CODIMENSION ONE TOTALLY GEODESIC FOLIATIONS OF H^n

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(Received June 27, 1983)

Introduction. The characterization of isometric immersions of codimension 1 between hyperbolic spaces was proposed by Nomizu in [7], where this problem is compared with the corresponding ones for space forms of positive and zero curvature.

In [4], Ferus classified the umbilic-free immersions $H^n \to H^{n+1}$ sharing a given relative nullity foliation T_0, which was determined by the arbitrary choice of an orthogonal trajectory. As the leaves of T_0 are hyperspaces (complete totally geodesic hypersurfaces) in H^n, such a curve will be enough to determine T_0 uniquely but the converse is not true, for two orthogonal trajectories of a hyperspace foliation need not be related by a congruence (rigid motion of H^n). This paper offers a classification of hyperspace foliations of H^n, up to congruence, which includes non-smooth foliations too. Such generality is needed in the study of immersions that may have umbilics (see [1]).

Basic results of Riemannian geometry assumed here will be found in Kobayashi-Nomizu [6]. Several other facts, more specific of hyperbolic geometry, were less readily available at least in the form needed here. Sections 2 and 3 deal with this material. Preparation of those sections was made easier by [2]. The author wishes to thank K. Nomizu for suggesting the problem that originated this work and for a great deal of further assistance and advice.

1. Notation and Terminology. We shall deal with smooth ($=C^\infty$) manifolds endowed with linear connections. Given such a manifold H, a geodesic will be assumed to have the largest possible domain. Since H will almost always be (geodesically) complete, a geodesic will then be a *map*

$$\sigma: R \to H$$

whose velocity vector is parallel. The *set* $\sigma(R)$ will be referred to as the *path of* σ. A (geodesic) segment is the image, under a geodesic, of a finite interval whereas a (geodesic) ray is the restriction of a geodesic to an interval of type $(-\infty, a]$ or $[a, \infty)$. Unless otherwise stated, a

ray has the domain $\bar{R}_+ = [0, \infty)$. We will use $R_+ = (0, \infty)$.

Given a vector $Z \in TH$, the Z-ray is the ray τ with starting velocity Z. We write $\dot{\tau}(0) = Z$. In particular, if σ is a geodesic, its positive ray is the $\dot{\sigma}(0)$-ray and the negative one is the $(-\dot{\sigma}(0))$-ray.

A complete auto-parallel hypersurface in H will be called a hyperspace of H. If the connection is Riemmanian, auto-parallel is equivalent to totally geodesic.

Let $\pi: TH \to H$ be the tangent bundle of H. The map EXP $= (\pi, \exp): TH \to H \times H$ will be called capital exponential map. Its domain is actually smaller than TH if the connection is not complete. As usual, $\exp_p: T_pH \to H$ is defined by restriction of exp.

A complete simply-connected Riemannian manifold of negative curvature will be called an Hadamard manifold. In such a manifold, given two points p and q, there exists a unique geodesic path through them. We define a geodesic γ_{pq} by requiring that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(d(p, q)) = q$. Here d is the Riemannian distance so that γ_{pq} has unit speed. The capital exponential map of an Hadamard manifold H is a diffeomorphism and so is $\exp_p$ for any $p \in H$.

The symbols g, $\| \; \|$, and d, respectively, will be reserved for the Riemmanian metric, the associated norm, and the induced (Riemannian) distance on M.

2. Hadamard Manifolds.

Let H be an Hadamard manifold and let $p \in H$. Let $\sigma: R \to H$ be a unit-speed geodesic such that $p \notin \sigma(R)$ and let us define $Q: R^2 \to H$ by requiring that, for each $u \in R$, $Q(\cdot, u)$ be the unique geodesic such that $Q(0, u) = p$ and $Q(1, u) = \sigma(u)$. We will call the map Q the pencil over σ with vertex p.

Let us write $r(u) = d(p, \sigma(u))$. Then $r > 0$ and, since $\exp_p: T_pH \to H$ is a diffeomorphism, we may write:

$$Q(t, u) = \exp(\mathrm{tr}\,(u)Z(u))$$

where $Z: R \to S_p$ is a (smooth) curve on the unit sphere $S_p = S_pH$ of T_pH. Restricted to the open set where $t \neq 0$, Q is an injective immersion. Indeed, if $Q(t, u) = Q(t', u')$ with $tt' \neq 0$, then $r(u)Z(u) = (t'/t)r(u')Z(u')$ and $\sigma(u')$ is in the path of $Q(\cdot, u)$. But so is $\sigma(u)$ and it follows that $u = u'$ and $t = t'$. Next we evaluate

$$Q_t = \frac{\partial Q}{\partial t} = (\exp_{p*})_{trZ}\,(rZ)$$

$$Q_u = \frac{\partial Q}{\partial u} = (\exp_{p*})_{trZ}(t(\dot{r}Z + r\dot{Z}))\;.$$

The partials Q_t and Q_u must be independent at a point of the form $(1, u)$ because $p \notin \sigma(R)$. But then Z and $\dot{Z}$ are linearly independent, (i.e., $\dot{Z} \neq 0$) and it follows that Q_* is non-singular for $t \neq 0$, as desired. For the purpose of the next two lemmas let us establish the following notation: K_Q will be the Gaussian curvature of the surface $Q(R_+ \times R)$ or equivalently, the curvature of the induced metric on $R_+ \times R$. Also, given $a < b$ in R, we let Φ_a^b be the length of the curve Z (in S_p) from a to b, i.e.,

$$\Phi_a^b = \int_a^b \| \dot{Z}(u) \| \, du .$$

LEMMA 2.1. *Let $a < b$ be given in R. For $\varepsilon > 0$, let Q_ε be the regular simplex obtained by restricting Q to the set $\{(t, u); a \leq u \leq b, \varepsilon/r(u) \leq t \leq 1\}$. Let φ, α, β be the respective angles at p, $\sigma(a)$, $\sigma(b)$ of the geodesic triangle determined by those three points. Then*

$$\lim_{\varepsilon \to 0} \int_{Q_\varepsilon} K_Q dA = \Phi_a^b + \alpha + \beta - \pi$$

$$\geq \varphi + \alpha + \beta - \pi$$

where dA is the element of area associated with the orientation induced by Q_ε.

PROOF. The inequality part comes from spherical geometry. We observe that φ is the spherical distance $\rho(Z(a), Z(b))$ whereas Φ_a^b is the length of a curve in S_p joining those points.

For the equality part, we apply the Gauss-Bonnet theorem to the simplex Q_ε. The only boundary segment that contributes to the geodesic curvature term is the "fourth" one

$$\delta(u) = Q(\varepsilon/r(u), u)$$

which is traversed in the opposite sense. We have

$$\dot{\delta}(u) = Q_u - (\varepsilon \dot{r}/r^2) Q_t$$

$$= (\exp_*)_{\varepsilon Z} (\varepsilon \dot{Z}) .$$

In particular, $Q_t(\varepsilon/r, u)$ is orthogonal to $\dot{\delta}(u)$ for all u. Thus, the four inner angles are $\pi/2$, α, β, and $\pi/2$. Therefore, we have

$$\int_{Q_\varepsilon} K_Q dA + k(\varepsilon) = (\alpha + \beta + \pi) - 2\pi$$

where

$$k(\varepsilon) = -\int_a^b g(\nabla_u(\dot{\delta}/\|\dot{\delta}\|), -Q_t/r)|_{t=\varepsilon/r(u)} \, du$$

with signs adjusted to take into account the orientations of Q_ε and δ.

Using the orthogonality of $\mathring{\delta}$ and Q_t once more, we transform the integrand into

$$g(\mathring{\delta}/\|\mathring{\delta}\|, \nabla_u(Q_t/r))|_{t=\varepsilon/r}$$

and, since $Q_t/r = \exp_*(Z)$, its covariant derivative is smooth everywhere in R^2, in particular, bounded in $[0, 1] \times [a, b]$. Such a bound applies also to the above integrand and it will enable us to use Lebesgue's dominated convergence theorem. We have

$$\frac{\mathring{\delta}}{\|\mathring{\delta}\|} = \frac{(\exp_*)_{\varepsilon Z}(\dot{Z})}{\|(\exp_*)_{\varepsilon Z}(\dot{Z})\|} \to \frac{\dot{Z}}{\|\dot{Z}\|} \qquad \text{as} \quad \varepsilon \to 0$$

and we already observed that $\dot{Z} \neq 0$. Similarly, we conclude that $\nabla_u(Q_t/r)$ converges to $dZ/du = \dot{Z}$ as $\varepsilon \to 0$. Thus the integrand converges pointwise to $\|\dot{Z}\|$ and we obtain $\lim k(\varepsilon) = -\Phi_a^b$, as needed. $\qquad\square$

For the next lemma we will need the sectional curvature of H. We let $QK(t, u)$ be the sectional curvature of the plane section spanned by $Q_t(t, u)$ and $Q_u(t, u)$.

