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A MODIFIED FORM OF THE VARIATION-OF-CONSTANTS FORMULA FOR EQUATIONS WITH INFINITE DELAY

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. For equations with finite delay, the variation-of-constants formula was given in Halanay's book [2]. Banks [1] pointed out a mistake in this book and presented the correct result. Since equations with finite delay were mainly considered, the results were derived under the restrictive hypotheses: the kernel function $\eta(t, \theta)$ of the linear operator $L(t, \cdot)$ (cf. Theorem 2.1) is constant for sufficiently small $\theta < 0$.

In the present paper, we start from the following hypotheses: $L(t, \phi)$ is continuous and the phase space for ϕ is the general space for equations with infinite delay introduced by Hale and Kato [4]. From the first hypothesis, the Borel measurability of $\eta(t, \theta)$ is naturally induced; from the second, the constant property of $\eta(t, \theta)$ mentioned above cannot be assumed (see Theorem 2.1). The equation related to the fundamental matrix is reduced to the standard equation with infinite delay (Proposition 3.1):

$$x'(t) = L(t, x_t) + h(t) , \tag{1.1}$$

where h is locally integrable. The representation of solutions in Theorem 3.3, which is already announced in [5], has a form that is somewhat different from the variation-of-constants formula given in [1], [2], [3]. For the special phase space $\mathscr{C}_\gamma$ defined in Section 4, our formula is rewritten in a form analogous to the variation-of-constants formula. However, it contains a new term depending on the "exponential limit of the initial function at $-\infty$". Finally, we remark that the present result is an extension of the work for autonomous equations [6] to the case of nonautonomous equations.

2. Representation of linear operators. For a function $x: (-\infty, a) \to C^n$, let $x_t: (-\infty, 0] \to C^n$, $t < a$, be defined by $x_t(\theta) = x(t + \theta)$ for θ in $(-\infty, 0]$. Suppose $\mathscr{B}$ is a linear space of functions $\phi, \psi, \cdots$, mapping $(-\infty, 0]$ into C^n, with a semi-norm $|\phi|, |\psi|, \cdots$ having the following properties.

(H_1) If $x: (-\infty, \sigma + A) \to C^n$, $A > 0$, is continuous on $[\sigma, \sigma + A)$ and $x_\sigma \in \mathscr{B}$, then x_t also lies in $\mathscr{B}$ and x_t is a continuous function of t in $[\sigma, \sigma + A)$.

(H_2) There exist measurable functions $K(t)$ and $M(t)$ of $t \geqq 0$, non-negative and locally bounded, such that

$$|x_t| \leqq K(t-\sigma) \sup\{|x(s)|: \sigma \leqq s \leqq t\} + M(t-\sigma)|x_\sigma|$$

for $\sigma \leqq t < \sigma + A$ and x having the properties in (H_1).

(H_3) $|\phi(0)| \leqq K|\phi|$ for ϕ in $\mathscr{B}$ and some constant K.

Hypothesis (H_1) implies that the space $\mathscr{B}$ contains the space $\mathscr{C}$ of all continuous functions mapping $(-\infty, 0]$ into C^n with compact support. To state the following representation theorem, a definition is needed. A function f mapping $(-\infty, 0]$ into a finite dimensional Banach space, locally of bounded variation on $(-\infty, 0]$, is said to be normalized if $f(0) = 0$ and it is continuous to the left in the interior of $(-\infty, 0]$.

THEOREM 2.1. *Suppose $L: J \times \mathscr{B} \to C^n$, where J is an interval, is a continuous mapping such that $L(t, \phi)$ is linear in $\phi \in \mathscr{B}$ for each t in J. Then there exists an $n \times n$ matrix function $\eta(t, \theta)$ for (t, θ) in $J \times (-\infty, 0]$, locally of bounded variation for θ in $(-\infty, 0]$, such that*

$$L(t, \phi) = \int_{-\infty}^{0} [d_\theta \eta(t, \theta)]\phi(\theta) \quad \text{for} \quad \phi \in \mathscr{C} \tag{2.1}$$

$$\mathrm{Var}\,(\eta(t, \cdot), [-r, 0]) \leqq c|L(t, \cdot)|K(r) \quad \text{for} \quad r > 0\,, \tag{2.2}$$

where c is a constant dependent on the norm of C^n and the integral in (2.1) *is the improper Riemann-Stieltjes integral (cf.* [6, *Theorem* 3.5]). *If $\eta(t, \theta)$ in Relation* (2.1) *is normalized in θ, then it is determined uniquely by L and is Borel measurable for (t, θ) in $J \times (-\infty, 0]$.*

