ is a Cartan subalgebra of $\mathfrak{k}$, there exists an element $k \in K$ such that $\mathrm{Ad}(k)\cdot[X, Y] \in \mathfrak{b}_1$. We set $\tilde{H} = \mathrm{Ad}(k)\cdot[X, Y]$ and $H = \mathrm{Ad}(g^{-1})\cdot\tilde{H}$. Then by Lemma 2.7, we obtain $\mathrm{rank}\, \rho([X, Y]) = \mathrm{rank}\, \rho(\tilde{H}) = \dim G/K - \dim_C \mathrm{Ker}(\mathrm{ad}\, \tilde{H}|_{\mathfrak{m}^c}) \leqq \dim G/K - \mathrm{rank}\, G + \mathrm{rank}\, K = 2c_0(G/K)$. Hence we have $c(G/K) = (1/2) \max_{X,Y \in \mathfrak{m}} \mathrm{rank}\, \rho([X, Y]) \leqq c_0(G/K)$. q.e.d.

PROPOSITION 3.3. *Let $\Gamma = \{\beta_1, \cdots, \beta_s\}$ be a subset of Δ^+ satisfying* *the conditions in Proposition* 2.2 *and assume that the set* Γ *satisfies the following two conditions*:

(1) *For each* $\alpha \in \Delta \backslash \Delta_{\sharp}$ *there exists* $\beta_i \in \Gamma$ *such that* $(\alpha, \beta_i) \neq 0$.

(2) *For each* $\alpha \in \Delta \cap \mathfrak{b}$, $\beta_i \in \Gamma$ *it holds* $\alpha \pm \beta_i \notin \Delta$.

Then $c(G/K) = c_0(G/K)$.

PROOF. Let $a_1, \cdots, a_s$ be complex numbers that are linearly independent over the field $\boldsymbol{Q}$ of rational numbers. We set $X = \sum_{i=1}^{s} a_i Z_{\beta_i}$, $Y = \sum_{i=1}^{s} Z_{-\beta_i}$ and $H = [X, Y]$. Since $\pm\beta_i \in \Delta_{\sharp}(-)$ (Proposition 2.8, Lemma 2.6), it follows that $\tilde{X} = \mathrm{Ad}(g)\cdot X \in \mathfrak{m}^c$, $\tilde{Y} = \mathrm{Ad}(g)\cdot Y \in \mathfrak{m}^c$ and $\tilde{H} = \mathrm{Ad}(g)\cdot H = [\tilde{X}, \tilde{Y}]$. By a simple calculation we obtain $H = \sum_{i=1}^{s} 2\sqrt{-1}a_i/(\beta_i, \beta_i)\cdot\beta_i \in \mathfrak{a}_0^c$ and hence $\tilde{H} \in \mathfrak{b}_0^c \subset \mathfrak{b}_1^c$. Therefore $c(G/K) \geqq (1/2)\,\mathrm{rank}_{\boldsymbol{C}}\,\rho^c([\tilde{X}, \tilde{Y}]) = (1/2)\,\mathrm{rank}_{\boldsymbol{C}}\,\rho^c(\tilde{H}) = (1/2)\,\mathrm{rank}_{\boldsymbol{C}}\,(\mathrm{ad}\,\tilde{H}|_{\mathfrak{m}^c}) = c_0(G/K) - (1/2)(\sharp\kappa_1(H) + \sharp\kappa_2(H))$ (Lemma 2.7).

We now show that $\kappa_1(H) = \kappa_2(H) = \emptyset$, then we have $c(G/K) \geqq c_0(G/K)$. This together with Lemma 3.2 proves the proposition. Let $\alpha \in \Delta$ satisfy $(\alpha, H) = 0$. Then since $a_1, \cdots, a_s$ are linearly independent over $\boldsymbol{Q}$ and since $2(\alpha, \beta_i)/(\beta_i, \beta_i) \in \boldsymbol{Z}$, it follows that $(\alpha, \beta_i) = 0$ for all $\beta_i \in \Gamma$. Thus by the condition (1), we know that $\alpha \notin \Delta\backslash\Delta_{\sharp}$, i.e., $\alpha \in \Delta_{\sharp} \subset \mathfrak{a}_0 + \mathfrak{b}$. This means that $\kappa_2(H) = \emptyset$. We next show that $\alpha \in \Delta \cap \mathfrak{b}$. In fact since $\Gamma = \{\beta_1, \cdots, \beta_s\}$ forms a basis of $\mathfrak{a}_0$, the $\mathfrak{a}_0$-component of α is equal to 0. Hence $\alpha \in \Delta \cap \mathfrak{b}$. Therefore by the condition (2) and by Proposition 2.8, we know that $\alpha \notin \Delta_{\sharp}(-)$. This implies that $\kappa_1(H) = \emptyset$. Thus the proof is completed. q.e.d.

The conditions in Proposition 3.3 are satisfied for many Riemannian symmetric spaces. In fact we have

PROPOSITION 3.4. *Let* G/K *be a simply connected irreducible Riemannian symmetric space of compact type, which is not isomorphic to any of the following spaces*:

(1°) *Compact simple Lie groups*, ***AII*** $SU(2(n+1))/Sp(n+1)$ $(n \geqq 1)$, ***DI, II*** $SO(p+q)/SO(p)\times SO(q)$ $(p, q = \text{odd},\ p \geqq q+2)$, ***EIV*** E_6/F_4.

(2°) ***BI, IIS*** $O(p+q)/SO(p)\times SO(q)$ $(p = \text{even},\ q = \text{odd},\ p \geqq q+3)$, ***CII*** $Sp(p+q)/Sp(p)\times Sp(q)$ $(p \geqq q \geqq 1)$, ***FII*** $F_4/\mathrm{Spin}(9)$.

Then the set $\Gamma = \{\beta_1, \cdots, \beta_s\}$ *selected in Appendix* 1 *satisfies the conditions* (1) *and* (2) *in Proposition* 3.3. *In particular* $c(G/K) = c_0(G/K)$.

PROOF. We note that since G/K is not isomorphic to any spaces listed in (1°), we have $s(G/K) \neq 0$ (see Table 1 in Appendix 1). First we suppose that $s(G/K) = \mathrm{rank}\,G/K$. Then we have $\mathrm{rank}\,G = \mathrm{rank}\,K$ and hence $\mathfrak{a}_1 = \{0\}$. Therefore the condition (1) of Proposition 3.3 is automatically satisfied. We next consider the case where $s(G/K) <$

rank G/K. By Table 1 in Appendix 1, we know that such spaces are limited to the following three types:

$$\textit{AI}\ SU(n+1)/SO(n+1) \quad (n \geqq 1)\ ;$$
$$\textit{DI}\ SO(p+q)/SO(p)\times SO(q) \quad (p=q=\text{odd})\ ;$$
$$\textit{EI}\ E_6/Sp(4)\ .$$

Then we have rank $G =$ rank G/K and hence $\mathfrak{t}=\mathfrak{a}$. Consequently we have $\theta\alpha=-\alpha$ for each $\alpha\in\Delta$. Now let us suppose there exists $\alpha\in\Delta^+$ satisfying $(\alpha, \beta_i)=0$ for any $\beta_i\in\Gamma$. Then it follows that $\alpha\pm\beta_i\notin\Delta\cup\{0\}$. (It is known that for a simple Lie algebra $\mathfrak{g}^c$ of type $[A_l]$, $[D_l]$ or $[E_l]$ the α-series containing three roots are of the form $-\alpha, 0, \alpha$.) Then the set $\Gamma'=\{\alpha\}\cup\Gamma$ satisfies the conditions of Proposition 2.2. But this contradicts the maximality of the set Γ (see Remark after Proposition 2.3). This shows the property (1) of Proposition 3.3.

For the verification of the property (2), see Remark at the end of Appendix 1. We know that since G/K is not isomorphic to any spaces listed in (2°), $\alpha\pm\beta_i\notin\Delta\cup\{0\}$ for any $\alpha\in\Delta\cap\mathfrak{b}$, $\beta_i\in\Gamma$. q.e.d.

We remark that any Hermitian symmetric space is not contained in neither (1°) nor (2°) and hence the proof of Theorem 1.4 is completed for Hermitian symmetric spaces.

In the subsequent sections we prove Theorem 1.4 for each G/K listed in (1°) and (2°) of Proposition 3.4.

4. Proof of Theorem 1.4. (Compact simple Lie groups, *AII* and *EIV*). In this section we treat the cases $G/K=$ compact simple Lie groups, *AII* $SU(2(n+1))/Sp(n+1)$ $(n\geqq 2)$ and *EIV* E_6/F_4. We remark that for these spaces $s(G/K)=0$ and hence X and Y which we defined in the proof of Proposition 3.3 reduce to 0. We use the same notations as in § 2 and § 3.

4.1. Compact simple Lie groups. Let M^* be a compact simple Lie group with a biinvariant Riemannian metric and let $\mathfrak{m}^*$ be the Lie algebra of M^*. Then as is well known that M^* may be represented by the Riemannian symmetric space G/K, where $G=M^*\times M^*$, $K=\{(x, x)\,|\,x\in M^*\}$ and the involution θ of G is given by $\theta(x, y)=(y, x)$ for $(x, y)\in G$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). Then we have $\mathfrak{g}=\mathfrak{m}^*\oplus\mathfrak{m}^*$, $\mathfrak{k}=\{(X, X)\in\mathfrak{g}\,|\,X\in\mathfrak{m}^*\}$. The differential of θ at the identity of G, denoted by the same letter θ, is given by $\theta(X, Y)=(Y, X)$ for $(X, Y)\in\mathfrak{m}^*\oplus\mathfrak{m}^*=\mathfrak{g}$. We define an inner product $(\ ,\ )$ of $\mathfrak{g}$ by $((X_1, Y_1), (X_2, Y_2))=-\{B(X_1, X_2)+B(Y_1, Y_2)\}$ for $(X_i, Y_i)\in\mathfrak{g}$ $(i=1, 2)$. As is easily observed, $(\ ,\ )$ is invariant under θ and the orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to (,) is given by $\mathfrak{m} = \{(X, -X) \in \mathfrak{g} \mid X \in \mathfrak{m}^*\}$. Let $\mathfrak{a}^*$ be a Cartan subalgebra of $\mathfrak{m}^*$ and set $\mathfrak{t} = \mathfrak{a}^* \oplus \mathfrak{a}^*$. Then we know that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ containing a maximal abelian subspace $\mathfrak{a} = \{(X, -X) \in \mathfrak{g} \mid X \in \mathfrak{a}^*\}$ of $\mathfrak{m}$ and $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$ is given by $\mathfrak{b} = \{(X, X) \in \mathfrak{g} \mid X \in \mathfrak{a}^*\}$. We define a linear order "$<$" in $\mathfrak{t}$ by the following law: $(H_1, H_2) > 0$ if and only if $H_1 \gg H_2$ or $H_1 = H_2 \gg 0$, where $\ll$ denotes an arbitrary linear order in $\mathfrak{a}^*$. Then it is easy to see that $<$ is a θ-order in $\mathfrak{t}$. Let Δ^* (resp. Δ) be the set of non-zero roots of $(\mathfrak{m}^*)^c$ (resp. $\mathfrak{g}^c$) with respect to $(\mathfrak{a}^*)^c$ (resp. $\mathfrak{t}^c$). Then we have $\Delta = \{\alpha^+, \alpha^- \mid \alpha \in \Delta^*\}$, where $\alpha^+ = (\alpha, 0)$, $\alpha^- = (0, \alpha)$. Clearly we have $\theta\alpha^+ = \alpha^-$ and $\theta\alpha^- = \alpha^+$ for $\alpha \in \Delta^*$.

PROOF OF THEOREM 1.4. Let $\Pi^* = \{\alpha_1, \cdots, \alpha_l\}$ ($l = \operatorname{rank} \mathfrak{m}^*$) denote the set of simple roots of $(\mathfrak{m}^*)^c$ with respect to $\ll$.