LEMMA 2.2. *Let a and b be as above and assume that one of them is a point of minimum for r. Let ξ be the corresponding value, i.e., $\xi = \text{dist}(p, \sigma(R))$. Let $\eta = \inf\{-QK(t, u); a \leq u \leq b, 0 \leq t \leq 1\}$. Then*

$$\varphi < \pi/(2 + \eta\xi^2),$$

borrowing notation from Lemma 2.1.

PROOF. We observe first that $QK \geq K_Q$ [6, v. II p. 26]. Therefore, with the notation of 2.1,

$$\int_{Q_\varepsilon} -K_Q dA \geq \int_a^b \int_{\varepsilon/r(u)}^1 -QK\|Q_t \wedge Q_u\| \, dt \, du$$

$$\geq \int_a^b \eta \int_{\varepsilon/r(u)}^1 tr^2 \|(\exp_*)_{trZ}(\dot{Z})\| \, dt \, du$$

and, since the curvature of H is negative, $\exp_*$ is length non-decreasing. Thus,

$$\int_{Q_\varepsilon} -K_Q dA \geq \eta \int_a^b \frac{\xi^2 - \varepsilon^2}{2} \|\dot{Z}(u)\| \, du = \frac{\eta}{2}(\xi^2 - \varepsilon^2)\Phi_a^b$$

and, taking limits as $\varepsilon \to 0$, we obtain

$$\pi - \alpha - \beta - \Phi_a^b \geq \eta\xi^2(\Phi_a^b/2).$$

One of α, β equals $\pi/2$ the other being positive. Hence

$$\pi/2 > (\eta\xi^2/2 + 1)\Phi_a^b > (\eta\xi^2 + 2)\varphi/2$$

as desired. ☐

For the next lemma we generalize the construction of Q, replacing p with a geodesic τ whose path does not meet that of σ and requiring that $Q(0, u) = \tau(u)$ instead of p. Again $r = \|Q_t\|$ depends only on u.

LEMMA 2.3. *The function* $r: \mathbf{R} \to \mathbf{R}_+$ *is strictly convex, i.e.,* $\ddot{r} > 0$. *In particular, it may have only one critical point, which is then its minimum.*

PROOF. Let us differentiate $r^2 = g(Q_t, Q_t)$ twice:

$$r\ddot{r} + \dot{r}^2 = g(\nabla_u^2 Q_t, Q_t) + \|\nabla_u Q_t\|^2$$
$$= g(R(Q_u, Q_t)Q_u, Q_t) + g(\nabla_t\nabla_u Q_u, Q_t) + \|\nabla_u Q_t\|^2 .$$

Bearing in mind that r and $\ddot{r}$ are independent of t, we integrate with respect to $t \in [0, 1]$:

$$r\ddot{r} = \int_0^1 -QK\|Q_t \wedge Q_u\| dt + \int_0^1 (\|\nabla_u Q_t\|^2 - \dot{r}(u)^2)dt .$$

Here we use $\nabla_u Q_u = 0$ for $t = 0$ and $t = 1$. The first integral above is positive because $QK < 0$ and the second one has a nonnegative integrand, namely:

$$\|\nabla_u Q_t\|^2 \geqq g(\nabla_u Q_t, Q_t/r)^2$$

by Schwartz's inequality. It follows that $\ddot{r} > 0$. ☐

If L is a complete totally geodesic submanifold of H and $p \notin L$, there exists $q \in L$ which is closest to p because L is a closed subset of H. Given any $q' \neq q$ in L, consider $\sigma = \gamma_{qq'}$. It is a curve in L so that 0 is a minimum point for $d(p, \sigma(\cdot))$. By Lemma 2.3, it is the unique minimum. It follows that $d(p, q') > d(p, q)$.

DEFINITION 2.4. With the above notation, the point q that realizes dist (p, L) is called the foot of p in L.

REMARK 2.5. We can also conclude that the geodesic (path) through p and q is the only one through p that is perpendicular to L. Indeed, it is straightforward that if a path starting at p meets L orthogonally, the point of intersection must be critical for the restriction of $d(\cdot, p)$ to L.

We mention next some results on triangles that are both well-known and easy to prove [5, p. 73]. Let a, b, c, be the (lengths of the) sides of a geodesic triangle and let α, β, γ, be the respective opposite angles.

Then we have the first law of cosines

$$c^2 \geqq a^2 + b^2 - 2ab \cos \gamma$$

and, as an easy consequence, the second law of cosines

$$c \leqq a \cos \beta + b \cos \alpha \ .$$

Moreover,

$$\alpha + \beta + \gamma \leqq \pi \ .$$

REMARK 2.6.   Those inequalities yield the following result: if (x_n), (y_n) are sequences and p is a point in H, let $X_n = \dot{\gamma}_{px_n}(0)$ and $Y_n = \dot{\gamma}_{py_n}(0)$. Assume that $d(p, x_n) \to \infty$ but $d(x_n, y_n)$ remains bounded as $n \to \infty$.   Then $\rho(X_n, Y_n) \to 0$.   Indeed, by the first law of cosines,

$$d(x_n, y_n)^2 \geqq (d(p, x_n) - d(p, y_n))^2$$
$$+ (1 - \cos \rho(X_n, Y_n))d(p, x_n)d(p, y_n)$$

and $|d(p, x_n) - d(p, y_n)|$ is bounded by $d(x_n, y_n)$ whereas $d(p, x_n)d(p, y_n)$ becomes arbitrarily large.   Therefore $\cos \rho(X_n, Y_n) \to 1$ i.e., $\rho(X_n, Y_n)$ converges to zero.

3.   Asymptotes.   Let us sharpen Lemma 2.3 as follows:

LEMMA 3.1.   *Assume that L is a complete totally geodesic submanifold of H.   The function* dist$(\cdot, L)$ *is convex, i.e., for any geodesic* $\sigma: R \to H$, dist$(\sigma(\cdot), L)$ *is a convex function on* R.

PROOF.   Let σ be given and let $a < b$ be real numbers such that $\sigma([a, b]) \cap L = \varnothing$.   Let $\tau: [a, b] \to L$ be the geodesic segment such that $\tau(a)$ is the foot of $\sigma(a)$ in L and $\tau(b)$, that of $\sigma(b)$.   By Lemma 2.3, the function $r: [a, b] \to R$ defined by $r(u) = d(\sigma(u), \tau(u))$, is convex (and smooth).   Given $t \in [0, 1]$, define u as $(1 - t)a + tb$.   Then

$$\text{dist}\,(\sigma(u), L) \leqq r(u) < (1 - t)r(a) + tr(b)$$

so that we have convexity in $[a, b]$.   If $\sigma(R) \cap L = \varnothing$, we are done while the case $\sigma(R) \subset L$ is trivial.   The remaining case is a single element intersection, say $\{\sigma(c)\}$ where c is the point of minimum for the function dist$(\sigma(\cdot), L)$.   The later is continuous and, by the preceding argument, convex on $(-\infty, c)$ and on (c, ∞).   Thus it is (globally) convex. $\square$

DEFINITION 3.2.   Let L and M be complete totally geodesic submanifolds of H.   We say that M is properly asymptotic to L if:

(i)   There exist $p \in M$, $0 \neq X \in T_pM$, and sequences (t_n) in R_+ and (σ_n) of geodesic rays such that $\sigma_n(t_n) \in L$ and $\dot{\sigma}_n(0) \to X$.

(ii) $L \cap M = \varnothing$.

The meaning of this definition will become clear in 3.4. Note that we must have $t_n \to \infty$; if a subsequence of (t_n) were convergent, then the X-ray itself would meet L, contradicting (ii). We let σ be the X-ray. Let q be any point of L and let τ_n be the unit-speed ray starting at q and going through $\sigma_n(t_n)$. By picking a subsequence, if necessary, we may assume that $\dot\tau_n(0)$ converges in $T_q L$. The limit defines a ray τ in L. To simplify the statement of the next lemma, let us assume that $\|X\| = 1$ in Definition 3.2.

LEMMA 3.3. *In the above notation, $d(\sigma(\cdot), \tau(\cdot))$ is bounded.*

PROOF. Let $a = d(p, q)$ and let $t > 0$. Then, for n large enough, we have $d(\sigma_n(0), p) \leqq 1$ and $t_n \geqq t$. By Lemma 3.1, we have then

$$\text{dist}\,(\sigma_n(t), \tau_n(\mathbf{R})) \leqq \text{dist}\,(\sigma_n(0), \tau_n(\mathbf{R})) \leqq a + 1$$

because dist $(p, \tau_n(\mathbf{R})) \leqq d(p, q) = a$. When $n \to \infty$, we have $\sigma_n(t) \to \sigma(t)$, so that dist $(\sigma(t), \tau(\mathbf{R})) \leqq a + 1$.