PROOF. The first result of the theorem is a direct consequence of Proposition 3.3 and Theorem 3.5 in [6]; Hypothesis (H_0) in [6] is not used to derive these results. Suppose $\eta(t, \theta)$ is normalized in θ and Relation (2.1) holds. For $m = 1, 2, \cdots$, define a function $\chi^m(t)$ by the relation $\chi^m(t) = -I,\ -m(t + 1/m)I,\ 0$ for t in $[0, \infty)$ $(-1/m, 0)$, $(-\infty, -1/m]$, respectively, where I is the $n \times n$ identity matrix and 0 the $n \times n$ null matrix. Then by integration by parts one has

$$\lim_{m\to\infty} L(t, \chi^m_{-\theta}) = \lim_{m\to\infty} m \int_{\theta - 1/m}^{\theta} \eta(t, r)dr = \eta(t, \theta) \quad \text{for} \quad \theta < 0\,,$$

since $\eta(t, \theta)$ is continuous to the left for $\theta < 0$. This shows that $\eta(t, \theta)$ is determined uniquely by L. Furthermore, it is Borel measurable for

(t, θ) in $J \times (-\infty, 0)$ as a limit function of the sequence of continuous functions $L(t, \chi^m_{-\theta})$. Therefore $\eta(t, \theta)$ is Borel measurable for (t, θ) in $J \times (-\infty, 0]$ since $\eta(t, 0) = 0$ for t in J. q.e.d.

3. Fundamental matrix and representation of solutions. Consider a linear functional differential equation with infinite delay (1.1), where $L: R \times \mathscr{B} \to C^n$ satisfies the assumption in Theorem 2.1 and $h: R \to C^n$ is locally integrable. Since the operator norm $|L(t)| = \sup\{|L(t, \phi)|: |\phi| = 1\}$ is a lower semi-continuous function for t in R, it is Borel measurable; furthermore, from Lemma 3.1 in [7], $|L(t)|$ is locally bounded for t in R. The normalized function $\eta(t, \theta)$, therefore, is locally bounded for (t, θ) in $R \times (-\infty, 0]$; in fact,

$$(3.1) \qquad |\eta(t, \theta)| \leqq c|L(t)|K(-\theta) \quad \text{for} \quad (t, \theta) \in R \times (-\infty, 0] \, .$$

For the following discussion, we set $\eta(t, \theta) = 0$ for $\theta > 0$. Following the arguments similar to the proofs of Theorems 2.1 and 2.4 in [4], one sees that for every (σ, ϕ) in $R \times \mathscr{B}$, there exists a unique solution $x(t, \sigma, \phi, h)$ of Equation (1.1) with $x_\sigma = \phi$ which is locally absolutely continuous for t in $[\sigma, \infty)$ and which satisfies Equation (1.1) a.e. for t in $[\sigma, \infty)$.

To introduce the fundamental matrix of Equation (1.1), we consider the equation

$$(3.2) \qquad \begin{aligned} x'(t) &= \int_{\sigma-t}^{0} d_\theta \eta(t, \theta) x_t(\theta) + g(t) \quad \sigma \leqq t \, , \\ x(\sigma) &= a \, , \end{aligned}$$

where $g: [\sigma, \infty) \to C^n$ is locally integrable.

PROPOSITION 3.1. *Under the above assumptions for L, η and g, Equation* (3.2) *is reduced to Equation* (1.1) *with initial condition $x_\sigma = 0$. Thus for every a in C^n there exists uniquely a locally absolutely continuous function $x(t)$ for $t \geqq \sigma$ such that $x(\sigma) = a$ and that the first relation of* (3.2) *holds a.e. in* $[\sigma, \infty)$.