Let $a_1, \cdots, a_l$ be complex numbers that are linearly independent over $\boldsymbol{Q}$. We then set $X = \sum_{i=1}^{l} a_i(Z_{\alpha_i^+} - \theta Z_{\alpha_i^+})$, $Y = \sum_{i=1}^{l} (Z_{-\alpha_i^+} - \theta Z_{-\alpha_i^+})$ and set $H = [X, Y]$. Then we have $X \in \mathfrak{m}^c$, $Y \in \mathfrak{m}^c$ and by a simple calculation we obtain $H = 2\sqrt{-1} \sum_{i=1}^{l} a_i/(\alpha_i^+, \alpha_i^+) \cdot (\alpha_i^+ + \theta\alpha_i^+) \in \mathfrak{b}^c$. (Note that $\alpha_i^+ - \alpha_j^+, \alpha_i^- - \alpha_j^- \notin \Delta \cup \{0\}$ $(i \neq j)$, $\alpha_i^+ \pm \alpha_j^- \notin \Delta \cup \{0\}$.) By Lemma 2.7 we have $\operatorname{rank}_{\boldsymbol{C}} \rho^c(H) = \operatorname{rank}_{\boldsymbol{C}} (\operatorname{ad} H)|_{\mathfrak{m}^c} = 2c_0(G/K) - (\sharp\kappa_1(H) + \sharp\kappa_2(H))$. (Note that since $s(G/K) = 0$, it follows that $g = e$ and hence $\operatorname{Ad}(g) \cdot H = H$, $\operatorname{Ad}(g) \cdot Z_\alpha = Z_\alpha$.) We now show that $\kappa_1(H) = \kappa_2(H) = \emptyset$, then it holds $c(G/K) \geqq c_0(G/K)$. This together with Lemma 3.2 proves $c(G/K) = c_0(G/K)$. Let $\alpha^+ = (\alpha, 0) \in \Delta$ satisfy $(\alpha^+, H) = 0$. Since $a_1, \cdots, a_l$ are linearly independent over $\boldsymbol{Q}$ and since $2(\alpha^+, \alpha_i^+)/(\alpha_i^+, \alpha_i^+) \in \boldsymbol{Z}$, $(\alpha^+, \theta\alpha_i^+) = 0$, it follows that $(\alpha^+, \alpha_i^+) = -B(\alpha, \alpha_i) = 0$ for all $\alpha_i \in \Pi^*$. But it is impossible because $\{\alpha_i\}$ forms a basis of $\mathfrak{a}^*$ and hence we have $(\alpha^+, H) \neq 0$. Similarly we can prove that $(\alpha^-, H) \neq 0$ for all $\alpha^- \in \Delta$. Hence we have $\kappa_1(H) = \kappa_2(H) = \emptyset$. q.e.d.

REMARK. Since $c(\boldsymbol{R}^n) = 0$, it can be easily seen that the equality $c(M^*) = (1/2)(\dim M^* - \operatorname{rank} M^*)$ holds for any compact Lie group M^* with a biinvariant Riemannian metric.

4.2. *AII* $SU(2(n+1))/Sp(n+1)$ $(n \geqq 2)$. In the following arguments we assume that $n \geqq 2$. In the case $n = 1$, $G/K = SU(4)/Sp(2)$ is isomorphic to $SO(6)/SO(5)$ and we treat this case in § 6.

Let $\Pi = \{\alpha_1, \cdots, \alpha_{2n+1}\}$ denote the simple roots of $\mathfrak{su}(2(n+1))^c$. Then the Satake diagram of G/K and the restriction of θ on $\mathfrak{t}$ are given as follows:

$$\begin{cases}\theta\alpha_{2i} = -(\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}) & (1 \leqq i \leqq n)\,, \\ \theta\alpha_{2i-1} = \alpha_{2i-1} & (1 \leqq i \leqq n+1)\,. \end{cases}$$

We first prove

LEMMA 4.1. *Let* $\alpha \in \Delta$. *Then for some* i $(1 \leqq i \leqq n)$ *it holds* $(\alpha, \alpha_{2i} + \theta\alpha_{2i}) \neq 0$.

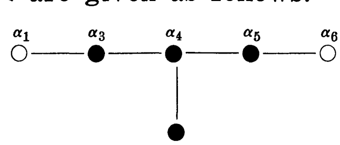
PROOF. As is well known that there exists a basis $\{\lambda_1, \cdots, \lambda_{2n+2}\}$ of $\mathfrak{t} \oplus \boldsymbol{R}$ such that $(\lambda_i, \lambda_j) = c\delta_{ij}$ $(c \in \boldsymbol{R}\backslash\{0\})$, $\mathfrak{t} = \{\sum_{i=1}^{2n+2} a_i\lambda_i \mid \sum_{i=1}^{2n+2} a_i = 0\}$ and $\alpha_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq 2n+1)$ (Bourbaki [4]).

Let α be written in the form $\alpha = \pm(\lambda_p - \lambda_{q+1})$ $(1 \leqq p \leqq q \leqq 2n+1)$. Since $\alpha_{2i} + \theta\alpha_{2i} = -(\alpha_{2i-1} + \alpha_{2i+1}) = -(\lambda_{2i-1} - \lambda_{2i} + \lambda_{2i+1} - \lambda_{2i+2})$, it holds $(\alpha, \alpha_p + \theta\alpha_p) \neq 0$ in case $p =$ even, $q \neq p+1$. We have $(\alpha, \alpha_{p-2} + \theta\alpha_{p-2}) \neq 0$ or $(\alpha, \alpha_{p+2} + \theta\alpha_{p+2}) \neq 0$ in case $p =$ even, $q = p+1$. (Note that we are assuming $n \geqq 2$.) Similarly we have $(\alpha, \alpha_{p-1} + \theta\alpha_{p-1}) \neq 0$ in case $p =$ odd > 1 and $(\alpha, \alpha_2 + \theta\alpha_2) \neq 0$ in case $p = 1$, $q \neq p+1$. Finally we have $(\alpha, \alpha_4 + \theta\alpha_4) \neq 0$ in case $p = 1$, $q = 2$, proving the lemma. q.e.d.

We remark that Lemma 4.1 does not hold if $n = 1$.

PROOF OF THEOREM 1.4. Let $a_1, \cdots, a_n$ be complex numbers that are linearly independent over $\boldsymbol{Q}$. We set $X = \sum_{i=1}^n a_i(Z_{(-1)^{i-1}\alpha_{2i}} - \theta Z_{(-1)^{i-1}\alpha_{2i}})$, $Y = \sum_{i=1}^n (Z_{(-1)^i\alpha_{2i}} - \theta Z_{(-1)^i\alpha_{2i}})$ and $H = [X, Y]$. Then we have $X \in \mathfrak{m}^c$, $Y \in \mathfrak{m}^c$. Since $\alpha_{2i} \pm \alpha_{2j} \notin \Delta \cup \{0\}$ $(i \neq j)$, $\alpha_{2i} \pm \theta\alpha_{2j} \notin \Delta \cup \{0\}$ $(i \neq j-1, j+1)$, $\alpha_{2i} + \theta\alpha_{2i+2}$, $\alpha_{2i} + \theta\alpha_{2i-2} \notin \Delta \cup \{0\}$, we obtain $H = 2\sqrt{-1}\sum_{i=1}^n (-1)^{i-1}a_i/(\alpha_{2i}, \alpha_{2i})\cdot(\alpha_{2i} + \theta\alpha_{2i}) \in \mathfrak{h}^c$. Let $\alpha \in \Delta$ satisfy $(\alpha, H) = 0$. Then since $a_1, \cdots, a_n$ are linearly independent over $\boldsymbol{Q}$ and since $2(\alpha, \alpha_{2i} + \theta\alpha_{2i})/(\alpha_{2i}, \alpha_{2i}) \in \boldsymbol{Z}$, we have $(\alpha, \alpha_{2i} + \theta\alpha_{2i}) = 0$ for all α_{2i} $(1 \leqq i \leqq n)$. But it is impossible (Lemma 4.1). Hence we have $\kappa_1(H) = \kappa_2(H) = \emptyset$. Therefore by Lemma 2.7 we obtain $\operatorname{rank}_C \rho^c(H) = \operatorname{rank}_C (\operatorname{ad} H)|_{\mathfrak{m}^c} = c_0(G/K)$. This together with Lemma 3.2 shows that $c(G/K) = c_0(G/K)$. q.e.d.

4.3. ***EIV*** E_6/F_4. Let $\Pi = \{\alpha_1, \cdots, \alpha_6\}$ be the set of simple roots of $\mathfrak{e}_6^c$ with respect to a θ-order in $\mathfrak{t}$. Then the Satake diagram of G/K and the restriction of θ on $\mathfrak{t}$ are given as follows:



$$\begin{cases}\theta\alpha_1 = -(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)\,, \\ \theta\alpha_6 = -(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)\,, \\ \theta\alpha_i = \alpha_i \quad (2 \leqq i \leqq 5)\,. \end{cases}$$

We first prove

LEMMA 4.2. *Let α be a non-zero root satisfying* $(\alpha, \alpha_1 + \theta\alpha_1) = (\alpha, \alpha_6 + \theta\alpha_6) = 0$. *Then* $\alpha \in \Delta \cap \mathfrak{b}$.

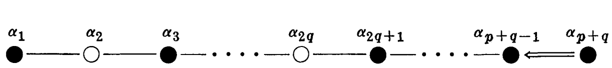
PROOF. By the Dynkin diagram of $\mathfrak{e}_6^c$, we know that $(\alpha_i, \alpha_i) = 2c$ $(1 \leqq i \leqq 6)$, $(\alpha_1, \alpha_3) = (\alpha_3, \alpha_4) = (\alpha_2, \alpha_4) = (\alpha_4, \alpha_5) = (\alpha_5, \alpha_6) = -c$ and otherwise $(\alpha_i, \alpha_j) = 0$, where c is a positive constant. We express $\alpha = \sum_{i=1}^{6} m_i\alpha_i$. Then since $\alpha_1 + \theta\alpha_1 = -(\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5)$ and $\alpha_6 + \theta\alpha_6 = -(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)$, we obtain $2m_1 - 2m_3 + m_6 = m_1 - 2m_5 + 2m_6 = 0$ from $(\alpha, \alpha_1 + \theta\alpha_1) = (\alpha, \alpha_6 + \theta\alpha_6) = 0$. Hence we know that $m_1 =$ even, $m_6 =$ even. On the other hand, since the highest root of $\mathfrak{e}_6^c$ is $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, it follows that $|m_1| \leqq 1$, $|m_6| \leqq 1$. Therefore we have $m_1 = m_6 = 0$ and hence $\alpha \in \Delta \cap \mathfrak{b}$. q.e.d.

PROOF OF THEOREM 1.4. Let a_1, a_2 be two non-zero complex numbers such that $a_1/a_2 \notin \boldsymbol{Q}$. We set $X = a_1(Z_{\alpha_1} - \theta Z_{\alpha_1}) + a_2(Z_{-\alpha_6} - \theta Z_{-\alpha_6})$ and $Y = (Z_{-\alpha_1} - \theta Z_{-\alpha_1}) + (Z_{\alpha_6} - \theta Z_{\alpha_6})$ and set $H = [X, Y]$. Then we have $X \in \mathfrak{m}^c$, $Y \in \mathfrak{m}^c$. Since $\alpha_1 - \theta\alpha_1$, $\alpha_1 + \alpha_6$, $\alpha_1 + \theta\alpha_6$, $\alpha_6 - \theta\alpha_6 \notin \Delta \cup \{0\}$, we have $H = 2\sqrt{-1}\{a_1/(\alpha_1, \alpha_1)\cdot(\alpha_1 + \theta\alpha_1) - a_2/(\alpha_6, \alpha_6)\cdot(\alpha_6 + \theta\alpha_6)\} \in \mathfrak{b}^c$. Let $\alpha \in \Delta$ satisfy $(\alpha, H) = 0$. Since $2(\alpha, \alpha_i + \theta\alpha_i)/(\alpha_i, \alpha_i) \in \boldsymbol{Z}$ $(i = 1, 6)$, the equality $(\alpha, H) = 0$ implies $(\alpha, \alpha_1 + \theta\alpha_1) = (\alpha, \alpha_6 + \theta\alpha_6) = 0$. Then by Lemma 4.2, we have $\alpha \in \Delta \cap \mathfrak{b} = \Delta_\sharp$, which implies $\kappa_2(H) = \emptyset$. (Note that $\mathfrak{a}_0 = \{0\}$ in this case.) By Remark (1) at the end of § 2, we have $\Delta_\sharp(-) = \emptyset$ and hence $\kappa_1(H) = \emptyset$. Therefore by Lemma 2.7 we obtain $\operatorname{rank}_C \rho^c(H) = \operatorname{rank}_C (\operatorname{ad} H)|_{\mathfrak{m}^c} = 2c_0(G/K)$. This together with Lemma 3.2 proves $c(G/K) = c_0(G/K)$. q.e.d.

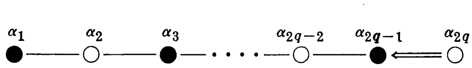
5. Proof of Theorem 1.4. (*CII* and *FII*).

5.1. *CII* $Sp(p + q)/Sp(p)\times Sp(q)$ $(p \geqq q \geqq 1)$. In the following arguments we assume that $(p, q) \neq (1, 1)$. In the case $(p, q) = (1, 1)$, $G/K = Sp(2)/Sp(1) \times Sp(1)$ is isomorphic to $SO(5)/SO(4)$ and we will treat this case in § 6.