Next, let $\tau(t_1)$ be the foot of $\sigma(t)$ in $\tau(\mathbf{R})$. Then

$$d(\sigma(t), \tau(t)) \leqq a + 1 + d(\tau(t_1), \tau(t)) = a + |t - t_1| + 1$$

and

$$|t - t_1| = |d(p, \sigma(t)) - d(q, \tau(t_1))|$$
$$\leqq d(p, q) + d(\sigma(t), \tau(t_1))$$

so that $d(\sigma(t), \tau(t))$ is bounded by $3a + 2$. $\qquad\square$

As a consequence of 3.3, we may assume in the definition (3.2) that the points $\sigma_n(0)$ all coincide with p. Indeed, we may replace the original sequence (σ_n) with the sequence of rays going from p to the points $\tau(n)$. We must still have convergence of the initial velocities to X by Remark 2.6.

For the rest of this section, let us assume that the curvature of H is bounded away from zero. Equivalently (by means of a scale factor), we will assume $K \leqq -1$.

PROPOSITION 3.4. *Let L and M be complete totally geodesic disjoint submanifolds of H. The following are equivalent conditions:*
    (i) *M is properly asymptotic to L.*
    (ii) *There exists a goedesic σ in M such that* dist $(\sigma(t), L) \to 0$ *as* $t \to \infty$.
    (iii) dist $(L, M) = 0$.

PROOF. (i) $\Rightarrow$ (ii) Let us consider σ, τ in L as in the preceding

lemma.   Let $\beta = \text{dist}\,(\sigma(R),\,\tau(R))$.   Given $\delta > \beta$, there exists a point z in $\tau(R)$ such that $\text{dist}\,(z,\,\sigma(R)) < \delta$.   Let y be the foot of z in $\sigma(R)$ and let w, in turn, be the foot of y in $\tau(R)$.   Let $Z = \dot{\gamma}_{yz}(0)$, $W = \dot{\gamma}_{yw}(0)$. Applying the second law of cosines to the triangle with vertices z, y, and w, we obtain

$$d(y,\,w) \leqq d(z,\,y)\cos\rho(Z,\,W)\,.$$
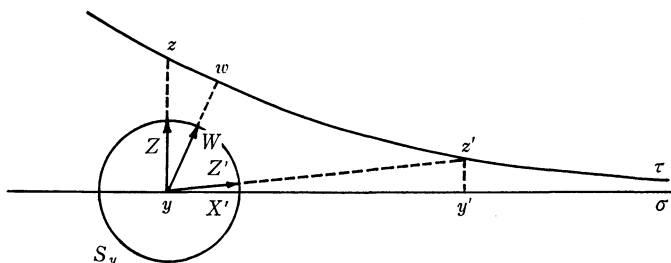
It then follows that $\cos\rho(Z,\,W) \geqq \beta/\delta$.

Now let $z' \in \tau(R)$ and let y' be its foot in $\sigma(R)$.   Let $X' = \dot{\gamma}_{yy'}(0)$ so that X' is the velocity vector of σ at y.   Further let $Z' = \dot{\gamma}_{yz'}(0)$.   Then $\rho(Z,\,X') = \pi/2$.   Therefore,

$$\rho(X',\,Z') \geqq \pi/2 - \rho(Z,\,Z') \geqq \pi/2 - \rho(Z,\,W) - \rho(W,\,Z')$$

and, by Lemma 2.2, $\rho(W,\,Z') < \pi/(2 + \beta^2)$ as long as $d(y,\,w) \geqq \beta$.   Thus

$$(*) \qquad\qquad \rho(X',\,Z') \geqq \frac{\pi\beta^2}{2(2 + \beta^2)} - \cos^{-1}(\beta/\delta)\,.$$

Since $d(y,\,y')$ and $d(y,\,z')$ can be arbitrarily large and $d(z',\,y')$ is bounded, it follows that $\rho(X',\,Z')$ can be made arbitrarily small (Remark 2.6). Thus, the right hand side of $(*)$ cannot be positive.   This forces $\beta = 0$, otherwise δ could be chosen very close to β and $\cos^{-1}(\beta/\delta)$ would be as small as needed to make positive the right hand side of $(*)$.   It follows that $\text{dist}\,(\sigma(R),\,L) = 0$ as well.   The proposed limit results now from convexity.

(ii) $\Rightarrow$ (iii) Immediate.

(iii) $\Rightarrow$ (i) Choose any $p \in M$ and let (x_n) be a sequence in L such that $\text{dist}\,(x_n,\,M) \to 0$ as $n \to \infty$.   Let σ_n be the unit-speed ray at p that goes through x_n and we may assume that $(\dot{\sigma}_n(0))$ converges to $X \in T_pH$, for we can always pick a convergent subsequence.   Since L does not meet M, (x_n) cannot be bounded, so that $t_n \to \infty$.   If σ is the X-ray, then $\lim \text{dist}\,(\sigma(t_n),\,M) = 0$ so that indeed $X \in T_pM$. $\square$

The following result is now immediate:

COROLLARY 3.5. *If M is properly asymptotic to L, then L is properly asymptotic to M.* □

REMARK 3.6. During the proof of 3.4 we obtained:

(a) Given p and σ as in 3.3, any $q \in L$ will be the starting point of a ray τ asymptotic to σ. Consequently, the point p in the definition is by no means unique; using 3.5 we see that any point in M will do.

(b) L and M are properly asymptotic if, and only if, they are disjoint and each contains a geodesic path asymptotic to the other one.

(c) Zero distance may be replaced by boundedness in (ii) of Proposition 3.4.

We are now in position to introduce:

DEFINITION 3.7. Two complete totally geodesic submanifolds of H are asymptotic if one of them contains the other, or if they are properly asymptotic.

Thus, asymptoticity will be reflexive and symmetric as a relation between complete totally geodesic submanifolds although not transitive, as can easily be seen in H^2.

In order to extend the notion of asymptoticity to rays we adopt a different approach.

DEFINITION 3.8. Let σ and $\tau: [0, \infty) \to H$ be rays. We say that τ is asymptotic to σ if $\mathrm{dist}\,(\tau(\cdot), \sigma(R))$ is a bounded function.

We observe at once that the actual parametrizations of σ and τ are irrelevant in the above definition. Also, the roles of σ and τ may be reversed. If σ and τ have the same speed, then the requirement of the definition is equivalent to the boundedness of $d(\sigma(\cdot), \tau(\cdot))$, by 3.3. We can also use the results of this section in a rather straightforward way to obtain properties of the relation of asymptoticity on rays. In particular, if σ and τ are rays of the same speed, 3.4(iii) says that they are asymptotic if, and only if, $d(\sigma(t), \tau(t))$ tends to zero as $t \to \infty$. It follows that asymptoticity is an equivalence relation on the collection of geodesic rays. Finally, we concatenate the two notions by declaring that a ray σ and a complete totally geodesic submanifold L are asymptotic if there exists a ray in L asymptotic to σ.

PROPOSITION 3.9. *Let τ be a geodesic ray in H. Given p in H, there exists a unique unit-speed ray σ starting at p and asymptotic to τ.*

PROOF. To prove uniqueness, if σ and σ_1 fulfill the above conditions, then σ is asymptotic to σ_1 by transitivity. Since $d(\sigma(t), \sigma_1(t))$ is bounded

for $t \to \infty$, we may invoke 2.6 to conclude that $\rho(\dot{\sigma}(0), \dot{\sigma}_1(0))$ is arbitrarily small. Thus $\dot{\sigma}(0) = \dot{\sigma}_1(0)$ whence $\sigma_1 = \sigma$.

For existence, we let $\sigma_n = \gamma_{p\tau(n)}$. Then, some subsequence of (σ_n) will converge to some $X \in T_p H$. Let $\dot{\tau}(0) = X$. $\square$

PROPOSITION 3.10. *Let σ and τ be nonasymptotic rays in H. There exists a unique geodesic path S that is asymptotic to both σ and τ.*

PROOF. By transitivity, we may replace σ (or τ) with an asymptotic ray. Then we may use 3.9 and assume $\sigma(0) = \tau(0)$. Without loss of generality, we may also assume that both are unit-speed rays.

For each integer $n \geq 1$, let S_n be the geodesic path through $\sigma(n)$ and $\tau(n)$ and let q_n be the foot of p in S_n. By uniqueness of the foot (2.3) q_n is distinct from either $\sigma(n)$ or $\tau(n)$. Note that this is also true—in a simple way—when $p \in S_n$ (i.e., when $\dot{\sigma}(0) = -\dot{\tau}(0)$). Thus, $\gamma_n = \gamma_{q_n \tau(n)}$ is a well-defined unit-speed geodesic.

Since $d(p, \gamma_n(\cdot))$ is strictly convex, and since we have $\tau(n)$ in the positive ray of γ_n, $\sigma(n)$ must be in the negative one. Again the case $\dot{\sigma}(0) = \dot{\tau}(0)$ should receive separate attention but it is simple. By 2.1, we have

$$\rho(\dot{\sigma}(0), \dot{\tau}(0)) < 2\pi/(2 + a_n^2)$$

where $a_n = d(p, q_n)$. Since σ and τ are non-asymptotic, the left hand side is positive. Consequently, (a_n) is a bounded sequence. It follows that $\dot{\gamma}_n(0)$ has a subsequence converging to a vector X. It is a simple matter to verify that the X-geodesic γ will provide the desired path S. The positive ray of γ is asymptotic to τ whereas the negative one is asymptotic to σ.