PROOF. Suppose $x(t)$ is a solution having the above properties. If we set $y(t) = 0$ for $t < \sigma$ and $y(t) = x(t) - a$ for $t \geqq \sigma$, then $y(t)$ satisfies

$$y'(t) = \int_{\sigma-t}^{0} d_\theta \eta(t, \theta) y_t(\theta) - \eta(t, \sigma - t)a + g(t)$$

a.e. in $t \geqq \sigma$. Since y_t lies in $\mathscr{C}$ with $\operatorname{supp} y_t \subset [\sigma - t, 0]$ for $t \geqq \sigma$, this relation is reduced to the equation $y'(t) = L(t, y_t) - \eta(t, \sigma - t)a + g(t)$ a.e. in $t \geqq \sigma$. Since $\eta(t, \theta)$ is Borel measurable and locally bounded for (t, θ) in R^2, the function $\eta(t, \sigma - t)$, as a function of t, is also Borel measurable and locally bounded on R. Hence the equation for y has a unique solution such that $y_\sigma = 0$: this implies that $x(t) = y(t) + a$ for $t \geqq \sigma$ is a unique solution of Equation (3.2) having the desired properties. q.e.d.

The fundamental matrix $X(t, \sigma)$ for (t, σ) in R^2 is defined to be the solution of the equation

$$\frac{\partial X}{\partial t}(t, \sigma) = \int_{\sigma - t}^{0} d_\theta \eta(t, \theta) X(t + \theta, \sigma) \quad \text{a.e. in} \quad t \geqq \sigma ,$$

$$X(\sigma, \sigma) = I \quad \text{and} \quad X(t, \sigma) = 0 \quad \text{for} \quad t < \sigma .$$

In case of finite delay, it is well known that the fundamental matrix has a relation with a certain matrix solution of the formal adjoint equation

$$y(s) + \int_s^t y(\alpha)\eta(\alpha, s - \alpha) d\alpha = b \quad s \leqq t , \tag{3.3}$$

where y and b are in $(C^n)^*$, the space of n-dimensional row vectors. In our case, we will see that this relation is also valid.

The following proposition corresponds to Theorem 3.1, Chapter 6, [3] (see also [1, 2]); the difference is that $\eta(t, \cdot)$ now may not be constant on $(-\infty, -r]$ for any $r > 0$. Since the proof is omitted in [1], [2], [3], we give it briefly in the manner suggested in [2] along with the estimate for the variation of the solution.

PROPOSITION 3.2. *Given t in R and b in $(C^n)^*$, Equation* (3.3) *has a unique solution $y(s)$ for s in $(-\infty, t]$ which is locally of bounded variation. The total variation of y satisfies*

$$\operatorname{Var}(y, [s, t]) \leqq |b| \left\{ \exp\left(\int_s^t c|L(\alpha)|K^*(\alpha - s) d\alpha \right) - 1 \right\} \tag{3.4}$$

where $K^*(r) = \sup\{K(s): 0 \leqq s \leqq r\}$.

PROOF. Suppose $y(s)$ is Borel measurable and locally bounded for s in $(-\infty, t]$ and designate by $(\Omega y)(s)$ the integral term of Equation (3.3). Since $\eta(\alpha, s - \alpha) = 0$ for $\alpha \leqq s$, one has

$$(\Omega y)(s) = \int_\sigma^t y(\alpha)\eta(\alpha, s - \alpha) d\alpha \quad \text{for} \quad \sigma \leqq s \leqq t ,$$

and $\operatorname{Var}(\eta^\alpha, [\sigma, t]) = \operatorname{Var}(\eta^\alpha, [\sigma, \alpha]) \leqq c|L(\alpha)|K(\alpha - \sigma)$ for $\sigma \leqq \alpha \leqq t$, where $\eta^\alpha(s) = \eta(\alpha, s - \alpha)$. This leads to

$$\operatorname{Var}(\Omega y, [\sigma, t]) \leqq \int_\sigma^t |y(\alpha)| c |L(\alpha)| K(\alpha - \sigma) d\alpha , \tag{3.5}$$

which implies Ωy is locally of bounded variation on $(-\infty, t]$. Such a function Ωy is also Borel measurable and locally bounded on $(-\infty, t]$.