Let $\Pi = \{\alpha_1, \cdots, \alpha_{p+q}\}$ denote the set of simple roots of $\mathfrak{sp}(p + q)^c$ with respect to a θ-order in $\mathfrak{t}$. Then the Satake diagram of G/K and the restriction of θ to $\mathfrak{t}$ are given as follows:



$(p \geqq q + 1)$,



$(p = q)$;

$$\begin{cases} \theta\alpha_{2i} = -(\alpha_{2i-1} + \alpha_{2i} + \alpha_{2i+1}) \quad (1 \leqq i \leqq q-1) , \\ \theta\alpha_{2q} = \begin{cases} -(\alpha_{2q-1} + \alpha_{2q} + 2\sum_{k=2q+1}^{p+q-1} \alpha_k + \alpha_{p+q}) \quad (p \geqq q+1) , \\ -(2\alpha_{2q-1} + \alpha_{2q}) \quad (p = q) , \end{cases} \\ \theta\alpha_i = \alpha_i \quad (i = \text{odd} < 2q \text{ or } i > 2q) . \end{cases}$$

Then the set $\Gamma = \{\beta_1, \cdots, \beta_q\}$ $(s(G/K) = q)$ which we define in Appendix 1 is given by $\beta_i = \alpha_{2i-1} + 2\sum_{k=2i}^{p+q-1} \alpha_k + \alpha_{p+q}$ $(1 \leqq i \leqq q-1)$ and $\beta_q = \alpha_{2q-1} + 2\sum_{k=2q}^{p+q-1} \alpha_k + \alpha_{p+q}$ if $p \geqq q+1$, $\beta_q = \alpha_{2q-1} + \alpha_{2q}$ if $p = q$. (In fact in this case the set of simple roots $\{\alpha_1, \cdots, \alpha_{p+q}\}$ of $\mathfrak{sp}(p+q)^c$ and the set of simple roots $\{\gamma_1, \cdots, \gamma_q\}$ of the reduced root system Σ_* are related by $\gamma_i = (1/2)(\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1})$ $(1 \leqq i \leqq q-1)$ and $\gamma_q = \alpha_{2q-1} + 2\sum_{k=2q}^{p+q-1}\alpha_k + \alpha_{p+q}$.) As is well known there exists a basis $\{\lambda_1, \cdots, \lambda_{p+q}\}$ of $\mathfrak{t}$ such that $(\lambda_i, \lambda_j) = c\delta_{ij}$ $(c \in \boldsymbol{R}\backslash\{0\})$ and $\alpha_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq p+q-1)$ and $\alpha_{p+q} = 2\lambda_{p+q}$ (Bourbaki [4]). By utilizing this basis, we have:

$$\Delta = \{\pm(\lambda_i - \lambda_j) \ (1 \leqq i < j \leqq p+q), \ \pm(\lambda_i + \lambda_j) \ (1 \leqq i \leqq j \leqq p+q)\} ,$$

$$\begin{cases} \theta\lambda_{2i-1} = -\lambda_{2i} , \quad \theta\lambda_{2i} = -\lambda_{2i-1} \quad (1 \leqq i \leqq q) , \\ \theta\lambda_i = \lambda_i \quad (2q+1 \leqq i \leqq p+q) ; \end{cases}$$

$$\beta_i = \lambda_{2i-1} + \lambda_{2i} \quad (1 \leqq i \leqq q) .$$

Consequently $\Delta \cap \mathfrak{b} = \{\pm(\lambda_{2i-1} - \lambda_{2i})$ $(1 \leqq i \leqq q)$, $\pm(\lambda_{2q+i} - \lambda_{2q+j})$ $(1 \leqq i < j \leqq p-q)$, $\pm(\lambda_{2q+i} + \lambda_{2q+j})$ $(1 \leqq i \leqq j \leqq p-q)\}$.

First we prepare two lemmas.

LEMMA 5.1. $\Delta_\sharp(-) \cap \mathfrak{b} = \{\pm\alpha_{2i-1} \ (1 \leqq i \leqq q)\}$.

PROOF. Since $\operatorname{rank} G = \operatorname{rank} K$, we have $\Delta_\sharp = \Delta \cap (\mathfrak{a}_0 + \mathfrak{b}) = \Delta$. Then using Proposition 2.8 (2) and by the above table, we can easily obtain $\Delta_\sharp(-) \cap \mathfrak{b} = \{\pm(\lambda_{2i-1} - \lambda_{2i}) \ (1 \leqq i \leqq q)\}$. q.e.d.

By changing the sign of Y_{β_i} for suitable $\beta_i \in \Gamma$ (see Remark (2) at the end of § 2), we have

LEMMA 5.2. *$\{\pm\alpha_{2i} \ (1 \leqq i \leqq q)\} \subset \Delta_\sharp(-)$ if $p \geqq q+1$ and $\{\pm\alpha_{2i} \ (1 \leqq i \leqq q-1)\} \subset \Delta_\sharp(-)$ if $p = q$.*

PROOF. (1) We first note that $\theta\alpha_{2i} = -(\lambda_{2i-1} - \lambda_{2i+2})$ $(1 \leqq i \leqq q-1)$ and $\theta\alpha_{2q} = -(\lambda_{2q-1} + \lambda_{2q+1})$. Hence we have $\theta\alpha_{2i} \pm \beta_j \notin \Delta \cup \{0\}$ in case $j \neq i$, $i+1$. Therefore combining this fact with Lemma 2.10, we have

$$(*)_i \qquad \operatorname{Ad}(g^{-2}) \cdot Z_{\theta\alpha_{2i}} = \operatorname{Ad}(\exp \pi Y_{\beta_i}) \cdot \operatorname{Ad}(\exp \pi Y_{\beta_{i+1}}) \cdot Z_{\theta\alpha_{2i}} = \varepsilon_{\alpha_{2i}} Z_{\alpha_{2i}} \quad (1 \leqq i \leqq q-1) ,$$

$$(*)_q \qquad \operatorname{Ad}(g^{-2}) \cdot Z_{\theta\alpha_{2q}} = \operatorname{Ad}(\exp \pi Y_{\beta_q}) \cdot Z_{\theta\alpha_{2q}} = \varepsilon_{\alpha_{2q}} Z_{\alpha_{2q}} .$$

We now consider the equality $(*)_q$. Since $\theta\alpha_{2q}-\beta_q\notin\Delta\cup\{0\}$ and $2(\theta\alpha_{2q},\beta_q)/(\beta_q,\beta_q)=-1$, we can prove by the same method as in Lemma 2.9 (1) that $\mathrm{Ad}(\exp(-\pi Y_{\beta_q}))\cdot Z_{\theta\alpha_{2q}}=-\mathrm{Ad}(\exp(\pi Y_{\beta_q}))\cdot Z_{\theta\alpha_{2q}}$. Hence if we change the sign of Y_{β_q}, then from the above equalities the sign of $\varepsilon_{\alpha_{2q}}$ changes. We fix the sign of Y_{β_q} such that $\varepsilon_{\alpha_{2q}}=-1$. We next consider the equality $(*)_{q-1}$. Since $2(\theta\alpha_{2q-2},\beta_q)/(\beta_q,\beta_q)=1$, it follows that $\mathrm{Ad}(\exp\pi Y_{\beta_q})\cdot\theta\alpha_{2q-2}=\theta\alpha_{2q-2}-\beta_q\in\Delta$ (see Lemma 2.5). Hence $\mathrm{Ad}(\exp\pi Y_{\beta_q})\cdot Z_{\theta\alpha_{2q-2}}\in\mathfrak{g}^c_{\theta\alpha_{2q-2}-\beta_q}$. Since $\theta\alpha_{2q-2}-\beta_q-\beta_{q-1}\notin\Delta\cup\{0\}$ and $2(\theta\alpha_{2q-2}-\beta_q,\beta_{q-1})/(\beta_{q-1},\beta_{q-1})=-1$, we have $\mathrm{Ad}(\exp-\pi Y_{\beta_{q-1}})\cdot\mathrm{Ad}(\exp\pi Y_{\beta_q})\cdot Z_{\theta\alpha_{2q-2}}=-\mathrm{Ad}(\exp\pi Y_{\beta_{q-1}})\times\mathrm{Ad}(\exp\pi Y_{\beta_q})\cdot Z_{\theta\alpha_{2q-2}}$. Hence replacing $Y_{\beta_{q-1}}$ by $-Y_{\beta_{q-1}}$ if necessary, we have $\varepsilon_{\alpha_{2q-2}}=-1$. Applying the above arguments to the equalities $(*)_i$ $(1\leqq i\leqq q-2)$ successively, we have $\varepsilon_{\alpha_{2i}}=-1$ for $i=1,\cdots,q-2$. Then by Lemma 2.6 we have $\varepsilon_{-\alpha_{2i}}=-1$ $(1\leqq i\leqq q)$ and therefore $\{\pm\alpha_{2i}\,(1\leqq i\leqq q)\}\subset\Delta_\sharp(-)$.

(2) can be verified by a similar method and we omit the details. q.e.d.

In the following we assume that the sign of Y_{β_i} $(\beta_i\in\Gamma)$ is selected such that the properties of this lemma is satisfied.

REMARK. In the case $p=q$, if $\alpha_{2q-1}\in\Delta_\sharp(-)$, then it necessarily holds that $\pm\alpha_{2q}\notin\Delta_\sharp(-)$. In fact since $\alpha_{2q}=\beta_q-\alpha_{2q-1}$ and $\beta_q,\alpha_{2q-1}\in\Delta_\sharp(-)$, we have by Lemma 2.6 $\varepsilon_{\alpha_{2q}}=\varepsilon_{\beta_q-\alpha_{2q-1}}=\varepsilon_{\beta_q}\cdot\varepsilon_{-\alpha_{2q-1}}=(-1)\times(-1)=1$, implying that $\pm\alpha_{2q}\in\Delta_\sharp(+)$.

PROOF OF THEOREM 1.4. We first assume that $q\geqq 2$. Let $a_1,\cdots,a_q,b_1,\cdots,b_{q-1}$ be complex numbers that are linearly independent over $\boldsymbol{Q}$. We put $X=\sum_{i=1}^q a_iZ_{(-1)^{i-1}\beta_i}+\sum_{j=1}^{q-1}b_jZ_{(-1)^j\alpha_{2j}}$, $Y=\sum_{i=1}^q Z_{(-1)^i\beta_i}+\sum_{j=1}^{q-1}Z_{(-1)^{j-1}\alpha_{2j}}$ and $H=[X,Y]$. Then we have $\tilde{X}=\mathrm{Ad}(g)\cdot X\in\mathfrak{m}^c$, $\tilde{Y}=\mathrm{Ad}(g)\cdot Y\in\mathfrak{m}^c$ (Lemma 5.2) and $\tilde{H}=\mathrm{Ad}(g)\cdot H=[\tilde{X},\tilde{Y}]$. Since $(-1)^{i-1}\beta_i+(-1)^j\beta_j\notin\Delta\cup\{0\}$ $(i\neq j)$, $(-1)^i\alpha_{2i}+(-1)^j\beta_j\notin\Delta\cup\{0\}$ and $(-1)^i\alpha_{2i}+(-1)^{j-1}\alpha_{2j}\notin\Delta\cup\{0\}$ $(i\neq j)$, we have $H=2\sqrt{-1}\{\sum_{i=1}^q(-1)^{i-1}a_i/(\beta_i,\beta_i)\cdot\beta_i+\sum_{j=1}^q(-1)^jb_j/(\alpha_{2j},\alpha_{2j})\cdot\alpha_{2j}\}\in\mathfrak{t}^c=\mathfrak{a}_0^c+\mathfrak{b}^c$. Since $\Delta_\sharp=\Delta$ in this case, we have clearly $\kappa_2(H)=\varnothing$. Now we show that $\kappa_1(H)=\varnothing$. Let $\alpha\in\Delta_\sharp(-)$ satisfy the equality $(\alpha,H)=0$. Then since $a_1,\cdots,b_{q-1}$ are linearly independent over $\boldsymbol{Q}$, we have $(\alpha,\beta_i)=0$ $(1\leqq i\leqq q)$ and $(\alpha,\alpha_{2i})=0$ $(1\leqq i\leqq q-1)$. From the first part, it follows that $\alpha\in\Delta_\sharp(-)\cap\mathfrak{b}=\{\pm\alpha_{2i-1}\,(1\leqq i\leqq q)\}$, i.e., $\alpha=\pm\alpha_{2i-1}$ for some i $(1\leqq i\leqq q)$. However it is impossible because $(\alpha_{2i-1},\alpha_{2i})\neq 0$ or $(\alpha_{2i-1},\alpha_{2i-2})\neq 0$. Consequently we have $\kappa_1(H)=\varnothing$. Therefore by Lemma 2.7, we obtain $\mathrm{rank}_C\,\rho^c(\tilde{H})=\dim G/K-\dim_C\mathrm{Ker}(\mathrm{ad}\,\tilde{H})|_{\mathfrak{m}^c}=2c_0(G/K)$. This together with Lemma 3.2 proves that $c(G/K)=c_0(G/K)$.