For uniqueness, if γ and β are unit-speed geodesics such that their positive rays are asymptotic to τ and their negative rays, to σ, then $d(\gamma(t), \beta(\mathbf{R})) \to 0$ as $t \to \pm\infty$. By convexity, $\gamma(\mathbf{R}) = \beta(\mathbf{R})$. $\square$

We close this section with a consequence of 3.3 and 3.4:

PROPOSITION 3.11. *Let M and N be complete totally geodesic submanifolds of H. If $\text{dist}(M, N) > 0$, then there exists a unique geodesic path orthogonal to both M and N, to be called the common perpendicular to M and N.*

PROOF. Let $d(M, N) = a$. We claim that the set

$$A = \{y \in N; \text{dist}(y, M) \leq a + 1\}$$

is compact. We need only prove that it is bounded. Assume, for contradiction, that A is unbounded and fix $x \in M$. Then there exist sequences

(y_n) in N and (x_n) in M such that $d(y_n, x) \to \infty$ but $d(x_n, y_n) \leqq a + 1$. Let $Y_n = \dot{\gamma}_{xy_n}(0)$, $X_n = \dot{\gamma}_{xx_n}(0)$. We may assume, using a subsequence, if necessary, that (Y_n) converges to some $Y \in T_x H$. By 2.6, $\rho(X_n, Y_n) \to 0$, whence $Y \in T_x M$. It follows that M is asymptotic to N, in violation of the hypothesis $a > 0$.

Now A is clearly nonempty so that there exists $q \in A$ such that $\mathrm{dist}\,(q, M) = a$. Let p be the foot of q in M. Then q is, in turn, the foot of p in N, and the geodesic path S through p and q is orthogonal to both M and N.

It is clear that S is the only common perpendicular to M and N that contains either p or q. Consider then $p' \neq p$ in M and $q' \neq q$ in N. We construct the geodesic quadrilateral with vertices p, q, q', p', in this order. It has one side in M and another in N. By the formula of Gauss-Bonnet, the sum of internal angles must be strictly less than 2π, provided that the two sides not in either of M and N do not meet. Since the angles at p and q add to π, the side (segment) with endpoints p' and q' cannot be perpendicular to both M and N. In the extreme case where the segments meet, we have two triangles with a common vertex. Again these triangles cannot have two right angles. This proves uniqueness. □

Henceforth, H will denote an Hadamard manifold of dimension greater than 1. A k-dimensional foliation T_0 of H is an integrable distribution T_0 of k-dimensional subspaces or a rank k integrable sub-bundle of the tangent bundle of H. It is not assumed to be smooth. Therefore, integrability is meant in a most direct sense, namely: each point of H is contained in a maximal manifold, called a *leaf* through that point. A hyperspace foliation is one whose leaves are hyperspaces of H. In this case, the uniqueness of the leaf through each point is automatic, regardless of smoothness or even any continuity hypothesis.

4. Limit Units. Let T_0 be a hyperspace foliation of H and let us choose a point $p \in H$ and a unit vector $Y_0 \in T_p H$ such that $Y_0 \perp T_0(p)$. Note that Y_0 is one of two possible choices. Both Y_0 and p will remain fixed throughout the discussion of this section. Let L_0 be the leaf through p. Then L_0 determines two open half-spaces. Specifically, l_0^+ will be the component of $H - L_0$ which contains $\exp{(Y_0)}$ and l_0^- will be the other one. Since L_0 is complete and its complement has exactly two components, it follows easily that each one of these components is a (geodesically) convex subset of H.

If M is another leaf of T_0 such $M \subset l_0^+$, the positive half-space $\mathfrak{m}^+$

determined by M is the one which does not contain L_0. If $M \subset l_0^-$, we exchange "does" with "does not". Of course, the convention depends upon Y_0. On the other hand, it allows us to extend Y_0 to a unit vector field $Y_1 \perp T_0$, parallel along each leaf, by requiring that $\exp_q(Y) \in \mathfrak{m}^+$, if q is in a leaf M as above. We will use superscripts "+" and "−" to distinguish positive and negative half-spaces determined by a given leaf, once Y is fixed as a reference.

Given M and $\mathfrak{m}^+$ as above, we will often say that a set G is *in the positive side* of M to mean $G \subset \mathfrak{m}^+$. If, in addition, G' is in the negative side of M we will say that M *separates* G and G'.

PROPOSITION 4.1. *Given $a > 0$, there exists a unique leaf $L_a \subset l_0^+$ such that* $\operatorname{dist}(p, L_a) = a$.

PROOF. We begin with uniqueness. Let L and M be distinct leaves in the positive side of L_0. We may assume that L is in the negative side of M, since $L \cap M = \varnothing$. Thus, L separates L_0 and M. Hence, the perpendicular segment from p to M must meet L. It follows that there exists $q \in L$ such that $d(p, q) < \operatorname{dist}(p, M)$, so that L is (strictly) closer to p than M is. We conclude that there can be at most one leaf at distance a from p, in the positive side of L_0.

For existence, let D_a be the closed d-ball of radius a around p. Let $D_a^+ = D_a \cap l_a^+$. If L is a leaf through a point $q \in D_a^+$, then $0 \leq \operatorname{dist}(p, L) \leq a$. Thus the set

$$\{\operatorname{dist}(p, L);\ L \text{ is a leaf},\ L \cap D_a^+ \neq \varnothing\}$$

is non-empty and bounded by a. Let b be its supremum. We can be find a sequence (b_n) converging monotonically to b, such that, for each n, there exists a leaf of T_0 at distance b_n from p. We let q_n be the foot of p in that leaf and, selecting a subsequence if needed, we assume that (q_n) converges to some q in the compact set $\overline{D_a^+}$.

Fix an integer $k > 0$. For each $n > k$, $d(p, q_n) \geq d(p, q_k)$ so that q_n cannot be in the negative side of the leaf M through q_k. Consequently, q is in the closed positive half-space of M. This means that $\operatorname{dist}(p, L_b) \geq d(p, q_k)$, where L_b is the leaf through q. Hence $\operatorname{dist}(p, L_b) \geq b$. But $q \in L_b \cap \overline{D_a^+}$ and clearly $q \notin L_0$. It follows that q must be inside $L_b \cap D_a^+$ whence $\operatorname{dist}(p, L_b) = b$.

Finally, suppose $b < a$. Then, we can find b', $b < b' < a$, and let $q' = \gamma\gamma_{pq}(b')$. If L' is the leaf through q', let q' itself is evidence that L' and L_0 are separated by L_b. Hence we have $\operatorname{dist}(p, L') > b$, a contradiction. It follows that $b = a$ and L_b is the promised L_a. $\square$

Henceforth, we assume that the curvature of H is bounded from above by -1 as in §3. We recall that S_p is the unit sphere of T_pH (p and L_0 remain fixed) and ρ stands for spherical distance. Note that S_p is the boundary of $\exp_p^{-1}(D_1)$.

THEOREM 4.2. *There exists a unique vector $X^+ \in S_p$ having the following property: given a leaf $M \subset l_0^+$ and any $\varepsilon > 0$, there exists $Z \in S_p$ with $\rho(Z, X^+) < \varepsilon$ such that the Z-ray meets M, i.e., $\exp(tZ) \in M$ for some $t > 0$.*

PROOF. For each $t > 0$ we let q_t be the foot of p in L_t (notation as in 4.1) and let $X_t = \dot\gamma_{pq_t}(0)$. We will prove that $\lim_{t \to \infty} X_t$ exists (in S_p). Let

$$A_t = \{Z \in S_p; \text{ the } Z\text{-ray meets } L_t\}.$$

Clearly $X_t \in A_t$ for all $t \in \mathbf{R}$ and, by Lemma 2.2, the diameter of A_t is no larger than $2\pi/(t^2 + 2)$. Indeed, by that lemma,

$$A_t \subset \{Z \in S_p; \rho(Z, X_t) < \pi/(t^2 + 2)\}$$

so long as $K \leqq -1$.

Next, we observe that, if $t' > t$, then L_t separates L_0 and $L_{t'}$. Thus, any ray from p that meets $L_{t'}$ must also meet L_t and this implies that $A_{t'} \subset A_t$. Therefore, $(\bar{A}_t)_{t>0}$ is a nested system of compact subsets of S_p with diameters tending to zero as $t \to \infty$. It follows that $\bigcap_{t>0} \bar{A}_t$ is a singleton. Its element is, of course, $\lim X_t$ and we denote it by X^+.

Let M and $\varepsilon > 0$ be given. Then, we can find $t > 0$ such that $\rho(X_t, X^+) < \varepsilon$ and $t > \text{dist}(p, M)$. It follows that the X_t-ray meets M, which separates L_0 and L_t.