From this remark, one can define succesive approximations $y^m(s)$ for $m = 0, 1, 2, \cdots$ as $y^0(s) = b$ and $y^m(s) = b - (\Omega y^{m-1})(s)$ for $s \leqq t$. Then, from Inequality (3.1), one has successively

$$|y^1(s) - y^0(s)| \leqq \int_s^t c|b||L(\alpha)|K(\alpha - s)d\alpha$$

$$|y^2(s) - y^1(s)| \leqq \int_s^t c|L(\alpha)|K(\alpha - s)\left\{\int_\alpha^t c|b||L(u)|K(u - \alpha)du\right\}d\alpha ,$$

for $s \leqq t$. Since $K(r) \leqq K^*(r)$ for $r \geqq 0$ and $K^*(r)$ is nondecreasing, one can replace $K(\alpha - s)$ and $K(u - \alpha)$ in the above inequalities by $K^*(\alpha - s)$ and $K^*(u - s)$, respectively. Thus the following inequality is proved by induction:

$$|y^m(s) - y^{m-1}(s)| \leqq \frac{|b|}{m!}\left\{\int_s^t c|L(\alpha)|K^*(\alpha - s)d\alpha\right\}^m \quad s \leqq t , \tag{3.6}$$

for $m = 1, 2, \cdots$. Therefore $y^m(s)$ converges to a function $y(s)$ uniformly on every compact set of $(-\infty, t]$, and

$$|y(s)| \leqq |b| \exp\left\{\int_s^t c|L(\alpha)|K^*(\alpha - s)d\alpha\right\} \quad s \leqq t . \tag{3.7}$$

This implies that $y(s) = \lim_{m\to\infty} y^m(s) = \lim_{m\to\infty} (b - \Omega y^{m-1}(s)) = b - \Omega y(s)$, that is, $y(s)$ is the solution of Equation (3.3). Since $y^m(s)$ are all Borel measurable for $s \leqq t$, $y(s)$ is also Borel measurable for $s \leqq t$. Since $\mathrm{Var}\,(y, [\sigma, t]) = \mathrm{Var}\,(\Omega y, [\sigma, t])$, one obtains (3.4) by using (3.5) and (3.7).

Suppose $z(s)$ is a solution of (3.3) with $b = 0$, and set $A_\sigma = \sup\{|z(s)|: \sigma \leqq s \leqq t\}$. Then, following arguments similar to the proof of (3.6), one can show that, for $\sigma \leqq s \leqq t$ and $m = 1, 2, \cdots$, $|z(s)|$ is not greater than the right hand side of Inequality (3.6) with $|b|$ replaced by A_σ. Therefore $z(s) = 0$, in other words, the solution of (3.3) is unique for b. q.e.d.

Let $Y(\sigma, t)$ be the matrix solution of the system

$$\begin{aligned} &Y(\sigma, t) + \int_\sigma^t Y(\alpha, t)\eta(\alpha, \sigma - \alpha)d\alpha = I \quad \text{for} \quad \sigma \leqq t \\ &Y(\sigma, t) = 0 \quad \text{for} \quad \sigma > t . \end{aligned} \tag{3.8}$$

From Proposition 3.2, $Y(\sigma, t)$ is locally of bounded variation in σ. Now, suppose $x(t)$ is the solution of Equation (3.2). By integration by parts, one has

$$\int_\sigma^t [d_\alpha Y(\alpha, t)]x(\alpha) + \int_\sigma^t Y(\alpha, t)d_\alpha x(\alpha) = x(t) - Y(\sigma, t)a . \tag{3.9}$$

By the same argument as in Theorem 3.2 in Chapter 6 [3], the second term on the left hand side becomes

$$\int_\sigma^t Y(\alpha, t)\left\{\int_\sigma^t [d_s\eta(\alpha, s-\alpha)]x(s)\right\}d\alpha + \int_\sigma^t Y(\alpha, t)g(\alpha)d\alpha .$$

Using Theorem 2.1, one sees that the Riemann-Stieltjes integral in the first term is the limit of a sequence of Riemann sums which are all Borel measurable for α in $[\sigma, t]$ and whose norms are not greater than $c|L(\alpha)|K(\alpha-\sigma)\sup\{|x(s)|: \sigma \leqq s \leqq t\}$ for α in $[\sigma, t]$. From the bounded convergence theorem, the order of integration and limit operation can be interchanged; thus, one obtains

$$\int_\sigma^t Y(\alpha, t)d_\alpha x(\alpha) = \int_\sigma^t \left[d_s\left\{\int_\sigma^t Y(\alpha, t)\eta(\alpha, s-\alpha)\,d\alpha\right\}\right]x(s) + \int_\sigma^t Y(\alpha, t)g(\alpha)d\alpha , \quad \text{for} \quad \sigma \leqq t .$$