We next consider the case $q=1$. In this case, we have by Lemma 5.2 $\alpha_2\in\Delta_\sharp(-)$ because $p\geqq 2$. Let a,b be two non-zero complex numbers such that $a/b \notin \boldsymbol{Q}$. We put $X = aZ_{\beta_1} + bZ_{-\alpha_2}$, $Y = Z_{-\beta_1} + Z_{\alpha_2}$ and $H = [X, Y]$. Then we have $\tilde{X} = \mathrm{Ad}\,(g)\cdot X \in \mathfrak{m}^c$, $\tilde{Y} = \mathrm{Ad}\,(g)\cdot Y \in \mathfrak{m}^c$ and $\tilde{H} = \mathrm{Ad}\,(g)\cdot H = [\tilde{X}, \tilde{Y}]$. Since $\alpha_2 + \beta_1 \notin \Delta \cup \{0\}$, we have $H = 2\sqrt{-1}\{a/(\beta_1, \beta_1)\cdot\beta_1 - b/(\alpha_2, \alpha_2)\cdot\alpha_2\} \in \mathfrak{a}_0^c + \mathfrak{b}^c$. Then by a similar method as above, we can show that $\kappa_1(H) = \kappa_2(H) = \varnothing$. Hence we have $\mathrm{rank}_{\boldsymbol{C}}\, \rho^c(\tilde{H}) = \dim G/K - \dim_{\boldsymbol{C}} \mathrm{Ker}\,(\mathrm{ad}\,\tilde{H})|_{\mathfrak{m}^c} = 2c_0(G/K)$. Thus by Lemma 3.2 we have $c(G/K) = c_0(G/K)$. q.e.d.

5.2. ***FII*** $F_4/Spin(9)$. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ denote the set of simple roots of $\mathfrak{f}_4$ with respect to a θ-order. Then the Satake diagram of $G/K = F_4/Spin(9)$ and the restriction of θ to $\mathfrak{t}$ are given as follows:

$$\begin{cases} \theta\alpha_4 = -(\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4)\,, \\ \theta\alpha_i = \alpha_i \quad (i = 1, 2, 3)\,. \end{cases}$$

It can be easily checked that the set $\Gamma = \{\beta_1\}$ $(s(G/K) = 1)$ which we define in Appendix 1 is given by $\beta_1 = \gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. As is well known there exists a basis $\{\lambda_1, \cdots, \lambda_4\}$ of $\mathfrak{t}$ such that $(\lambda_i, \lambda_j) = c\delta_{ij}$ $(c \in \boldsymbol{R}\backslash\{0\})$ and $\alpha_1 = \lambda_2 - \lambda_3$, $\alpha_2 = \lambda_3 - \lambda_4$, $\alpha_3 = \lambda_4$ and $\alpha_4 = (1/2)(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$ (Bourbaki [4]). By utilizing this basis, we have:

$$\begin{aligned} \Delta = \{&\pm\lambda_i \ (1 \leqq i \leqq 4),\ \pm\lambda_i \pm \lambda_j \ (1 \leqq i < j \leqq 4), \\ &(1/2)(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)\}\,, \\ \theta\lambda_1 &= -\lambda_1\,, \qquad \theta\lambda_i = \lambda_i \quad (i = 2, 3, 4)\,. \end{aligned}$$

We now prove

LEMMA 5.3. $\Delta_\sharp(-) \cap \mathfrak{b} = \{\pm\alpha_3, \pm(\alpha_2 + \alpha_3), \pm(\alpha_1 + \alpha_2 + \alpha_3)\}$.

PROOF. We first note that $\beta_1 = \lambda_1$. Hence by Proposition 2.8, we have $\Delta_\sharp(-) \cap \mathfrak{b} = \{\pm\lambda_2, \pm\lambda_3, \pm\lambda_4\}$. This proves the lemma. q.e.d.

PROOF OF THEOREM 1.4. First note that $\theta\alpha_4 - \beta_1 = (1/2)(-3\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \notin \Delta \cup \{0\}$ and $2(\theta\alpha_4, \beta_1)/(\beta_1, \beta_1) = -1$. Hence replacing Y_{β_1} by $-Y_{\beta_1}$ if necessary, we have $\alpha_4 \in \Delta_\sharp(-)$ (cf. Lemma 2.9 and Lemma 5.2). Now let a, b be two non-zero complex numbers satisfying $a/b \notin \boldsymbol{Q}$. We set $X = aZ_{\beta_1} + bZ_{-\alpha_4}$, $Y = Z_{-\beta_1} + Z_{\alpha_4}$ and $H = [X, Y]$. Then we have $\tilde{X} = \mathrm{Ad}\,(g)\cdot X \in \mathfrak{m}^c$, $\tilde{Y} = \mathrm{Ad}\,(g)\cdot Y \in \mathfrak{m}^c$ and $\tilde{H} = \mathrm{Ad}\,(g)\cdot H = [\tilde{X}, \tilde{Y}]$. By a simple calculation we obtain $H = 2\sqrt{-1}\{a/(\beta_1, \beta_1)\cdot\beta_1 - b/(\alpha_4, \alpha_4)\cdot\alpha_4\}$. (Note that $\alpha_4 + \beta_1 \notin \Delta \cup \{0\}$.) Since $\mathrm{rank}\, G = \mathrm{rank}\, K$, we have $\kappa_2(H) = \varnothing$ in the same way as before. Now we show $\kappa_1(H) = \varnothing$. Let $\alpha \in \Delta_\sharp(-)$ satisfy $(\alpha, H) = 0$. Then since $a/b \notin \boldsymbol{Q}$, we have $(\alpha, \beta_1) = (\alpha, \alpha_4) = 0$.

From the equality $(\alpha, \beta_1) = 0$, it follows that $\alpha \in \Delta_{\sharp}(-) \cap \mathfrak{b} = \{\pm\alpha_3, \pm(\alpha_2 + \alpha_3), \pm(\alpha_1 + \alpha_2 + \alpha_3)\}$. But it is impossible because $(\alpha, \alpha_4) \neq 0$ for all $\alpha \in \Delta_{\sharp}(-) \cap \mathfrak{b}$. Hence we know that $\kappa_1(H) = \emptyset$. Therefore we have $\operatorname{rank}_C \rho^c(\tilde{H}) = \dim G/K - \dim_C \operatorname{Ker}(\operatorname{ad} \tilde{H})|_{\mathfrak{m}^c} = 2c_0(G/K)$ (Lemma 2.7), proving $c(G/K) = c_0(G/K)$ (Lemma 3.2). q.e.d.

6. Real Grassmann manifolds $SO(p+q)/SO(p)\times SO(q)$ $(p \geqq q+2 \geqq 3,$ $q = \text{odd})$.

6.1. In this section we determine the integers $c(G/K)$ for real Grassmann manifolds $G/K = SO(p+q)/SO(p) \times SO(q)$ $(p \geqq q+2 \geqq 3$, $q = \text{odd})$. (We have already determined the integers $c(G/K)$ in the cases $q = \text{even}$ or $p = q,\ q+1$ in § 3.) We first consider the upper bounds for the integers $c(G/K)$. We prove

LEMMA 6.1. *Let* $G/K = SO(p+q)/SO(p)\times SO(q)$ $(p \geqq q \geqq 1,\ q = \text{odd})$. *Then it holds* $c(G/K) \leqq (1/2)\min\{pq,\ pq-(p-2q)\}$.

PROOF. Let us denote by $M(m, n)$ the space of all $m \times n$ real matrices. Then we have

$$\mathfrak{g} = \mathfrak{o}(p+q) = \{X \in M(p+q,\ p+q) \mid {}^tX = -X\}\ ;$$

$$\mathfrak{k} = \left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in M(p, p),\ B \in M(q, q),\ {}^tA = -A,\ {}^tB = -B \right\}\ ;$$

$$\mathfrak{m} = \left\{\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \middle| X \in M(p, q) \right\}\ ,$$

and the linear isotropy representation $\rho\colon \mathfrak{k} \to \mathfrak{o}(\mathfrak{m})$ can be written as follows:

$$\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} = \begin{pmatrix} 0 & AX - XB \\ {}^tXA - B\,{}^tX & 0 \end{pmatrix}$$
$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{k}\,,\quad \begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix} \in \mathfrak{m}\,.$$

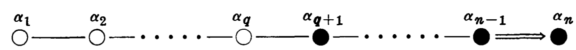
We write this equality in the form $\rho(A, B)(X) = AX - XB$, for simplicity. Let $X, Y \in M(p, q)$ and define $A(X, Y) \in \mathfrak{o}(p)$ and $B(X, Y) \in \mathfrak{o}(q)$ by

$$\left[\begin{pmatrix} 0 & X \\ -{}^tX & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y \\ -{}^tY & 0 \end{pmatrix}\right] = \begin{pmatrix} A(X, Y) & 0 \\ 0 & B(X, Y) \end{pmatrix},$$

i.e., $A(X, Y) = -X\,{}^tY + Y\,{}^tX$ and $B(X, Y) = -{}^tXY + {}^tYX$. We now show that it holds $\dim \operatorname{Ker} \rho(A(X, Y),\ B(X, Y)) \geqq p - 2q$ for every $X, Y \in M(p, q)$. In the following we fix $X, Y \in M(p, q)$. Let U be the subspace of $M(p, 1)$ consisting of all $u \in M(p, 1)$ satisfying ${}^tXu = {}^tYu = 0 \in M(q, 1)$. Then it is obvious that $\dim U \geqq p - 2q$. On the other hand, since $q =$ odd, there exists a non-zero $v_0 \in M(q, 1)$ such that ${}^tv_0 \cdot B(X, Y) = 0$. Then the linear map $\phi: M(p, 1) \to M(p, q)$ defined by $\phi(u) = u\,{}^tv_0$ for $u \in M(p, 1)$ is clearly injective. Since $A(X, Y)\phi(u) = \phi(u)B(X, Y) = 0$ for each $u \in U$, the image $\phi(U)$ is contained in $\operatorname{Ker} \rho(A(X, Y), B(X, Y))$, which implies that $\dim \operatorname{Ker} \rho(A(X, Y), B(X, Y)) \geqq \dim U \geqq p - 2q$. Therefore $c(G/K) \leqq (1/2)(\dim G/K - \max\{0, p - 2q\}) = (1/2)\min\{pq, pq - (p - 2q)\}$, proving the lemma. q.e.d.

6.2. ***BI, II*** $SO(p + q)/SO(p) \times SO(q)$ $(p \geqq q + 3 \geqq 4,\ p = \text{even},\ q = \text{odd})$.

Let $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ $(n = (1/2)(p + q - 1))$ be the set of simple roots of $\mathfrak{o}(p + q)^c$ with respect to a θ-order. Then the Satake diagram of G/K and the restriction of θ to $\mathfrak{t}$ are given as follows:



$$\begin{cases} \theta\alpha_i = -\alpha_i \quad (1 \leqq i \leqq q - 1)\,, \\ \theta\alpha_q = -\left(\alpha_q + 2\sum_{k=q+1}^{n} \alpha_k\right)\,, \\ \theta\alpha_{q+i} = \alpha_{q+i} \quad (1 \leqq i \leqq n - q)\,. \end{cases}$$

The set $\Gamma = \{\beta_1, \cdots, \beta_q\}$ $(s(G/K) = q)$ which we define in Appendix 1 is given by $\beta_i = \alpha_{2i-1}$ $(1 \leqq i \leqq (1/2)(q - 1))$, $\beta_i = \alpha_{2i-1} + 2\sum_{k=2i}^{n}\alpha_k$ $((1/2)(q - 1) + 1 \leqq i \leqq q - 1)$ and $\beta_q = \sum_{k=q}^{n}\alpha_k$. (In fact the set of simple roots $\{\gamma_1, \cdots, \gamma_q\}$ of the reduced root system Σ_* and the set $\{\alpha_1, \cdots, \alpha_n\}$ are related by $\gamma_i = \alpha_i$ $(1 \leqq i \leqq q - 1)$ and $\gamma_q = \sum_{k=q}^{n}\alpha_k$.)

