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THE IRREDUCIBILITY OF AN AFFINE HOMOGENEOUS CONVEX DOMAIN

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Introduction. An affine homogeneous convex domain Ω in the n-dimensional real number space $\boldsymbol{R}^n$ is said to be *reducible* if it is affinely equivalent to a direct product of affine homogeneous convex domains. Otherwise, it is said to be *irreducible*. By using the characteristic function φ_Ω of Ω, we have a Riemannian metric $g_\Omega =$ Hessian of $\log \varphi_\Omega$ on Ω, which is called the *canonical metric* of Ω. The canonical metric is invariant under the group $G(\Omega)$ of all affine automorphisms of Ω (cf. [18], [16]).

With respect to the canonical metric, an affine homogeneous convex domain is a reducible homogeneous Riemannian manifold if it is a reducible convex domain ([16]). It is natural to raise the question whether the irreducibility of a convex domain Ω implies that of the Riemannian manifold (Ω, g_Ω) or not. A homogeneous convex cone is a special case of an affine homogeneous convex domain. It is known that a homogeneous convex cone in $\boldsymbol{R}^n (n \geqq 2)$ is always reducible as a Riemannian manifold ([7], [15]). However, for affine homogeneous convex domains other than homogeneous convex cones, the answer is affirmative. The main purpose of the present paper is to prove this fact. After reviewing results of [18] in §1 and preparing some lemmas in §2, we will prove the main result in §3 (Theorem 3.1).

In §4, we will study Riemannian geometric relations between an affine homogeneous convex domain Ω and the tube domain $D(\Omega)$ over it. It is known that the canonical metric of Ω coincides with the metric induced from the Bergman metric of $D(\Omega)$ (cf. [6]). By using this and a result of [3], we will prove that a tube domain $D(\Omega)$ is irreducible with respect to the Bergman metric if and only if Ω is an irreducible convex domain (Theorem 4.4).

The Bergman metric of an arbitrary homogeneous bounded domain in a complex number space is Einstein (cf. e.g., [5], [12]). In the case of affine homogeneous convex domains, an elementary domain is the only irreducible domain whose canonical metric is Einstein. This fact will be proved in §5 (Theorem 5.1).

In the present paper, the theory of affine homogeneous convex domains and T-algebras developed by E. B. Vinberg ([18], [19]) plays an important role, and also, the results obtained in [14]-[17] will be used. The same terminologies and notation as those in the previous papers will be kept.

1. Preliminaries. In this section, we will briefly recall the fundamental correspondence between affine homogeneous convex domains and T-algebras due to E. B. Vinberg. The full description for them can be found in [18].

Let $\mathfrak{A} = \sum_{1 \leqq i,j \leqq r} \mathfrak{A}_{ij}$ be a T-algebra of rank r provided with an involution $*$. We now employ the following notation:

$$n_{ij} = \dim \mathfrak{A}_{ij}\ , \quad n_i = 1 + \frac{1}{2} \sum_{k \neq i} n_{ik}$$

and

$$e_i = \frac{1}{2\sqrt{n_i}} e_{ii} \quad (1 \leqq i,\, j \leqq r)\ ,$$

where $e_{ii} = 1$ is the unit element of the subalgebra $\mathfrak{A}_{ii} = \boldsymbol{R}$ of $\mathfrak{A}$. General elements of $\mathfrak{A}_{ij}$ will be denoted as $x_{ij}, y_{ij}, z_{ij}, \cdots$, and also an element x of $\mathfrak{A}$ will be represented by the matrix $x = (x_{ij})$, where x_{ij} is the $\mathfrak{A}_{ij}$-component of x. The unit element e of $\mathfrak{A}$ is given by $e = (\delta_{ij})$, where δ_{ij} means the Kronecker delta.

Let us define subsets $T = T(\mathfrak{A})$, $V = V(\mathfrak{A})$ and $X = X(\mathfrak{A})$ of $\mathfrak{A}$ by

$$(1.1) \quad \begin{aligned} &T = \{t = (t_{ij}) \in \mathfrak{A};\ t_{ii} > 0 (1 \leqq i \leqq r),\ t_{ij} = 0 (1 \leqq j < i \leqq r)\}\ , \\ &V = \{tt^*;\ t \in T\} \quad \text{and} \quad X = \{x \in \mathfrak{A};\ x^* = x\}\ , \end{aligned}$$

respectively. Then the set V is a homogeneous convex cone in the real vector space X and the set T is a connected Lie group acting linearly and simply transitively on V.