Finally, for uniqueness, let $W \in S_p$ be such that $\rho(X^+, W) > 0$. Then we can certainly find $t > 0$ such that $4\pi < (t^2 + 2)\rho(X^+, W)$. If Z is any vector in A_t, $\rho(Z, X^+) \leqq \rho(Z, X_t) + \rho(X_t, X^+) < 2\pi/(t^2 + 2)$. It follows that $\rho(Z, W) > 2\pi/(t^2 + 2)$. Hence W cannot have the required property. $\square$

REMARK 4.3. During the proof of 4.2, we obtained: if we are given $Z_t \in A_t$, for each $t \in \mathbf{R}$, then $Z_t \to X^+$ when $t \to \infty$. Equivalently, if $y_t \in L_t$ are given, then $\dot\gamma_{py_t}(0) \to X^+$ as $t \to \infty$.

DEFINITION 4.4. We will call X^+ the *positive limit unit* at p. Similarly, we have a *negative limit unit* X^-. Of course, the names "positive" and "negative" depend on the choice of Y_0.

PROPOSITION 4.5. *Let $q \in H$ and let M be the leaf through q. Let Z^+ be the positive limit unit at q. Then the X^+-ray and the Z^+-ray are asymptotic.*

PROOF. We consider two cases. First, let us assume that the X^+-ray meets all leaves L_t for $t > 0$. In this case let the intersection be $\{x_n\}$ for each L_n, $n \geq 1$, and let $W_n = \dot{\gamma}_{qx_n}(0)$. For sufficiently large n, L_n will be on the positive side of M. By Remark 4.3, $W_n \to Z^+$ as $n \to \infty$. Thus Z^+ defines a ray asymptotic to the X^+-ray.

Second, let us assume that the X^+-ray does not meet some leaf L_b with $b > 0$. Then it cannot meet any L_t with $t > b$.

Let σ be the X^+-ray. Given integers $n \geq m \geq b$, we let x_n^m be the point of intersection of L_m with the X_n-ray. Then, the sequence $(X_n^m)_{n \geq 1}$ can be used to apply the definition of proper asymptoticity (def. 3.2) to $\sigma(R)$ and L_m. From Proposition 3.4, we can obtain $z_m \in L_m$ such that dist$(z_m, \sigma(R)) < 1/m$.

Let $\tau_m = \gamma_{qz_m}$ and let τ be the Z^+-ray. Then $\dot{\tau}_m(0) \to Z^+$ as long as dist$(q, L_m) \to \infty$. For each $t > 0$, we can find an integer m such that $d(q, z_m) > t$. Using the convexity of dist$(\tau_m(\cdot), \sigma(R))$, we have:

$$\text{dist}(\tau_m(t), \sigma(R)) < \text{dist}(q, \sigma(R)) + 1/m$$
$$\leq d(p, q) + 1/m$$

for large enough m. Taking limits, we obtain:

$$\text{dist}(\tau(t), \sigma(R)) \leq d(p, q) .$$

Thus τ and σ are asymptotic.   □

Evidently, we can do the same with negative limit units at p and q. In particular, if X^+ and X^- coincide at p, all other points will have coincident limit units. In the opposite case, there exists a unique geodesic path S which is asymptotic to the X^+-ray and to the X^--ray, and this S is independent of the choice of p. In this situation,

DEFINITION 4.6. Any unit-speed geodesic having S for path will be called a *waist curve*. If T_0 admits no waist curve (the case $X^+ = X^-$) we say that it is *tight*. We say that it is *loose* when it is not tight and it has no leaf that contains S.

Thus, in a loose foliation, the waist path S (or any waist curve) is transversal to all leaves (in particular it may fail to meet some of the leaves.)

REMARK 4.7. During the proof of Proposition 4.5, we proved that if the X^+-ray fails to meet some leaf L in the positive side of L_0, then it must be asymptotic to L.

5. Constant Curvature. We now restrict our attention to space

forms. Specifically, in this section we will have $H = H^n$ for some $n > 1$. Then hyperspaces are plentiful. In fact, they can be obtained merely by exponentiation of hyperplanes in tangent spaces.

Let us sharpen some of the results obtained in Section 2 while studying the pencil Q over a unit-speed geodesic σ with vertex p. We have $r(u) = d(p, \sigma(u))$ for $u \in R$, and we assume that $\sigma(0)$ is the foot of p in $\sigma(R)$. Bearing in mind that $Q_u(0, u) = 0$, we decompose the Jacobi field $Q_u(\cdot, u)$ according to tangential and normal components:

$$Q_u = ta(u)E_1 + m(u) \sinh (tr(u))E_2$$

where $E_1 = Q_t/r$ and E_2 is uniquely determined by: the frame $\{E_1, E_2\}$ is orthonormal and $m(u) > 0$, $\forall u$. This frame along Q is defined on all of R^2 and parallel along each geodesic $Q(\cdot, u)$. Its orientation is consistent with that induced by Q for $t > 0$.

To determine a, we differentiate $g(Q_t, Q_u) = tar$. Since ∇ is symmetric (torsion zero) and $r^2 = g(Q_t, Q_t)$, we have

$$ar = g(Q_t, \nabla_u Q_t) = r\dot{r}$$

so that $a = \dot{r}$. For m, we use $Q_t \perp E_2$;

$$g(\nabla_u E_2, E_1) = \frac{1}{r}g(\nabla_u E_2, Q_t) = -\frac{1}{r}g(\nabla_u Q_t, E_2)$$

and, using again the symmetry of ∇, we arrive at:

$$g(\nabla_u E_2, E_1) = -m \cosh (tr) \ .$$

Now the vanishing of the acceleration of σ yields:

$$\begin{cases} \ddot{r} - m^2 \sinh r \cosh r = 0 \\ 2m\dot{r} \cosh r + \dot{m} \sinh r = 0 \ . \end{cases}$$

We solve the second equation for m, recalling that $\dot{r}(0) = 0$ and $\|\dot{\sigma}\| = 1$;

$$m(u) = \frac{\sinh r_0}{\sinh^2 r(u)}$$

where $r_0 = r(0)$. Solving the first equation,

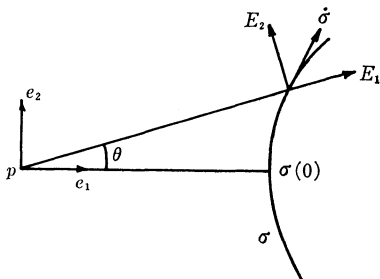$$\dot{r}^2 = 1 - \frac{\sinh^2 r_0}{\sinh^2 r}$$

or

$$(\dot{r} \sinh r)^2 = \cosh^2 r - \cosh^2 r_0$$

and then we obtain

$$\cosh r = \cosh r_0 \cosh u \ .$$

To study the behavior of the frame $\{E_1, E_2\}$ along the u-axis (i.e., at p), we let $e_i = E_i(0, 0)$, $i = 1, 2$,

and we define $\theta \colon \mathbf{R} \to \mathbf{R}$ by requiring that $\theta(0) = 0$ and

$$E_1(0, u) = \cos \theta(u)e_1 + \sin \theta(u)e_2 \,.$$

For $t = 0$ we have $\nabla_u E_1 = m(u)E_2$ and then

$$\dot\theta(u) = m(u) = \frac{\sinh r_0}{\cosh^2 u \sinh^2 r_0 + \sinh^2 u}$$

whence

$$\theta(u) = \tan^{-1}\!\left(\frac{\tanh u}{\sinh r_0}\right).$$

LEMMA 5.1. *Let L be a hyperspace in $H = \mathbf{H}^n$. Let $p \notin L$ and let $X \in S_pH$. Let q be the foot of p in L and let $e_1 = \dot\gamma_{pq}(0)$. Then the X-ray meets L if, and only if, $g(X, e_1) > \tanh(d(p, q))$. The very same ray is asymptotic to L if, and only if, the above inequality becomes an equality.*

PROOF. Let N be a complete totally geodesic surface tangent to e_1 and X (it is uniquely determined unless $X = \pm e_1$). Then $L \cap N$ is a geodesic path which we parametrize as a unit-speed geodesic σ such that $\sigma(0) = q$. Of course, the foot of p in $\sigma(\mathbf{R})$ is still q and the X-ray meets L if, and only if, it meets $\sigma(\mathbf{R})$. But then, it meets $\sigma(\mathbf{R})$ if, and only if, X is in the range of $E_1(0, \cdot)$. This range is defined by the equation $|\tan \theta| < 1/\sinh r_0$, i.e., $\cos \theta > \tanh r_0$.

As for asymptoticity, if it occurs, then the X-ray must be *properly* asymptotic to σ since $p \notin L$. Thus we must have $g(X, e_1) \leq \tanh r_0$. On the other hand, the very definition of proper asymptoticity (3.2) forces the reversal of the inequality. $\square$

Let T_0 be a hyperspace foliation of an open subset G of H and let J be an open interval. Let $\tau \colon J \to G$ be a unit-speed geodesic segment

transversal to T_0, that is, to the leaves of T_0. Given $s \in J$, the leaf $L(s)$ through $\tau(s)$ is completely determined by the unit vector $Y(s)$ at $\tau(s)$ that is orthogonal to $L(s)$ and such that $g(Y(s), \dot{\tau}(s)) > 0$. We have then a unit vector field along τ from which all leaves of T_0 that meet $\tau(J)$ may be recovered. To study the "covariant variation" of Y along τ, we introduce a generalization of the spherical distance ρ; if $a, b \in J$, let

$$\rho_\tau(Y(a), Y(b)) = \rho(\tau_s^a(Y(a)), \tau_s^b(Y(b)))$$

for any $s \in J$. Here, τ_s^a stands for parallel displacement along τ and the choice of s is immaterial.