As is well known there exists a basis $\{\lambda_1, \cdots, \lambda_n\}$ of $\mathfrak{t}$ such that $(\lambda_i, \lambda_j) = c\delta_{ij}$ $(c \in \boldsymbol{R}\backslash\{0\})$ and $\alpha_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq n - 1)$ and $\alpha_n = \lambda_n$. Utilizing this basis, we have:

$$\Delta = \{\pm(\lambda_i - \lambda_j)\ (1 \leqq i < j \leqq n),\ \pm(\lambda_i + \lambda_j)\ (1 \leqq i < j \leqq n),\ \pm\lambda_i\ (1 \leqq i \leqq n)\}\,,$$

$$\theta\lambda_i = -\lambda_i \quad (1 \leqq i \leqq q)\,, \qquad \theta\lambda_i = \lambda_i \quad (q + 1 \leqq i \leqq n)\,,$$

$$\begin{cases} \beta_i = \lambda_{2i-1} - \lambda_{2i} \quad (1 \leqq i \leqq (1/2)(q - 1))\,, \\ \beta_i = \lambda_{2i-1} + \lambda_{2i} \quad ((1/2)(q - 1) + 1 \leqq i \leqq q - 1)\,, \\ \beta_q = \lambda_q\,. \end{cases}$$

Consequently we have $\mathfrak{a} = \mathfrak{a}_0 = \sum_{i=1}^{q}\boldsymbol{R}\beta_i = \sum_{i=1}^{q}\boldsymbol{R}\lambda_i$, $\mathfrak{b} = \sum_{i=q+1}^{n}\boldsymbol{R}\lambda_i$. Moreover since $\operatorname{rank} G = \operatorname{rank} K$, we have $\Delta_\sharp = \Delta$.

LEMMA 6.2. $\Delta_\sharp(-) \cap \mathfrak{b} = \{\pm\sum_{k=q+i}^{n}\alpha_k\ (1 \leqq i \leqq n - q)\}$.

PROOF. By Proposition 2.8, we easily have $\Delta_\sharp(-)\cap\mathfrak{b}=\{\pm\lambda_{q+i}\ (1\leqq i\leqq n-q)\}$. This implies the lemma. q.e.d.

We now put $\xi_i=\sum_{k=2i}^{q+i-1}\alpha_k\ (=\lambda_{2i}-\lambda_{q+i})\in\Delta_\sharp$ for $1\leqq i\leqq m$, where $m=\min\{(1/2)(q-1),\ n-q\}$. Then we have $\xi_i\pm\xi_j\notin\Delta\cup\{0\}\ (i\neq j)$, $\theta\xi_i=-(\sum_{k=2i}^{q+i-1}\alpha_k+2\sum_{k=q+i}^{n}\alpha_k)\ (=-(\lambda_{2i}+\lambda_{q+i}))\ (1\leqq i\leqq m)$.

By changing the sign of Y_{β_j} for suitable $\beta_i\in\Gamma$ (see Remark (2) at the end of § 2), we have

LEMMA 6.3. $\{\pm\xi_i\ (1\leqq i\leqq m)\}\subset\Delta_\sharp(-)$.

PROOF. For simplicity, we put $\bar{i}=(1/2)(q-1)+i$ for $1\leqq i\leqq (1/2)(q-1)$. Then it is easy to see that $\xi_i\pm\beta_j,\ \theta\xi_i\pm\beta_j\notin\Delta\cup\{0\}$ if $j\neq i,\bar{i}$ and hence $\mathrm{Ad}(\exp tY_{\beta_j})\cdot Z_{\theta\xi_i}=Z_{\theta\xi_i}$ for $j\neq i,\bar{i}$. Therefore combining with Lemma 2.10, we have

$$\mathrm{Ad}(g^{-2})\cdot\theta Z_{\xi_i}=\mathrm{Ad}(\exp\pi Y_{\beta_i})\cdot\mathrm{Ad}(\exp\pi Y_{\beta_{\bar{i}}})\cdot Z_{\theta\xi_i}=\varepsilon_{\xi_i}Z_{\theta\xi_i}\quad(1\leqq i\leqq m)\,.$$

Since $2(\theta\xi_i,\beta_{\bar{i}})/(\beta_{\bar{i}},\beta_{\bar{i}})=-1$ for $1\leqq i\leqq m$, it follows from Lemma 2.5 that $\mathrm{Ad}(\exp\pi Y_{\beta_{\bar{i}}})\cdot\theta\xi_i=\theta\xi_i+\beta_{\bar{i}}$ and hence $\mathrm{Ad}(\exp\pi Y_{\beta_{\bar{i}}})\cdot Z_{\theta\xi_i}\in\mathfrak{g}_{\theta\xi_i+\beta_{\bar{i}}}$. On the other hand, since $2(\theta\xi_i+\beta_{\bar{i}},\beta_i)/(\beta_i,\beta_i)=1$ and $\theta\xi_i+\beta_{\bar{i}}+\beta_i\notin\Delta\cup\{0\}$, it follows from Lemma 2.9 (1) that $\mathrm{Ad}(\exp-\pi Y_{\beta_i})\cdot Z_{\theta\xi_i+\beta_{\bar{i}}}=-\mathrm{Ad}(\exp\pi Y_{\beta_i})\cdot Z_{\theta\xi_i+\beta_{\bar{i}}}$. Hence replacing Y_{β_i} by $-Y_{\beta_i}$ if necessary, we have $\varepsilon_{\xi_i}=-1$. Therefore by Lemma 2.6 we have $\pm\xi_i\in\Delta_\sharp(-)$. q.e.d.

In the following we fix the sign of Y_{β_i} such that $\pm\xi_j\in\Delta_\sharp(-)$ for $i=1,\cdots,m$.

PROOF OF THEOREM 1.4. Let $a_1,\cdots,a_{(1/2)(q-1)},b_1,\cdots,b_{(1/2)(q-1)},c,d_1,\cdots,d_m$ be complex numbers that are linearly independent over $\boldsymbol{Q}$. We put

$$X=\sum_{i=1}^{(1/2)(q-1)}(a_iZ_{(-1)^i\beta_i}+b_iZ_{(-1)^{i+1}\beta_{\bar{i}}})+cZ_{\beta_q}+\sum_{j=1}^{m}d_jZ_{(-1)^j\xi_j}\,,$$
$$Y=\sum_{i=1}^{(1/2)(q-1)}(Z_{(-1)^{i+1}\beta_i}+Z_{(-1)^i\beta_{\bar{i}}})+Z_{-\beta_q}+\sum_{j=1}^{m}Z_{(-1)^{j+1}\xi_j}$$

and $H=[X,Y]$. Then we have $\tilde{X}=\mathrm{Ad}(g)\cdot X\in\mathfrak{m}^c$, $\tilde{Y}=\mathrm{Ad}(g)\cdot Y\in\mathfrak{m}^c$ (Lemma 6.3) and $\tilde{H}=\mathrm{Ad}(g)\cdot H=[\tilde{X},\tilde{Y}]$. By a simple calculation we obtain

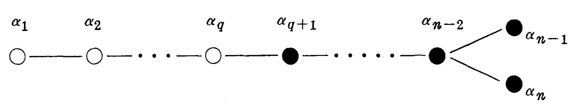
$$H=2\sqrt{-1}\Big[\sum_{i=1}^{(1/2)(q-1)}(-1)^i\{a_i/(\beta_i,\beta_i)\cdot\beta_i-b_i/(\beta_{\bar{i}},\beta_{\bar{i}})\cdot\beta_{\bar{i}}\}+c/(\beta_q,\beta_q)\cdot\beta_q$$
$$+\sum_{j=1}^{m}(-1)^jd_j/(\xi_j,\xi_j)\cdot\xi_j\Big]$$

because $\beta_i\pm\beta_j\notin\Delta\cup\{0\}\ (i\neq j)$, $(-1)^i\beta_i+(-1)^{j+1}\xi_j$, $(-1)^{i+1}\beta_{\bar{i}}+(-1)^{j+1}\xi_j$, $\beta_q\pm\xi_j\notin\Delta\cup\{0\}$ and $\xi_i\pm\xi_j\notin\Delta\cup\{0\}\ (i\neq j)$. Since $\Delta_\sharp=\Delta$, we have

$\kappa_2(H) = \emptyset$. We now determine the set $\kappa_1(H)$. Let $\alpha \in \Delta_\sharp(-)$ satisfy $(\alpha, H) = 0$. Then since $a_1, \cdots, d_m$ are linearly independent over $\boldsymbol{Q}$, we have $(\alpha, \beta_i) = 0$ $(1 \leqq i \leqq q)$ and $(\alpha, \xi_j) = 0$ $(1 \leqq j \leqq m)$. From the first part we have $\alpha \in \Delta_\sharp(-) \cap \mathfrak{b} = \{\pm \sum_{k=q+i}^{n} \alpha_k \ (1 \leqq i \leqq n-q)\} = \{\pm \lambda_{q+i} \ (1 \leqq i \leqq n-q)\}$. Consequently from the second part we have $(\alpha, \lambda_{q+i}) = 0$ $(1 \leqq i \leqq m)$. Hence we have $\alpha = \pm \lambda_{q+m+i}$ for some $i = 1, \cdots, n-q-m$. This shows that $\kappa_1(H) = \{\pm \sum_{k=q+m+i}^{n} \alpha_k \ (1 \leqq i \leqq n-q-m)\}$. In particular we have $\sharp \kappa_1(H) = 2(n-q-m)$. Therefore $\operatorname{rank}_C \rho^c(\tilde{H}) = \dim G/K - \dim_C \operatorname{Ker}(\operatorname{ad} \tilde{H})|_{\mathfrak{m}^c} = pq - 2(n-q-m) = \min\{pq, pq-(p-2q)\}$ (Lemma 2.7). This together with Lemma 6.1 proves that $c(G/K) = (1/2)\min\{pq, pq-(p-2q)\}$. q.e.d.

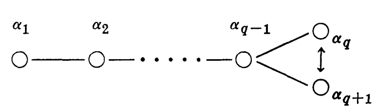
6.3. ***DI, II*** $SO(p+q)/SO(p) \times SO(q)$ $(p \geqq q+2 \geqq 3,\ p = \text{odd},\ q = \text{odd})$. Let $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ $(n = (1/2)(p+q))$ denote the set of simple roots of $\mathfrak{o}(p+q)^c$ with respect to a θ-order. Then the Satake diagram of G/K and the restriction of θ to $\mathfrak{t}$ are given as follows:

(i) $n \geqq q+2$



$$\begin{cases} \theta\alpha_i = -\alpha_i \quad (i < q), \qquad \theta\alpha_i = \alpha_i \quad (i > q), \\ \theta\alpha_q = -\left(\alpha_q + 2\sum_{k=q+1}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n\right). \end{cases}$$

(ii) $n = q+1$



$$\begin{cases} \theta\alpha_i = -\alpha_i \quad (i < q), \\ \theta\alpha_q = -\alpha_{q+1}, \qquad \theta\alpha_{q+1} = -\alpha_q. \end{cases}$$

The set $\Gamma = \{\beta_1, \cdots, \beta_{q-1}\}$ $(s(G/K) = q-1$ in this case) which we define in Appendix 1 is given by $\beta_i = \alpha_{2i-1}$ $(1 \leqq i \leqq (1/2)(q-1))$, $\beta_{(1/2)(q-1)+i} = \alpha_{2i-1} + 2\sum_{k=2i}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n$ $(1 \leqq i \leqq (1/2)(q-1))$. (In fact the set of simple roots $\{\gamma_1, \cdots, \gamma_q\}$ of the reduced root system Σ_* and the set $\{\alpha_1, \cdots, \alpha_n\}$ are related by $\gamma_i = \alpha_i$ $(1 \leqq i \leqq q-1)$ and $\gamma_q = \sum_{k=q}^{n-2} \alpha_k + (1/2)(\alpha_{n-1} + \alpha_n)$.)