Therefore, using the fact that Y satisfies Equation (3.8), one can rewrite Relation (3.9) as

$$x(t) = Y(\sigma, t)a + \int_\sigma^t Y(\alpha, t)g(\alpha)d\alpha \quad \sigma \leqq t .$$

If one takes $a = I$ and $g(t) = 0$ for $\sigma \leqq t$, one obtains $X(t, \sigma) = Y(\sigma, t)$ for all (t, σ) in R^2; consequently,

$$x(t) = X(t, \sigma)a + \int_\sigma^t X(t, \alpha)g(\alpha)d\alpha \quad \sigma \leqq t . \tag{3.10}$$

To demonstrate the main theorem, we introduce linear operators $S(t): \mathscr{B} \to \mathscr{B}$, $t \geqq 0$, by $[S(t)\phi](\theta) = \phi(t+\theta)$ for $t + \theta \leqq 0$ and $[S(t)\phi](\theta) = \phi(0)$ for $t + \theta > 0$. Hypothesis (H_1) guarantees that $S(t)\phi$ is continuous in t for each fixed ϕ in $\mathscr{B}$.

THEOREM 3.3. *Suppose $L: R \times \mathscr{B} \to C^n$ is continuous, $L(t, \phi)$ is linear for ϕ in $\mathscr{B}$ and $h: [\sigma, \infty) \to C^n$ is locally integrable. Then for every ϕ in $\mathscr{B}$ the solution $x(t, \sigma, \phi, h)$ of Equation* (1.1) *such that $x_\sigma = \phi$ is given by*

$$x(t, \sigma, \phi, h) = \phi(0) + \int_\sigma^t X(t, s)L(s, S(s-\sigma)\phi)ds + \int_\sigma^t X(t, s)h(s)ds \quad \textit{for} \quad t \geqq \sigma . \tag{3.11}$$

PROOF. From the superposition principle, it follows that $x(t, \sigma, \phi, h) = x(t, \sigma, \phi, 0) + x(t, \sigma, 0, h)$ for $t \geqq \sigma$. Since $x(t, \sigma, 0, h)$ is a solution of Equation (3.2) with $a = 0$ and $g = h$, it is equal to the third term on the right hand side of Relation (3.11). Now consider the function

$z(t) = x(t, \sigma, \phi, 0) - y(t)$, where, $y(t) = \phi(t - \sigma)$ for $t \leqq \sigma$ and $y(t) = \phi(0)$ for $t > \sigma$. Then $z(t)$ satisfies $z'(t) = L(t, z_t) + L(t, y_t)$ a.e. in $t \geqq \sigma$, and $z_\sigma = 0$. Since $y_t = S(t - \sigma)\phi$ for $t \geqq \sigma$, from Formula (3.10), it follows that $z(t)$ is equal to the second term on the right hand side of Relation (3.11). Therefore, the relation $x(t, \sigma, \phi, h) = \phi(0) + z(t) + x(t, \sigma, 0, h)$ becomes Relation (3.11) for $t \geqq \sigma$. q.e.d.

4. Applications to equations with the phase space $\mathscr{C}_\gamma$. The representation formula (2.1) is applicable to functions in $\mathscr{C}$. Is it valid for other functions in $\mathscr{B}$? A partial information is given in Theorem 4.4 in [6]; for the space $\mathscr{C}_\gamma$ defined below, however, a complete answer was obtained in [5]. In this section, we summarize this result with some comments.

For any fixed γ in R, let $\mathscr{C}_\gamma$ be the space of continuous functions $\phi: (-\infty, 0] \to C^n$ such that $\tilde{\phi}(-\infty) = \lim_{\theta \to -\infty} e^{-\gamma\theta}\phi(\theta)$ exists in C^n. It is a Banach space with the norm $|\phi| = \sup\{e^{-\gamma\theta}|\phi(\theta)|: \theta \leqq 0\}$, and it satisfies Hypothesis (H_1, H_2, H_3). Changing independent variables, one knows that $\mathscr{C}_\gamma$ is isomorphic to the space $C([-1, 0], C^n)$, the space of continuous functions mapping $[-1, 0]$ into C^n. This observation yields the following result due to Hagemann ([5, Lemma]). If $L: R \times \mathscr{C}_\gamma \to C^n$ is continuous and $L(t, \phi)$ is linear for ϕ in $\mathscr{C}_\gamma$, then there exist matrix functions $\Lambda(t)$ and $\eta(t, \theta)$ such that $\eta(t, \theta)$ is locally of bounded variation for θ in $(-\infty, 0]$ and that