THEOREM 5.2. *Let T_0, τ, and Y be as above. Then,*

$$\limsup_{a \to b} \frac{\rho_\tau(Y(a), Y(b))}{|b - a|} \leqq g(Y(b), \dot{\tau}(b)) \ .$$

PROOF. Let $a \neq b$ be given in R and let again $L(s)$ denote the leaf through $\tau(s)$. Let $p = \tau(a)$ and let q be the foot of p in $L(b)$.

Let $\sigma = \gamma_{q\tau(b)}$ or, if $q = \tau(b)$, choose any unit-speed geodesic σ in $L(b)$, starting at q. Let N be the complete totally geodesic surface determined by p and σ. If we displace $Y(b)$ parallelly along σ to $T_q H$, we end up with a vector tangent to the segment from p to q. Thus $Y(b)$ must be tangent to N.
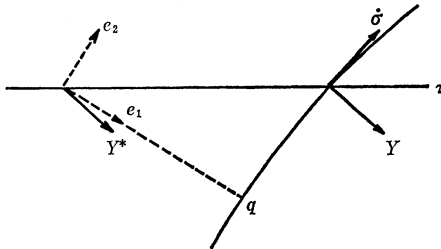
Let $u_1 = d(q, \tau(b))$ and let us consider the pencil Q over σ, with vertex p. We have $\sigma(u) = \tau(b)$ while $Y(b)$ is a linear combination of the vectors $E_2(u_1, 1)$ and $E_1(u_1, 1) = \pm\dot{\tau}(b)$ ($\pm$ according to $a < b$ or $a > b$).

Now we have a vector orthogonal to $Y(b)$ in $T_{\tau(b)}N$ namely

$$\dot{\sigma}(u_1) = \dot{r}(u_1)E_1(1, u_1) + m(u_1) \sinh(r(u_1))E_2(1, u_1) \ ,$$

and we know that $g(Y, \dot{\tau})$ is positive. Therefore we can obtain an explicit expression for $Y(b)$. Indeed, $\tau_a^b(Y(b))$ is the vector

$$Y^* = \pm m_1 E_1(0, u_1) - \dot{r}(u_1)E_2(0, u_1)$$

where $m_1 = m(u_1) \sinh (r(u_1))$ is just $g(Y(b), \dot{\tau}(b))$. Let us assume $a < b$ for the other case is analogous. Then

$$g(Y^*, e_1) = m_1 \cos \theta(u_1) + \dot{r}(u_1) \sin \theta(u_1)$$
$$= \cos (\theta(u_1) - \cos^{-1} m_1)$$

and we observe that the inner angles of the triangle having p, q, and $\tau(b)$ for vertices are, respectively, $\theta(u_1)$, $\sin^{-1} m_1$, and $\pi/2$. It follows that $\theta(u_1) \leqq \pi/2 - \sin^{-1} m_1$ so that $\cos^{-1} m_1 \geqq \theta(u_1)$. Thus

$$\rho(Y^*, e_1) = \cos^{-1} m_1 - \theta(u_1) .$$

Consider now $Y(a)$ and let $c = g(Y(a), e_1)$. Then $c > 0$ because $q = \exp (r_0 e_1)$ is in the positive side of $L(a)$. If $c < 1$, we can define a unit vector X by

$$e_1 = c Y(a) + \sqrt{1 - c^2}\, X$$

and $X \perp Y(a)$. The X-ray is in $L(a)$, which means that it does not meet $L(b)$. Thus $g(X, e_1) \leqq \operatorname{tahn} r_0$ whence

$$g(Y(a), e_1) \geqq \operatorname{sech} r_0 .$$

Of course, this inequality holds also when $c = 1$, i.e., when $Y(a) = e_1$. It is equivalent to $L(a) \cap L(b) = \varnothing$. We apply now the triangle inequality of the distance ρ;

$$\rho_\tau(Y(a), Y(b)) \leqq \cos^{-1} m_1 - \theta(u_1) + \cos^{-1} (\operatorname{sech} r_0)$$

and, by use of the explicit expressions for θ and r, the right hand side becomes $\psi(b - a)$, where

$$\psi(v) = \tan^{-1} (m_1 \sinh v) - \tan^{-1} \left(\frac{\sqrt{1 - m_1^2}}{m_1 \cosh v} \right)$$
$$+ \tan^{-1} \left(\frac{\sqrt{1 - m_1^2}}{m_1} \right)$$

which is a differentiable function such that $\psi(0) = m_1$. The proposed inequality follows now from the definition of derivative. ☐

We observe at this point that hyperspace foliations have been defined without many restrictions. In fact, a hyperspace foliation on a subset G of H is merely a partition of G into hyperspaces: the leaves must be disjoint and their union must be all of G.

COROLLARY 5.3. *A hyperspace foliation on a subset G of H is necessarily continuous (as a distribution).*

PROOF. Given any $q \in G$, let τ be a unit-speed geodesic not in the leaf through $\tau(0) = q$. Then τ will be transversal to all of the leaves

(although it may fail to meet some of them). If we define a unit vector field Y as in 5.2, then $g(Y, \dot{\tau}) \leqq 1$ and it follows that Y is uniformly continuous. The orthogonal complements of the vectors $Y(t)$ define a continuous subbundle $\pi: B \to R$ of the pullback $\tau^*(TH)$ and, since τ is an embedding, we may (and will) identify B with a topological submanifold of TH. Let $E: B \to H$ be the restriction of the exponential map to this submanifold; it maps the fibre over each $t \in R$ diffeomorphically onto the leaf $L(t)$ through $\tau(t)$ and, since the leaves are disjoint, B is injective. Given $a < b$ in R, the leaf $L(a)$ determines half-spaces $l_a^{\pm}$ where the $\pm$ refers to $Z(a)$, in the sense of Section 4; similarly, $L(b)$ determines a half-space pair $l_b^{\pm}$. The open set $G = l_a^+ \cap l_b^-$ is the (disjoint) union of the leaves $L(t)$, with $a < t < b$. Indeed, if $z \in G$, the function $q(Y(\cdot), \dot{\gamma}_{\tau(\cdot)z}(0))$ is continuous on $[a, b]$ and its values on the endpoints have opposite signs. Thus it must vanish at some point c in (a, b), that is, $z \in L(c)$. Setting $\pi_1 \circ E = \pi$ defines $\pi_1: E(B) \to R$ and it has just been proved that $\pi_1^{-1}((a, b))$ is the open set G. It follows that π_1 is continuous; in fact, $E(B) = \pi_1^{-1}(R)$ is open in H. Now

$$E^{-1} = (\mathrm{EXP})^{-1} \circ (\pi_1, \text{identity})$$

and E maps B homeomorphically onto $E(B)$. This proves the continuity of T_0, as a subbundle of TH. $\qquad \square$

LEMMA 5.4. *Let L be a hyperspace of H and let $p \notin L$. If σ is a ray in L, there exists a unique hyperspace M through p which is asymptotic to both σ and L.*

PROOF. For uniqueness, suppose that M_1 and M_2 are hyperspaces as specified and let τ be the unit-speed ray starting at p and asymptotic to σ, so that $\tau(R) \subset M_1 \cap M_2$. We define a unit field Z_i along τ by requiring that $Z_i \perp M_i$ and that M_i not separate L and $\exp(Z_i)$, $i = 1, 2$ (thus Z_i is parallel along τ).

For each $t > 0$, let $b = b(t)$ be the distance from $\tau(t)$ to L and let $Z_0 = Z_0(t)$ be the initial velocity of the unit-speed ray that starts at $\tau(t)$ and is perpendicular to L. Now observe that neither M_1 nor M_2 may meet L as $p \notin L$ implies proper asymptoticity. Just as in the proof of 5.2, we have

$$\rho(Z_i, Z_0) \leqq \cos^{-1}(\mathrm{sech}\, b)\ .$$

But then $\rho(Z_1, Z_2)$ gets arbitrarily small as $t \to \infty$ because $b \to 0$, and $\rho(Z_1, Z_2)$ is constant by parallellism. We have then $Z_1 = Z_2$ whence $M_1 = M_2$.