As is well known there exists a basis $\{\lambda_1, \cdots, \lambda_n\}$ of $\mathfrak{t}$ such that $(\lambda_i, \lambda_j) = c\delta_{ij}$ $(c \in \boldsymbol{R}\backslash\{0\})$ and $\alpha_i = \lambda_i - \lambda_{i+1}$ $(1 \leqq i \leqq n-1)$ and $\alpha_n = \lambda_{n-1} + \lambda_n$. Utilizing this basis, we have:

$$\Delta = \{\pm(\lambda_i - \lambda_j)\ (1 \leqq i < j \leqq n),\ \pm(\lambda_i + \lambda_j)\ (1 \leqq i < j \leqq n)\},$$

$$\begin{cases} \theta\lambda_i = -\lambda_i \quad (1 \leqq i \leqq q), \qquad \theta\lambda_{q+i} = \lambda_{q+i} \quad (1 \leqq i \leqq n-q), \\ \beta_i = \lambda_{2i-1} - \lambda_{2i} \quad (1 \leqq i \leqq (1/2)(q-1)), \\ \beta_{(1/2)(q-1)+i} = \lambda_{2i-1} + \lambda_{2i} \quad (1 \leqq i \leqq (1/2)(q-1)). \end{cases}$$

Consequently we have $\mathfrak{a}_0 = \sum_{i=1}^{q-1} \boldsymbol{R}\beta_i = \sum_{i=1}^{q-1} \boldsymbol{R}\lambda_i$, $\mathfrak{a}_1 = \boldsymbol{R}\lambda_q$ and $\mathfrak{b} = \sum_{i=1}^{n-q} \boldsymbol{R}\lambda_{q+i}$. Then we can easily show the following

LEMMA 6.4. (1) $\Delta^+ \backslash \Delta_\sharp = \{\lambda_q \pm \lambda_{q+i}\ (1 \leqq i \leqq n-q)\}$.
(2) $\Delta_\sharp(-) \cap \mathfrak{b} = \varnothing$.

Now let us set $\xi_i = \sum_{k=2i}^{q+i-1} \alpha_k\ (= \lambda_{2i} - \lambda_{q+i})$ for $1 \leqq i \leqq m$, where $m = \min\{(1/2)(q-1),\ n-q\}$. Then we can easily verify that $\xi_i \pm \xi_j \notin \Delta \cup \{0\}$ $(i \neq j)$, $\xi_i \in \Delta_\sharp$ $(1 \leqq i \leqq m)$ and $\theta\xi_i = -(\sum_{k=2i}^{q+i-1} \alpha_k + 2\sum_{k=q+i}^{n-2} \alpha_k + \alpha_{n-1} + \alpha_n) = -\lambda_{2i} - \lambda_{q+i}$ $(1 \leqq i \leqq m)$. Moreover we have $\theta\xi_i \pm \beta_j \notin \Delta \cup \{0\}$ $(j \neq i, \bar{i})$, $\theta\xi_i + \beta_i + \beta_{\bar{i}} \notin \Delta \cup \{0\}$ $(1 \leqq i \leqq m)$ and $2(\theta\xi_i, \beta_j)/(\beta_j, \beta_j) = \delta_{ij} - \delta_{\bar{i}j}$ $(1 \leqq i \leqq m,\ 1 \leqq j \leqq q-1)$, where $\bar{i} = (1/2)(q-1)+i$ $(1 \leqq i \leqq (1/2)(q-1))$. Then by a similar method developed in the proof of Lemma 6.3, we may assume that $\pm\xi_i \in \Delta_\sharp(-)$ $(1 \leqq i \leqq m)$.

PROOF OF THEOREM 1.4. (a) The case $n - q \leqq (1/2)(q-1)$, (i.e., $p \leqq 2q-1$). In this case we have $m = n-q$. Let $a_1, \cdots, a_{(1/2)(q-1)}$, $b_1, \cdots, b_{(1/2)(q-1)}$, $c_1, \cdots, c_m$ be complex numbers that are linearly independent over $\boldsymbol{Q}$. We set

$$X = \sum_{i=1}^{(1/2)(q-1)} (a_i Z_{(-1)^i\beta_i} + b_i Z_{(-1)^{i+1}\beta_{\bar{i}}}) + \sum_{j=1}^{m} c_j Z_{(-1)^j\xi_j},$$

$$Y = \sum_{i=1}^{(1/2)(q-1)} (Z_{(-1)^{i+1}\beta_i} + Z_{(-1)^i\beta_{\bar{i}}}) + \sum_{j=1}^{m} Z_{(-1)^{j+1}\xi_j},$$

and set $H = [X, Y]$. Then we have $\tilde{X} = \mathrm{Ad}(g)\cdot X \in \mathfrak{m}^c$, $\tilde{Y} = \mathrm{Ad}(g)\cdot Y \in \mathfrak{m}^c$ $(\pm\xi_i \in \Delta_\sharp(-))$ and $\tilde{H} = \mathrm{Ad}(g)\cdot H = [\tilde{X}, \tilde{Y}]$. By a simple calculation we obtain

$$H = 2\sqrt{-1}\Big[\sum_{i=1}^{(1/2)(q-1)} (-1)^i\{a_i/(\beta_i, \beta_i)\cdot\beta_i - b_i/(\beta_{\bar{i}}, \beta_{\bar{i}})\cdot\beta_{\bar{i}}\} + \sum_{j=1}^{m} (-1)^j c_j/(\xi_j, \xi_j)\cdot\xi_j\Big],$$

because $\beta_i \pm \beta_j \notin \Delta \cup \{0\}$ $(i \neq j)$, $(-1)^i\beta_i + (-1)^{j+1}\xi_j \notin \Delta \cup \{0\}$, $(-1)^{i+1}\beta_{\bar{i}} + (-1)^{j+1}\xi_j \notin \Delta \cup \{0\}$ and $\xi_i \pm \xi_j \notin \Delta \cup \{0\}$ $(i \neq j)$.

We now determine the sets $\kappa_1(H)$ and $\kappa_2(H)$. Let $\alpha \in \Delta$ satisfy $(\alpha, H) = 0$. Then since $a_1, \cdots, c_m$ are linearly independent over $\boldsymbol{Q}$, we have $(\alpha, \beta_i) = 0$ $(1 \leqq i \leqq q-1)$ and $(\alpha, \xi_j) = 0$ $(1 \leqq j \leqq m)$. From the first part we obtain $(\alpha, \lambda_i) = 0$ $(1 \leqq i \leqq q-1)$ and hence $\alpha \in \Delta \cap (\mathfrak{a}_1 + \mathfrak{b}) \subset$

$\sum_{i=q}^{n} \boldsymbol{R}\lambda_i$. Consequently from the second part we have $(\alpha, \lambda_{q+i}) = 0$ $(1 \leqq i \leqq m = n - q)$. Thus we have $\alpha \in \boldsymbol{R}\lambda_q$. But it is impossible because $\Delta \cap \boldsymbol{R}\lambda_q = \emptyset$. Hence we have $\kappa_1(H) = \kappa_2(H) = \emptyset$. Therefore $\mathrm{rank}_C\, \rho^c(\tilde{H}) = \dim G/K - \mathrm{rank}\, G + \mathrm{rank}\, K = pq - (1/2)(p+q) + (1/2)(p+q) - 1 = pq - 1$ (Lemma 2.7). This together with Lemma 6.1 implies that $c(G/K) = (1/2)(pq-1)$ in the case $p \leqq 2q - 1$.

(b) The case $n - q > (1/2)(q-1)$, (i.e., $p > 2q - 1$). In this case we have $m = (1/2)(q-1) < n - q$. We set $\eta = \sum_{k=q}^{n-2} \alpha_k + \alpha_n$ $(= \lambda_q + \lambda_n)$. Then we have $\eta \in \Delta^+ \backslash \Delta_\sharp$ and $\eta \pm \xi_i$, $\theta\eta \pm \eta$, $\theta\eta \pm \xi_i$, $\eta \pm \beta_i$, $\theta\eta \pm \beta_i \notin \Delta \cup \{0\}$. Let $a_1, \cdots, a_{(1/2)(q-1)}, b_1, \cdots, b_{(1/2)(q-1)}, c_1, \cdots, c_m, d$ be complex numbers that are linearly independent over $\boldsymbol{Q}$. We set

$$X = \sum_{i=1}^{(1/2)(q-1)} (a_i Z_{(-1)^i \beta_i} + b_i Z_{(-1)^{i+1}\beta_{\bar{i}}}) + \sum_{j=1}^{m} c_j Z_{(-1)^j \xi_j} + d(Z_\eta - \theta Z_\eta) ,$$

$$Y = \sum_{i=1}^{(1/2)(q-1)} (Z_{(-1)^{i+1}\beta_i} + Z_{(-1)^i \beta_{\bar{i}}}) + \sum_{j=1}^{m} Z_{(-1)^{j+1}\xi_j} + (Z_{-\eta} - \theta Z_{-\eta}) ,$$

and $H = [X, Y]$. Then we have $\tilde{X} = \mathrm{Ad}(g)\cdot X \in \mathfrak{m}^c$, $\tilde{Y} = \mathrm{Ad}(g)\cdot Y \in \mathfrak{m}^c$ and $\tilde{H} = \mathrm{Ad}(g)\cdot H = [\tilde{X}, \tilde{Y}]$. (Note that $\mathrm{Ad}(g)\cdot Z_{\pm\eta} = Z_{\pm\eta}$ and $\mathrm{Ad}(g)\cdot \theta Z_{\pm\eta} = \theta Z_{\pm\eta}$.) By a simple calculation, we obtain

$$H = 2\sqrt{-1} \sum_{i=1}^{(1/2)(q-1)} (-1)^i \{a_i/(\beta_i, \beta_i)\cdot\beta_i - b_i/(\beta_{\bar{i}}, \beta_{\bar{i}})\cdot\beta_{\bar{i}}\}$$
$$+ \sum_{j=1}^{m} (-1)^j c_j/(\xi_j, \xi_j)\cdot\xi_j + d/(\eta, \eta)\cdot(\eta + \theta\eta) .$$

We now determine the sets $\kappa_1(H)$ and $\kappa_2(H)$. Let $\alpha \in \Delta$ satisfy $(\alpha, H) = 0$. Then since $a_1, \cdots, d$ are linearly independent over $\boldsymbol{Q}$, we have $(\alpha, \beta_i) = 0$ $(1 \leqq i \leqq q-1)$, $(\alpha, \xi_j) = 0$ $(1 \leqq j \leqq m)$ and $(\alpha, \eta + \theta\eta) = 0$. From the first part we have $(\alpha, \lambda_i) = 0$ $(1 \leqq i \leqq q-1)$ and hence $\alpha \in \Delta \cap (\mathfrak{a}_1 + \mathfrak{b}) \subset \sum_{i=q}^{n} \boldsymbol{R}\lambda_i$. Consequently from the second and the third parts we obtain $(\alpha, \lambda_{q+i}) = 0$ $(1 \leqq i \leqq m = (1/2)(q-1))$ and $(\alpha, \lambda_n) = 0$. (Note that $\eta + \theta\eta = \lambda_q + \lambda_n - \lambda_q + \lambda_n = 2\lambda_n$.) Hence we have $\alpha \in \boldsymbol{R}\lambda_q + \sum_{i=m+1}^{n-q-1} \boldsymbol{R}\lambda_{q+i}$. Therefore if $\alpha \in \Delta_\sharp(-) \subset \Delta \cap (\mathfrak{a}_0 + \mathfrak{b})$, we have $\alpha \in \Delta_\sharp(-) \cap \mathfrak{b}$. But it is impossible because $\Delta_\sharp(-) \cap \mathfrak{b} = \emptyset$ (see Lemma 6.4). This shows that $\kappa_1(H) = \emptyset$. Now we assume that $\alpha \in \Delta^+ \backslash \Delta_\sharp$. Then by Lemma 6.4, we know that $\alpha = \lambda_q \pm \lambda_{q+i}$ for some i $(m+1 \leqq i \leqq n-q-1)$. Thus we have $\kappa_2(H) = \{\lambda_q \pm \lambda_{q+i}\ (m+1 \leqq i \leqq n-q-1)\}$ and hence $\sharp\kappa_2(H) = 2(n-q-m-1) = p - 2q - 1$. Therefore we have $\mathrm{rank}_C\, \rho^c(\tilde{H}) = \dim G/K - \mathrm{rank}\, G + \mathrm{rank}\, K - \sharp\kappa_2(H) = pq - (1/2)(p+q) + (1/2)(p+q) - 1 - (p - 2q - 1) = pq - (p - 2q)$. This together with Lemma 6.1 proves that $c(G/K) = (1/2)\{pq - (p-2q)\}$ in the case $p > 2q - 1$. q.e.d.

Thus the proof of Theorem 1.4 is completely finished.

Appendix 1. In this appendix we give the proof of Proposition 2.2. We retain the notations used in § 2.

Let $\alpha \in \Delta$. By $\bar{\alpha}$ we mean the $\mathfrak{a}$-component of α with respect to the decomposition $\mathfrak{t} = \mathfrak{a} + \mathfrak{b}$. Clearly we have $\bar{\alpha} = 0$ if and only if $\alpha \in \Delta \cap \mathfrak{b}$. We denote by Σ the set of all $\bar{\alpha}$ ($\alpha \in \Delta \backslash \mathfrak{b}$). The elements of Σ are called *restricted roots* of G/K. In general Σ does not form a reduced root system, but it is known that the subset Σ_* of Σ defined by $\Sigma_* = \{\psi \in \Sigma \mid 2\psi \notin \Sigma\}$ forms a reduced root system. Moreover since G/K is irreducible, Σ_* forms an irreducible reduced root system. Therefore Σ_* is isomorphic to a root system of some complex simple Lie algebra whose rank equals $l = \operatorname{rank} G/K$.