$$(4.1) \qquad L(t, \phi) = \Lambda(t)\tilde{\phi}(-\infty) + \lim_{R\to\infty} \int_{-R}^{0} [d_\theta \eta(t, \theta)]\phi(\theta) \quad \text{for} \quad \phi \in \mathscr{C}_\gamma \,.$$

If $\eta(t, \theta)$ is normalized in θ, then $\Lambda(t)$ and $\eta(t, \theta)$ are determined uniquely by L and they are Borel measurable.

Let $\omega(\gamma)(\theta)$, γ in R, be defined as $\omega(\gamma)(\theta) = \exp(\gamma\theta)$ for $\theta \leqq 0$. Then $\mathscr{C}_\gamma$ is considered as $\mathscr{C}_\gamma = \omega(\gamma)\mathscr{C}_0 = \{\omega(\gamma)\psi: \psi: (-\infty, 0] \to C^n$ is continuous and $\psi(\theta)$ approaches some vector in C^n as $\theta \to -\infty\}$. If $\Lambda(t) \neq 0$, then Formula (2.1) is valid only for ϕ in $\mathscr{C}_\gamma$ with $\tilde{\phi}(-\infty) = 0$, which are in the very restricted subclass of $\mathscr{C}_\gamma$. On the other hand, if Relation (4.1) holds with $\eta(t, \theta)$ normalized in θ, then Theorem 2.1 says that $\eta(t, \theta)$ is unique for L and is Borel measurable for (t, θ). If one set $\phi = \omega(\gamma)a$, a in C^n, then one can compute $\Lambda(t)a$ for every a in C^n; consequently

$$\Lambda(t) = L(t, \omega(\gamma)I) - \lim_{R\to\infty} \int_{-R}^{0} d_\theta \eta(t, \theta)e^{\gamma\theta} \quad \text{for} \quad t \text{ in } R \,.$$

This also shows that $\Lambda(t)$ is unique for L and it is Borel measurable.

Finally, if one applies Representation (4.1) to $L(s, S(s - \sigma)\phi)$ in Formula (3.11), one obtains the following [5, Theorem]

$$x(t, \sigma, \phi, h) = X(t, \sigma)\phi(0) + \left[\int_\sigma^t X(t, s)\Lambda(s)e^{\gamma(s-\sigma)}ds\right]\tilde{\phi}(-\infty)$$
$$+ \lim_{R\to\infty}\int_{-R}^0 d_\theta\left[\int_\sigma^t X(t, s)\eta(s, \sigma+\theta-s)\,ds\right]\phi(\theta)$$
$$+ \int_\sigma^t X(t, s)h(s)ds \quad \text{for} \quad t \geqq \sigma, \ \phi \text{ in } \mathscr{C}_\gamma .$$

This is really an extension of Formula (9) in [1] (see also [3, Theorem 3.2]). In addition to the formal prolongation of the interval of the Riemann-Stieltjes integral, there appears a new term dependent on the limit $\tilde{\phi}(-\infty)$, "the exponential limit of ϕ at $-\infty$".

References

[1] H. T. Banks, Representations for solutions of linear functional differential equations, J. Differential Equations 5 (1969), 399-409.

[2] A. Halanay, "Differential Equations: Stability, Oscillations, Time Lags", Academic Press, New York, 1966.

[3] J. K. Hale, "Theory of Functional Differential Equations", Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[4] J. K. Hale and J. Kato, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11-41.

[5] T. Hagemann and T. Naito, Functional differential equations with infinite delay on the space $\mathscr{C}_\gamma$, Proc. EQUADIFF 82, Lecture Notes in Math. 1017, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983, 207-214.

[6] T. Naito, Fundamental matrix of linear autonomous retarded equations with infinite delay, Tôhoku Math. J. 32 (1980), 539-556.

[7] K. Sawano, Exponential asymptotic stability for functional differential equations with infinite retardations, Tôhoku Math. J. 31 (1979), 363-382.

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