For existence, we again let τ be the unit-speed ray starting at p and
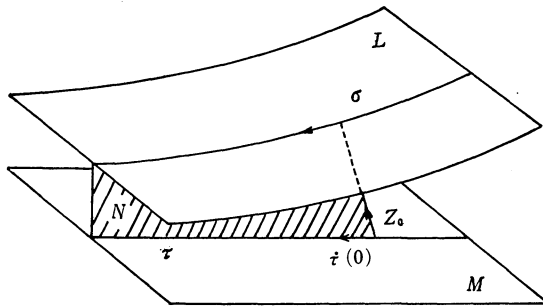
asymptotic to σ, hence, to L. By transitivity, σ may be replaced at will with any other ray in L that belongs to the same class of asymptoticity. Thus we assume that $\sigma(0)$ is the foot of p in L, in order to simplify notation. We again let $Z_0 = Z_0(0) \in S_p$ be the initial velocity of the corresponding perpendicular. Let N be the complete totally geodesic surface tangent to both Z_0 and $\dot{\tau}(0)$. It must contain $\sigma(R)$ because there exists a unique unit-speed ray in N starting at p and asymptotic to τ. This ray must be σ because it is a ray in H as well and we can apply uniqueness from 3.9.

We define M by stipulating that $T_p M$ is spanned by $(T_p N)^\perp$ and $\dot{\tau}(0)$. Then we already have $\tau(R) \subset M$ so that we need only prove that M and L are disjoint.

Given $W \in S_p M$, we write;

$$W = g(W, \dot{\tau}(0))\dot{\tau}(0) + W_0$$

where $W_0 \perp N$. Then $g(W, Z_0) \leqq g(\dot{\tau}(0), Z_0) \leqq \tanh(d(p, \tau(0)))$ so that the W-ray does not meet L. Thus $L \cap M = \varnothing$. ◻

THEOREM 5.5. *Let J be an interval, possibly infinite, and let $\tau: J \to H$ be a geodesic (segment). Let Z be a unit vector field along τ such that $g(Z, \dot{\tau}) > 0$ and, for any $b \in J$,*

$$\limsup_{a \to b} \frac{\rho_\tau(Z(a), Z(b))}{|b - a|} \leqq g(Z(b), \dot{\tau}(b)) .$$

Then, for any $s \neq t$ in J, the hyperspaces of H orthogonal to $Z(s)$ and $Z(t)$ are disjoint. Moreover, if $J = R$, the hyperspaces orthogonal to Z extend to a unique foliation of H having τ as waist.

PROOF. Let $L(s)$ be the hyperspace orthogonal to $Z(s)$, for each $s \in J$. Let us assume, for a contradiction, that $L(a_1) \cap L(a_2) \neq \varnothing$ for $a_1 < a_2$ in J.

We consider first the two-dimensional case, in which the hyperspaces

$L(s)$ are but geodesic paths so that $L(a_1) \cap L(a_2)$ is necessarily a singleton $\{y\}$. Let B_i be the segment between $\tau(a_i)$ and y, $i = 1, 2$.

Now, given any $s \in (a_1, a_2)$, $L(s)$ separates $\tau(a_1)$ and $\tau(a_2)$. Thus, at least one of $L(s) \cap B_1$ and $L(s) \cap B_2$ is nonempty. Let its only element be called y_s. Since Z is continuous, y_s is continuously dependent on s. We let

$$A_i = \{s \in (a_1, a_2); \, y_s \in B_i\}, \qquad i = 1, 2.$$

Then A_1 and A_2 are closed subspaces of (a_1, a_2) whose union is the whole interval. Thus, at least one of them, say A_2, has nonempty interior. We let $b = \sup A_2$. Unless $b = a_2$, we have $b \in A_2$ and then b must also belong to A_1. Thus $L(b)$ contains y. The latter conclusion holds (trivially) when $b = a_2$ too.

Now there exists a strictly increasing sequence (b_k) in A_2 such that $b_k \to b$. For each k, $L(b_k)$ runs from $\tau(b_k)$ to a point in B_2. Thus it must
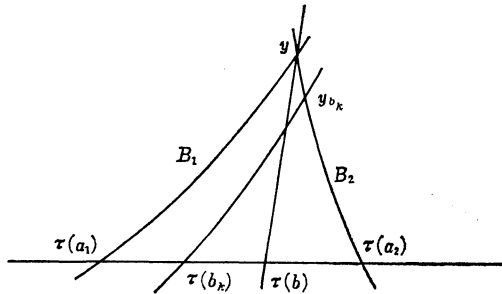
meet $L(b)$ at a point between $\tau(b)$ and y. In other words, $L(b_k) \cap L(b)$ contains a point no farther from $\tau(b)$ than is y.

Returning to the general case, we can choose $y \in L(a_1) \cap L(a_2)$ and apply the preceding argument to the complete totally geodesic surface determined by τ and y. We obtain a sequence (b_k) converging to $b \in J$ such that $b_k < b$ for all $k \geq 1$. For this sequence, there exists a $C > 0$ such that dist $(\tau(b), L(b_k) \cap L(b)) \leq C$, for all k.

To estimate $\rho_\tau(Z(b), Z(b_k))$ let us fix $b_k = a$ and let us invoke the construction of Theorem 5.2. Here we have a unit vector $X \perp Z(a)$ such that the X-ray meets $L(b)$ at a point whose distance from $\tau(b)$ is no larger than C. Thus

$$\tan \rho(X, e_1) \leq \frac{\tanh C}{m_1 \sinh(|b - a|)}.$$

For convenience, let $m_0 = m_1/\tanh C$ so that $m_0 > m_1$. Then,

$$g(X, e_1) \geqq \frac{m_0 \sinh (|b - a|)}{\sqrt{1 + m_0^2 \sinh^2 (|b - a|)}}$$

and $g(X, e_1)^2 + g(Z(a), e_1)^2 \leqq 1$ so that

$$\rho(Z(a), e_1) \geqq \tan^{-1} (m_0 \sinh (|b - a|))$$

and then, as in 5.2, we obtain

$$\rho_\tau(Z(a), Z(b)) \geqq \tan^{-1} (m_0 \sinh (|b - a|)) - \tan^{-1} \frac{\sqrt{1 - m_1^2}}{m_1}$$

$$+ \tan^{-1} \frac{\sqrt{1 - m_1^2}}{m_1 \cosh (|b - a|)} \ .$$

Replacing a with b_k and letting $k \to \infty$, we obtain

$$\limsup_{t \to b} \frac{\rho_\tau(Z(t), Z(b))}{|b - t|} \geqq m_0 > m_1 \ ,$$

a contradiction that proves the first part.

For the second part, $J = R$ and, just as in 5.3, we denote by $\pi: B \to R$ the continuous subbundle orthogonal to Z in $\tau^* TH$, while B is identified with an open subset of TH and $E: B \to H$ is the restriction of exp to B. Given $s \in R$, E maps the fibre over s onto the hyperspace $L(s)$ orthogonal to $Z(s)$ and, by the first part of this proof, E is injective. Keeping the notation $l_a^\pm$ for the half-space pair determined by $L(a)$, we conclude, as in 5.3, that E will map the foliation of B by fibres onto a hyperspace foliation of $E(B)$ and, if $a < b$, $\pi^{-1}(a, b)$ will be mapped onto an open set of the type $l_a^+ \cap l_b^-$, having $L(a) \cup L(b)$ for boundary. The full image $E(B)$ is also open in H but, in general, that will be just a proper subset of H.

Set $p = \tau(0)$ and $c_s = \text{dist} \, (p, L(s))$, each s in R. Then c_s increases (strictly) with $|s|$. If c_s is unbounded for $s < 0$ then $c_s \to \infty$ when $s \to -\infty$ and, for each $y \in l_0^-$, there exists $s < 0$ such that $d(p, y) < c_s$. Thus the segment from p to y does not meet $L(s)$ whence $y \in E(\pi^{-1}(-\infty, 0))$ and $l_0^- \subset E(B)$. Similarly, if $c_s \to \infty$ as $s \to +\infty$, then $l_0^+ \subset E(B)$. If both are true, then $E(B)$ is all of H and no extension of T_0 is necessary.

Now assume $E(B) \neq H$. Then c_s must be bounded on at least one half of the real axis; say $\lim_{s \to \infty} c_s = c \in R^+$. The feet of p in the leaves within l_0^+ form a bounded set and there exists a sequence (q_k) in l_0^+ such that q_k is the foot of p in the respective leaf, $q_k \to q \in H$, and $d(p, q_k) \to c$. As we pointed out above, c_s is monotone for $s > 0$ so that $c > c_s$, $\forall s > 0$. Thus $q \notin E(B)$ and we must find q in the boundary $\partial(E(B))$.

Let Y be the only unit vector field orthogonal to T_0 such that

$Y \circ \tau = Z$. Note that $Y(q_k) \to \dot{\gamma}_{pq}(c)$ because $Y(q_k) = \dot{\gamma}_{pq_k}(d(p, q_k))$. If T_0 has an extension, continuity forces the hyperplane at q to be $\dot{\gamma}_{pq}(c)^\perp$. Let M be the corresponding hyperspace and assume that $E(B) \cap M$ contains some point y. If $b = d(q, y)$, the sequence of geodesics $\sigma_k = \gamma_{q_k y}$ is such that $\dot{\sigma}_k(0) \in T_0(q_k)$ and $\dot{\sigma}_k(0) \to \dot{\gamma}_{qy}(0)$. Hence $\sigma_k(b) \to y$ and $\dot{\gamma}_{qy}(b) \in T_0(y)$ so that the path of γ_{qy} is in the leaf through y contradicting $q \notin E(B)$.