For each $\psi \in \Sigma$, let us set $\Delta(\psi) = \{\alpha \in \Delta \backslash \mathfrak{b} \mid \bar{\alpha} = \psi\}$. We call the cardinality $m(\psi) = \sharp \Delta(\psi)$ the *multiplicity* of ψ.

We first prove

LEMMA. *Let* $\psi \in \Sigma$. *Then*:

(1) $\psi \in \Delta$ *if and only if* $m(\psi)$ *is odd.*

(2) *If* $2\psi \in \Sigma$, *then* $2\psi \in \Delta$.

(3) *Let* $\psi' \in \Sigma$ *satisfy* $|\psi'| = |\psi|$. *Then* $m(\psi') = m(\psi)$.

(4) *Assume that* $\psi \in \Delta$. *Then it holds* $m(\psi) > 1$ *if and only if* $\alpha \pm \psi \in \Delta$ *for some* $\alpha \in \Delta \cap \mathfrak{b}$.

PROOF. The proofs of the assertions (1) and (2) can be found in Helgason [9], Chap. X, Exercises. We now prove the assertions (3) and (4).

(3) Let K_0 (resp. K_0') be the centralizer (resp. normalizer) of $\mathfrak{a}$ in K and let $\mathfrak{k}_0$ be the Lie algebra of K_0. Since the Weyl group K_0'/K_0 of the pair (G, K) acts transitively on each subset Σ of the same length, there exists an element $k_1 \in K_0'$ such that $\psi' = \operatorname{Ad}(k_1)\cdot\psi$ (see Helgason [9]). On the other hand, since $\mathfrak{b}$ and $\operatorname{Ad}(k_1)\cdot\mathfrak{b}$ are maximal abelian subalgebra of $\mathfrak{k}_0$, there exists an element $k_0 \in K_0$ such that $\operatorname{Ad}(k_0)\cdot\mathfrak{b} = \operatorname{Ad}(k_1)\cdot\mathfrak{b}$. Put $k = k_0^{-1}\cdot k_1$. Then we have $\psi' = \operatorname{Ad}(k)\cdot\psi$ and $\operatorname{Ad}(k)\cdot\mathfrak{t} = \mathfrak{t}$. Therefore we have $\operatorname{Ad}(k)\cdot\Delta = \Delta$ and hence $\Delta(\psi') = \operatorname{Ad}(k)\cdot\Delta(\psi)$. This implies $m(\psi') = m(\psi)$.

(4) First we assume that $m(\psi) > 1$. Let $\beta \in \Delta(\psi)\backslash\{\psi\}$ and put $\alpha = \beta - \psi$. Then we have $\alpha \in \Delta \cap \mathfrak{b}$, because $(\beta, \psi) = (\psi, \psi) > 0$ and $\bar{\alpha} = \bar{\beta} - \bar{\psi} = \psi - \psi = 0$. Since $\alpha - \psi = \theta(\alpha + \psi) = \theta\beta \in \Delta$, we have $\alpha \pm \psi \in \Delta$. Conversely if we assume that $\alpha \pm \psi \in \Delta$ for some $\alpha \in \Delta \cap \mathfrak{b}$, then we have $m(\psi) > 1$ because $\bar{\psi} \pm \bar{\alpha} = \psi$. q.e.d.

In a natural way a linear order in $\mathfrak{a}$ is induced from the θ-order in $\mathfrak{t}$. We denote by Σ_*^+ the set of positive restricted roots in Σ_*. As usual we say that an element in Σ_*^+ is *simple* if and only if it cannot be written

TABLE 1

	G/K	rank G/K	$s(G/K)$	Dynkin diagram of Π_*	$m(\gamma_i)$
AI	$SU(n+1)/SO(n+1)$ $(n\geqq 1)$	n	$[(n+1)/2]$	o—o— ··· —o—o γ_1 γ_2 γ_{n-1} γ_n	1
AII	$SU(2(n+1))/Sp(n+1)$ $(n\geqq 1)$	n	0	o—o— ··· —o—o γ_1 γ_2 γ_{n-1} γ_n	4
AIII	$SU(p+q)/S(U(p)\times U(q))$ $(p\geqq q\geqq 1)$	q	q	o—o— ··· —o⇐o $(q\geqq 2)$ γ_1 γ_2 γ_{q-1} γ_q o $(q=1)$ γ_1	$2\ (i<q)$ $1\ (i=q)$ 1
BI	$SO(p+q)/SO(p)\times SO(q)$ $(p+q=\text{odd},\ p>q\geqq 2)$	q	q	o—o— ··· —o⇒o γ_1 γ_2 γ_{q-1} γ_q	$1\ (i<q)$ $p-q\ (i=q)$
BII	$SO(p+1)/SO(p)$ $(p=\text{even},\ p\geqq 2)$	1	1	o γ_1	$p-1$
CI	$Sp(n)/U(n)$ $(n\geqq 1)$	n	n	o—o— ··· —o⇐o γ_1 γ_2 γ_{n-1} γ_n	1
CII	$Sp(p+q)/Sp(p)\times Sp(q)$ $(p\geqq q\geqq 1)$	q	q	o—o— ··· —o⇐o $(q\geqq 2)$ γ_1 γ_2 γ_{q-1} γ_q o $(q=1)$ γ_1	$4\ (i<q)$ $3\ (i=q)$ 3
DI	$SO(p+q)/SO(p)\times SO(q)$ $\left(\begin{matrix} p+q=\text{even},\ p\geqq q\geqq 2 \\ (p,q)\neq(2,2)\end{matrix}\right)$	q	$2[q/2]$	o—o— ··· —o⇒o $(p\geqq q+2)$ γ_1 γ_2 γ_{q-1} γ_q γ_{q-2} o—o— ··· —o—o $(p=q)$ γ_1 γ_2 | γ_{q-1} o γ_q	$1\ (i<q)$ $p-q\ (i=q)$ 1
DII	$SO(p+1)/SO(p)$ $(p=\text{odd},\ p\geqq 3)$	1	0	o γ_1	$p-1$
DIII	$SO(2n)/U(n)$ $(n\geqq 2)$	$[n/2]$	$[n/2]$	o—o— ··· —o⇐o $(n\geqq 4)$ 21 γ_2 $\gamma_{[n/2]}$ o $(n=2\text{ or }3)$ γ_1	$4\ (i<[n/2])$ $1\ (i=[n/2])$ 1
EI	$E_6/Sp(4)$	6	4	γ_4 o—o—o—o—o γ_1 γ_3 | γ_5 γ_6 o γ_2	1
EII	$E_6/SU(6)\cdot SU(2)$	4	4	o—o⇒o—o γ_1 γ_2 γ_3 γ_4	$1\ (i=1,2)$ $2\ (i=3,4)$
EIII	$E_6/Spin(10)\cdot SO(2)$	2	2	o⇒o γ_1 γ_2	$1\ (i=1)$ $6\ (i=2)$
EIV	E_6/F_4	2	0	o—o γ_1 γ_2	8
EV	$E_7/SU(8)$	7	7	γ_4 o—o—o—o—o—o γ_1 γ_3 | γ_5 γ_6 γ_7 o γ_2	1
EVI	$E_7/SO(12)\cdot SU(2)$	4	4	o—o⇒o—o γ_1 γ_2 γ_3 γ_4	$1\ (i=1,2)$ $4\ (i=3,4)$
EVII	$E_7/E_6\cdot SO(2)$	3	3	o—o⇐o γ_1 γ_2 γ_3	$8\ (i=1,2)$ $1\ (i=3)$
EVIII	$E_8/SO(16)$	8	8	γ_4 o—o—o—o—o—o—o γ_1 γ_3 | γ_5 γ_6 γ_7 γ_8 o γ_2	1
EIX	$E_8/E_7\cdot SU(2)$	4	4	o—o⇒o—o γ_1 γ_2 γ_3 γ_4	$1\ (i=1,2)$ $8\ (i=3,4)$
FI	$F_4/Sp(3)\cdot SU(2)$	4	4	o—o⇒o—o γ_1 γ_2 γ_3 γ_4	1
FII	$F_4/Spin(9)$	1	1	o γ_1	7
G	$G_2/SU(2)\times SU(2)$	2	2	o⇚o γ_1 γ_2	1

as a sum of any two elements of Σ_*^+. Hereafter we denote by $\Pi_* = \{\gamma_1, \cdots, \gamma_l\}$ ($l = \operatorname{rank} G/K$) the set of all simple restricted roots of Σ_*^+.

For each irreducible Riemannian symmetric space G/K of compact type with G simple we exhibit, in table 1, the rank of G/K, the integer $s(G/K)$, the Dynkin diagram of $\Pi_* = \{\gamma_1, \cdots, \gamma_l\}$ and the multiplicities $m(\gamma_i)$ ($\gamma_i \in \Pi_*$) (cf. Helgason [9] Chap. X, pp. 532–534).

We note that in case $M^* = G/K$ is a compact simple Lie group, i.e., in case $G = M^* \times M^*$, $K = \{(x, x) \in G \mid x \in M^*\}$ and $\theta(x, y) = (y, x)$, $x, y \in M^*$, we have $s(G/K) = \operatorname{rank} G/K - \operatorname{rank} G + \operatorname{rank} K = \operatorname{rank} M^* - 2 \operatorname{rank} M^* + \operatorname{rank} M^* = 0$.

We now define a subset $\Gamma_0 = \{\beta_1, \cdots, \beta_{s_0}\}$ of Σ_*^+ ($\subset \mathfrak{a}$) according to the type of Σ_* ($l = \operatorname{rank} \Sigma_*$) (see Table 2).

In view of the list of non-zero roots contained in each irreducible reduced root system Σ_* (see Bourbaki [4]), we can directly verify that

TABLE 2

Σ_*		s_0	β_i
$[A_l]$ ($l \geqq 1$)	o—o— ⋯ —o—o $\gamma_1\ \gamma_2\ \ \gamma_{l-1}\gamma_l$	$[(l+1)/2]$	$\beta_i = \gamma_{2i-1}$ $(1 \leqq i \leqq [(l+1)/2])$
$[B_l]$ ($l \geqq 2$)	o—o— ⋯ —o⇒o $\gamma_1\ \gamma_2\ \ \gamma_{l-1}\gamma_l$	l	$\beta_i = \gamma_{2i-1}$ $(1 \leqq i \leqq [l/2])$ $\beta_{[l/2]+i} = \gamma_{2i-1} + 2\sum_{k=2i}^{l} \gamma_k$ $(1 \leqq i \leqq [l/2]-1)$ $\beta_l = \begin{cases} \gamma_{l-1} + 2\gamma_l & (l = \text{even}) \\ \gamma_l & (l = \text{odd}) \end{cases}$
$[C_l]$ ($l \geqq 3$)	o—o— ⋯ —o⇐o $\gamma_1\ \gamma_2\ \ \gamma_{l-1}\gamma_l$	l	$\beta_i = 2\sum_{k=i}^{l-1} \gamma_k + \gamma_l$ $(1 \leqq i \leqq l)$
$[D_l]$ ($l \geqq 4$)	o—o— ⋯ —o—o with γ_{l-2} branching to γ_l $\gamma_1\ \gamma_2\ \ \gamma_{l-2}\ \gamma_{l-1}\ \gamma_l$	$2[l/2]$	$\beta_i = \gamma_{2i-1}$ $(1 \leqq i \leqq [l/2])$ $\beta_{[l/2]+i} = \gamma_{2i-1} + 2\sum_{k=2i}^{l-2} \gamma_k + \gamma_{l-1} + \gamma_l$ $(1 \leqq i \leqq [(l-2)/2])$ $\beta_{2[l/2]} = \begin{cases} \gamma_l & (l = \text{even}) \\ \gamma_{l-2} + \gamma_{l-1} + \gamma_l & (l = \text{odd}) \end{cases}$
$[E_6]$	o—o—o—o—o with γ_4 branching to γ_2 $\gamma_1\ \gamma_3\ \gamma_4\ \gamma_5\ \gamma_6$; γ_2	4	$\beta_1 = \gamma_2$, $\beta_2 = \gamma_3$, $\beta_3 = \gamma_5$, $\beta_4 = \gamma_2 + \gamma_3 + 2\gamma_4 + \gamma_5$
$[E_7]$	o—o—o—o—o—o with γ_4 branching to γ_2 $\gamma_1\ \gamma_3\ \gamma_4\ \gamma_5\ \gamma_6\ \gamma_7$; γ_2	7	$\beta_1 = \gamma_2$, $\beta_2 = \gamma_3$, $\beta_3 = \gamma_5$, $\beta_4 = \gamma_7$, $\beta_5 = \gamma_2 + \gamma_3 + 2\gamma_4 + \gamma_5$, $\beta_6 = \gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + \gamma_7$, $\beta_7 = 2\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 3\gamma_5 + 2\gamma_6 + \gamma_7$
$[E_8]$	o—o—o—o—o—o—o with γ_4 branching to γ_2 $\gamma_1\ \gamma_3\ \gamma_4\ \gamma_5\ \gamma_6\ \gamma_7\ \gamma_8$; γ_2	8	$\beta_1 = \gamma_2$, $\beta_2 = \gamma_3$, $\beta_3 = \gamma_5$, $\beta_4 = \gamma_7$, $\beta_5 = \gamma_2 + \gamma_3 + 2\gamma_4 + \gamma_5$, $\beta_6 = \gamma_2 + \gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + \gamma_7$, $\beta_7 = 2\gamma_1 + 2\gamma_2 + 3\gamma_3 + 4\gamma_4 + 3\gamma_5 + 2\gamma_6 + \gamma_7$, $\beta_8 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 6\gamma_4 + 5\gamma_5 + 4\gamma_6 + 3\gamma_7 + 2\gamma_8$
$[F_4]$	o—o⇒o—o $\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4$	4	$\beta_1 = \gamma_2$, $\beta_2 = \gamma_2 + 2\gamma_3$, $\beta_3 = \gamma_2 + 2\gamma_3 + 2\gamma_4$, $\beta_4 = 2\gamma_1 + 3\gamma_2 + 4\gamma_3 + 2\gamma_4$
$[G_2]$	o⇚o $\gamma_1\ \gamma_2$	2	$\beta_1 = \gamma_2$, $\beta_2 = 2\gamma_1 + \gamma_2$