By 4.7, we should also have M asymptotic to τ. To prove it, we let σ_k be the unit-speed ray from q_k to $\tau(t_k)$, where $t_k = \pi_1(q_k)$. Some subsequence of $(\dot{\sigma}_k(0))$ will converge to some vector in $T_q M$ which will, in turn, define a ray σ in M asymptotic to τ.

To complete the construction in l_0^+, we need to provide a leaf through each point in the exterior of $E(B)$ in l_0^+. Given such a point x, a leaf through it should not meet M but it should be asymptotic to τ if the latter is to be a waist curve. Therefore, we need a hyperspace M_x through x asymptotic to both M and τ. By Lemma 5.4, M_x is uniquely determined. We must now show that, if $x, y \in l_0^+ - E(B)$ and $M_x \neq M_y$, then M_x and M_y are disjoint.

Assume that M_x separates p and y (otherwise M_y separates p and x). Then there exists a unique hyperspace M' through y which is asymptotic to M_x and to τ. Just apply 5.4 to M_x and the ray starting at x and asymptotic to τ. But then M' is also asymptotic to σ and M_x separates it from M. Thus M' is asymptotic to M as well. It follows that $M' = M_y$ so that M_y does not meet M_x. The construction is now complete. It can also be applied to the negative side of L_0 if necessary. $\square$

If we restrict τ to a closed interval in the second part of the preceding proof, we obtain:

COROLLARY 5.6. *Let $J = [a, b]$ in 5.5. The hyperspaces $L(s)$ are the leaves of a foliation on a closed subset of H whose boundary is $L(a) \cup L(b)$.* $\square$

Let us fix a unit-speed geodesic τ and a parallel orthonormal frame v along τ such that $v \cdot e_n = \dot{\tau}$. We follow Kobayashi-Nomizu [6] in that an orthonormal frame at $p \in H$ is a linear isometry from R^n to $T_p H$, where R^n carries the standard inner product which renders orthonormal the canonical basis $\{e_1, \cdots, e_n\}$.

Then, the unit fields Z along τ satisfying the limit condition of Theorem 5.5 are in one-to-one correspondence with the set C of curves $\beta = (\beta^1, \cdots, \beta^n)$ on S^{n-1} such that $\beta^n > 0$ and, for all $b \in R$,

$$\limsup_{a \to b} \frac{\rho(\beta(a),\ \beta(b))}{|b - a|} < \beta^n(b) \ .$$

In order to classify the loose hyperspace foliations on H according to congruence, we need only consider those that have τ for waist because any geodesic path can be mapped upon $\tau(R)$ by some isometry.

Now suppose that Z and W are unit fields along τ satisfying the conditions of 5.5. Then they define, via the construction of 5.5, loose hyperspace foliations of H, having τ for waist. If these foliations are congruent, there exist $a \in R$ and an isometry $f \colon H \to H$ such that, for all $t \in R$, $f(\tau(t)) = \tau(t + a)$ and $f_* Z(t) = W(t + a)$, so long as $g(Z, \dot{\tau})$ and $g(W, \dot{\tau})$ are positive. We observe then that f is uniquely determined by a and the linear isometry $\varphi \in O(n)$ such that $v(t + a) \circ \varphi = f_* \circ v(t)$; this equation is independent of t because v is parallel. Since $\varphi \cdot e_n = e_n$, we need only consider the restriction ψ of φ to $R^{n-1} = (e_n)^\perp$.

We define an action of the group $R \times O(n - 1)$ in C:

$$((a, \psi) \cdot \beta)(s) = \psi \cdot \beta(s + a)$$

where $O(n - 1)$ is embedded in $O(n)$ in the usual way, i.e., $\psi \cdot e_n = e_n$. It follows that the classes of congruence of loose hyperspace foliations are in one-to-one correspondence with the orbits of the above action on C.

We complete the classification by proving that any two tight hyperspace foliations are congruent as well as any two that are neither tight nor loose. Consider first two hyperspaces L and M with respective half-space pairs $l^\pm$ and $\mathfrak{m}^\pm$ such that $L \subset \mathfrak{m}^-$ and $M \subset l^+$.

LEMMA 5.7. *Assume L and M to be (properly) asymptotic and let $p \in G = l^+ \cap \mathfrak{m}^-$. Then, a hyperspace N through p separates L and M if, and only if, it is asymptotic to both. Moreover, such an N is uniquely determined by p. All such N define a foliation on G.*

PROOF. If N separates L and M, then we must have $\mathrm{dist}\,(L, N) = \mathrm{dist}\,(N, M) = 0$, because L and M are at distance zero from each other. Thus N is asymptotic to both L and M. Conversely, if N is asymptotic to L and M, then it must be properly so because $p \notin L \cup M$. Since $p \in G$, N must separate L and M.

Let σ be a ray in L asymptotic to M. If a hyperspace N separates L and M, it separates σ and M as well. Thus we have $\mathrm{dist}\,(\sigma(R), N) = 0$ so that N is asymptotic to σ. But then, the requirement that p be contained in $N = N_p$ uniquely determines N_p, according to Lemma 5.4. To prove existence we will again invoke 5.4, taking N_p to be the unique

hyperspace through p asymptotic to L and σ. Let τ be the ray starting at p asymptotic to σ and let $p' \in M$. Then τ is a ray in N_p so that there exists unique a hyperspace M' through p' asymptotic to N_p and τ. But then M' is asymptotic to σ and N_p separates it from L so that M' is asymptotic to L as well. From the uniqueness part of 5.4 it follows that $M' = M$.

Finally, if $q \notin N$, then N_p separates q from either L or M. Let us say L, for instance. Then the very last of the above arguments can be applied with q replacing p' and N_q replacing M to show that L, N_p, N_q, are mutually asymptotic. Since $q \notin N_p$, we have $N_p \cap N_q = \varnothing$. $\square$

Fix a point p_c and a hyperspace L_c through p_c. Let X_c be a fixed unit vector tangent to L_c at p_c. From Lemma 5.6, it follows that there exists a unique hyperspace foliation T_0 on H whose leaves are all asymptotic to L_c and to the X_c-ray. In particular, the leaf through p_c is L_c. Now let T_0 be a tight foliation of H and let us choose a point $p \in H$. Then the limit units X^+ and X^- at p coincide with a single vector X which must be tangent to the leaf L through p. By Remark 4.7, the leaf through any point $q \in H$ can be none other than the unique hyperspace through q which is asymptotic to L and to the X-ray. It follows at once that T_0 is congruent to T_c. Indeed, any isometry of H mapping X to X_c and $T_0(p)$ to $T_c(p_c)$ will serve the purpose of congruence.

For the "neither-tight-nor-loose" (NTNL) case, we choose a unit vector $Y_c \perp T_c(p_c)$ and let T' be the foliation that agrees with T_c on the positive side of L_c while the leaves on the negative side are replaced by the hyperspaces asymptotic to L_c and to the $(-X_c)$-ray. Given any NTNL foliation T_0, choose p in the waist path and choose a unit vector Y orthogonal to $T_0(p)$. By Definition 4.6, the leaf L through p contains the waist. Then the limit units X^+ and X^- at p satisfy $X^+ = -X^-$ and are tangent to L. Just as in the tight case, any isometry of H mapping Y to Y_c and X to X_c, will be a congruence between T_0 and T'.

REMARK 5.8. In case the unit field Z of 5.5 is smooth, the limit condition becomes $\|\nabla_s Z\| \leqq g(Z, \dot{\tau})$. Now, given $s \in J$, we have

$$\dot{\tau}(s) = g(Z(s), \dot{\tau}(s))Z(s) + X$$

where X is some vector orthogonal to $Z(s)$. Let Y be the unit field on H which extends Z parallelly along each leaf. Since $\nabla_X Y = 0$,

$$\nabla_s Z = g(Z, \dot{\tau})(\nabla_Y Y) \circ \tau$$

and it follows that $\|\nabla_Y Y\| \leqq 1$ along τ. But we can find a segment

transversal to the leaves through any point of H. Thus the inequality is valid on all of H. This result is due to Ferus [4, Thm. 2].

REFERENCES

[1] H. BROWNE, Codimension one partial auto-parallel foliations and hyperbolic immersions, to appear.
[2] P. EBERLEIN AND B. O'NEILL, Visibility manifolds, Pacific J. Math. 46 (1973), 45-109.
[3] D. FERUS, On the completeness of nullity foliations, Michigan Math. J. 18 (1971), 61-64.
[4] D. FERUS, On isometric immersions between hyperbolic spaces, Math. Ann. 205 (1973), 193-200.
[5] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.
[6] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, vol. I-II, Interscience, New York, 1963, 1969.
[7] K. NOMIZU, Isometric immersions of the hyperbolic plane into the hyperbolic space, Math. Ann. 205 (1973), 181-192.

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