$\beta_i \pm \beta_j \notin \Sigma_* \cup \{0\}$ $(i \neq j)$. We can also check that all $\beta_i \in \Gamma_0$ have the same length in every case except for $[B_l]$ $(l = \text{odd})$, $[G_2]$. We note that in the case Σ_* is of type $[B_l]$ $(l = \text{odd})$ it holds $|\beta_1| = \cdots = |\beta_{l-1}| = \sqrt{2}|\beta_l|$.

PROOF OF PROPOSITION 2.2. We assume that $s(G/K) > 0$ in the following, i.e., G/K is not a compact simple Lie group, ***AII***, ***DII*** nor ***EIV*** (see Table 1).

(1°) The case where G/K is not of type ***DI*** $(q = \text{odd})$. In this case G/K has the following properties:

(i) $\beta_i \in \Delta^+$;

(ii) $\beta_i \pm \beta_j \notin \Delta \cup \{0\}$ $(i \neq j)$;

(iii) $\#\Gamma_0 = s(G/K)$.

Hence if we put $\Gamma = \Gamma_0$, the conditions (1) and (2) of Proposition 2.2 are satisfied. Now we prove the above properties.

Proof of (i). First suppose that Σ_* is not of type $[B_l]$ $(l = \text{odd})$, $[G_2]$. Since all $\beta_i \in \Gamma_0$ have the same length, we have $m(\beta_1) = \cdots = m(\beta_{s_0})$ (Lemma (3)). On the other hand, we know that there exists some $\beta_i \in \Gamma_0$ such that $\beta_i \in \Pi_*$, i.e., $\beta_i = \gamma_j$ for some j $(1 \leqq j \leqq l)$. Viewing Table 1, we know that $m(\beta_i)$ is odd for such a β_i. Hence $m(\beta_i)$ is odd for every $\beta_i \in \Gamma_0$, implying $\beta_i \in \Delta^+$ (Lemma (1), (3)). Next we consider the case where Σ_* is of type $[B_l]$ $(l = \text{odd})$ or $[G_2]$. In the case $[B_l]$ $(l = \text{odd})$, we have $|\beta_1| = \cdots = |\beta_{l-1}| = \sqrt{2}|\beta_l|$ and $\beta_1 = \gamma_1 \in \Pi_*$, $\beta_l = \gamma_l \in \Pi_*$. From Table 1, we know that both $m(\beta_1)$ and $m(\beta_l)$ are odd. Hence $m(\beta_i)$ is odd for every $\beta_i \in \Gamma_0$. This means $\beta_i \in \Delta^+$. In the case $[G_2]$ (this case occurs if and only if G/K is of type ***G***), we have $m(\gamma_1) = m(\gamma_2) = 1$ and hence $m(\beta_1) = m(\beta_2) = 1$, which implies $\beta_1, \beta_2 \in \Delta^+$.

Proof of (ii). Suppose that $\beta_i + \beta_j \in \Delta$ for some $\beta_i, \beta_j \in \Gamma_0$ $(i \neq j)$. Since $\beta_i + \beta_j \notin \Sigma_*$ and $\beta_i + \beta_j \in \Sigma$, we have $2(\beta_i + \beta_j) \in \Sigma$. Then by Lemma (2) we obtain $2(\beta_i + \beta_j) \in \Delta$. This contradicts the assumption $\beta_i + \beta_j \in \Delta$. Hence $\beta_i + \beta_j \notin \Delta \cup \{0\}$. Similarly we can show that $\beta_i - \beta_j \notin \Delta \cup \{0\}$. Thus we have $\beta_i \pm \beta_j \notin \Delta \cup \{0\}$ for $i \neq j$.

Proof of (iii). Assume that G/K is not of type ***AI*** $(n \geqq 2)$ nor ***EI***. Then from Table 1 we have $s(G/K) = \operatorname{rank} G/K$. Moreover in these cases we know that Σ_* is not of type $[A_l]$ $(l \geqq 2)$, $[D_l]$ $(l = \text{odd})$ nor $[E_6]$. Therefore we have $s_0 = l$, i.e., $\#\Gamma_0 = \operatorname{rank} G/K$, which implies $\#\Gamma_0 = s(G/K)$. For the space ***AI*** $(n \geqq 2)$ we have $\#\Gamma_0 = [(n+1)/2] = s(G/K)$ and in the case of ***EI*** we have $\#\Gamma_0 = 4 = s(G/K)$.

Thus the proofs of (i), (ii) and (iii) are completed.

(2°) The case where G/K is of type ***DI*** $(q = \text{odd})$. We have $s(G/K) = \operatorname{rank} G/K - 1 = q - 1$ and Σ_* is of type $[B_q]$ in the case $p \geqq$

$q+2$ and is of type $[D_q]$ in the case $p=q$. Thus if we put $\Gamma=\Gamma_0\backslash\{\beta_q\}$ in the case $p\geqq q+2$ and put $\Gamma=\Gamma_0$ in the case $p=q$, the set Γ has the properties (1) and (2) of Proposition 2.2. This can be easily verified by a similar argument as above. The details are left to the reader. q.e.d.

REMARK. The set Γ selected in the above proof possesses the following properties:

(a) If G/K is of type $\boldsymbol{DI}$ ($q=$ odd, $q\geqq q+2$), then it holds $(\alpha,\beta_i)=0$ for each $\alpha\in\Delta(\beta_q)$ and $\beta_i\in\Gamma$.

(b) If G/K is not of type $\boldsymbol{BI,II}$ ($q=$ odd, $p\geqq q+3$), $\boldsymbol{CII}$ nor $\boldsymbol{FII}$, then it holds $m(\beta_i)=1$ for each $\beta_i\in\Gamma$. Hence in these cases we have $\alpha\pm\beta_i\notin\Delta$ for each $\alpha\in\Delta\cap\mathfrak{b}$ and $\beta_i\in\Gamma$ (Lemma (4)). For the spaces $\boldsymbol{BI,II}$ ($q=$ odd, $p\geqq q+3$), $\boldsymbol{CII}$ and $\boldsymbol{FII}$ we have $m(\beta_q)=p-q$, $m(\beta_q)=3$ and $m(\beta_1)=7$, respectively. Therefore for these spaces we have $\alpha\pm\beta_i\in\Delta$ for some $\alpha\in\Delta\cap\mathfrak{b}$ and some $\beta_i\in\Gamma$ (Lemma (4)).

The proofs of these facts are easy (see Tables 1 and 2). The property (b) implies that the space G/K which is not isomorphic to any of the spaces listed in (2°) of Proposition 3.4 satisfies the condition (2) of Proposition 3.3.

Appendix 2. As an application of the modified Gauss equation (Lemma 1.1), we show here a theorem concerning global conformal immersions.

THEOREM (cf. Moore [20], Kobayashi and Nomizu [15]). *Let (M, g) be a compact n-dimensional Riemannian manifold with non-positive sectional curvature. Then (M, g) cannot be conformally immersed into the $(2n-2)$-dimensional Euclidean space $\boldsymbol{R}^{2n-2}$.*

PROOF. Suppose that there exists a conformal immersion of (M, g) into $\boldsymbol{R}^{2n-2}$. Let S^{2n-2} be the hypersphere in $\boldsymbol{R}^{2n-1}$ centered at the origin of $\boldsymbol{R}^{2n-1}$ with radius 1. Since $\boldsymbol{R}^{2n-2}$ is conformally equivalent to S^{2n-2} minus a point, there exists a conformal immersion $\boldsymbol{f}$ of (M, g) into $\boldsymbol{R}^{2n-1}$ whose image is contained in S^{2n-2}. Then we have $\langle\boldsymbol{f},\boldsymbol{f}\rangle=1$, $\langle\nabla\boldsymbol{f},\nabla\boldsymbol{f}\rangle=e^{2\rho}g$, where ρ is a function on M. Since M is compact, ρ attains its minimum value at some point $p\in M$. Then at p we have $\boldsymbol{\alpha}=\nabla\nabla\boldsymbol{f}$ and $\beta=e^{2\rho}(\nabla\nabla\rho)\geqq 0$. We now prove that it holds $\boldsymbol{\alpha}(W,W)\neq 0$ for any non-zero vector $W\in T_pM$. In fact by the equality $\langle\boldsymbol{f},\boldsymbol{f}\rangle=1$ we have $\langle\nabla_W\nabla_W\boldsymbol{f},\boldsymbol{f}\rangle=-\langle\nabla_W\boldsymbol{f},\nabla_W\boldsymbol{f}\rangle=-e^{2\rho}g(W,W)\neq 0$, meaning that $\boldsymbol{\alpha}(W,W)=\nabla_W\nabla_W\boldsymbol{f}\neq 0$. Let T_pM^c and $T_p^{\perp}M^c$ denote the complexifications of T_pM and $T_p^{\perp}M$, respectively. Then $\boldsymbol{\alpha}$ can be naturally extended to a $T_p^{\perp}M^c$-valued complex symmetric bilinear form on T_pM^c, denoted by the same letter $\boldsymbol{\alpha}$. Since $\dim T_p^{\perp}M=\dim T_pM-1$, there exists a non-zero vector $Z\in T_pM^c$ such that $\boldsymbol{\alpha}(Z,Z)=0$. Writing $Z=X+\sqrt{-1}\,Y$ ($X, Y\in T_pM$), we have $\alpha(X, X)=\alpha(Y, Y)$ and $\alpha(X, Y)=0$. By the fact proved above we know that X and Y are linearly independent and $\alpha(X, X)\neq 0$. We now set $k=g(X, Y)/g(X, X)$ and set $Y_1=Y-kX$. Then we have $g(X, Y_1)=0$, $\alpha(Y_1, Y_1)=(1+k^2)\alpha(X, X)$ and $\alpha(X, Y_1)=-k\alpha(X, X)$. Therefore by Lemma 1.1, we obtain

$$\begin{aligned}0 &\geqq -e^{2\rho}g(R(X, Y_1)X, Y_1)\\ &=\langle\alpha(X, X), \alpha(Y_1, Y_1)\rangle-\langle\alpha(X, Y_1), \alpha(Y_1, X)\rangle+\beta(X, X)g(Y_1, Y_1)\\ &\quad+g(X, X)\beta(Y_1, Y_1)-\beta(X, Y_1)g(Y_1, X)-g(X, Y_1)\beta(Y_1, X)\\ &=(1+k^2)\langle\alpha(X, X), \alpha(X, X)\rangle-k^2\langle\alpha(X, X), \alpha(X, X)\rangle\\ &\quad+\beta(X, X)g(Y_1, Y_1)+g(X, X)\beta(Y_1, Y_1)\\ &=\langle\alpha(X, X), \alpha(X, X)\rangle+\beta(X, X)g(Y_1, Y_1)+g(X, X)\beta(Y_1, Y_1)>0\,.\end{aligned}$$

This is a contradiction. q.e.d.

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