We next define subsets $T_0 = T_0(\mathfrak{A})$ and $X_0 = X_0(\mathfrak{A})$ of $\mathfrak{A}$ by

$$T_0 = \{t = (t_{ij}) \in T;\ t_{rr} = 1\} \quad \text{and} \quad X_0 = \{x = (x_{ij}) \in X;\ x_{rr} = 0\}\ ,$$

respectively. Let us put

$$(1.2) \qquad \Omega(\mathfrak{A}) = \{tt^*;\ t \in T_0\} = V \cap (X_0 + e)\ .$$

If the rank of $\mathfrak{A}$ is greater than one, then $\Omega(\mathfrak{A})$ is an affine homogeneous convex domain in the affine subspace $X_0 + e$ of the vector space X. Moreover, T_0 is a connected Lie subgroup of T acting simply transitively on $\Omega(\mathfrak{A})$ as affine transformations by

$$(s, tt^*) \in T_0 \times \Omega(\mathfrak{A}) \to (st)(st)^* \in \Omega(\mathfrak{A}) .$$

Conversely, every affine homogeneous convex domain is affinely equivalent to the domain $\Omega(\mathfrak{A})$ given by means of a T-algebra $\mathfrak{A}$.

The Lie algebra $\mathfrak{t}_0$ of T_0 can be identified with the following subspace of $\mathfrak{A}$:

$$\{(t_{ij}) \in \mathfrak{A};\ t_{ij} = 0 (1 \leqq j < i \leqq r) \quad \text{and} \quad t_{rr} = 0\}$$

provided with the bracket relation $[a, b] = ab - ba$. By identifying the tangent space of $\Omega(\mathfrak{A})$ at the point e with the vector space X_0, we have the following linear isomorphism:

$$a \in \mathfrak{t}_0 \to a + a^* \in X_0 .$$

Using this isomorphism and the canonical metric $g_{\Omega(\mathfrak{A})}$ at the point e, we have an inner product $\langle\ ,\ \rangle$ on $\mathfrak{t}_0$. With respect to this inner product, the condition $\langle e_i, e_i \rangle = 1$ holds and the Lie algebra $\mathfrak{t}_0$ is the orthogonal direct sum of the subspaces $\mathfrak{A}_{ii} (1 \leqq i \leqq r-1)$ an $\mathfrak{A}_{ij} (1 \leqq i < j \leqq r)$.

The connection function α and the curvature tensor R for the canonical metric are given by the following formulas (cf. [8]):

$$\alpha\colon \mathfrak{t}_0 \times \mathfrak{t}_0 \to \mathfrak{t}_0 ,$$

$$\langle \alpha(x, y), z \rangle = \frac{1}{2}(\langle [z, x], y \rangle + \langle [z, y], x \rangle + \langle [x, y], z \rangle) ; \tag{1.3}$$

$$R\colon \mathfrak{t}_0 \times \mathfrak{t}_0 \times \mathfrak{t}_0 \to \mathfrak{t}_0 ,$$

$$R(x, y, z) = R(x, y)z = \alpha(x, \alpha(y, z)) - \alpha(y, \alpha(x, z)) - \alpha([x, y], z)$$

for all $x, y, z \in \mathfrak{t}_0$.

From now on, for an affine homogeneous convex domain, we will consider exclusively the domain realized as the form (1.2) by means of a T-algebra.

2. Some lemmas on irreducible domains. In this section, we prepare some lemmas for later use. Let $\Omega = \Omega(\mathfrak{A})$ be an affine homogeneous convex domain and α the connection function for the canonical metric of Ω. For every $x \in \mathfrak{t}_0$, we now define a linear operator A_x on $\mathfrak{t}_0$ by $A_x(y) = \alpha(x, y)$. Let us denote by $\mathscr{A}$ the subspace $\{A_x;\ x \in \mathfrak{t}_0\}$ of $\mathfrak{gl}(\mathfrak{t}_0)$. Then it is known in [9] that the Lie algebra $\mathfrak{h}$ of the holonomy group for the canonical metric is equal to the smallest Lie subalgebra of $\mathfrak{gl}(\mathfrak{t}_0)$ such that $R(x, y) \in \mathfrak{h}$ for all $x, y \in \mathfrak{t}_0$ and $[\mathscr{A}, \mathfrak{h}] \subset \mathfrak{h}$. The homogeneous Riemannian manifold (Ω, g_Ω) is irreducible if and only if $\mathfrak{h}$ is irreducible as a set of linear operators on $\mathfrak{t}_0$.

We first prove the following

LEMMA 2.1. *For an affine homogeneous convex domain Ω, the holonomy algebra $\mathfrak{h}$ coincides with the Lie subalgebra of $\mathfrak{gl}(\mathfrak{t}_0)$ generated by the set $\mathscr{A}$. In particular, the Riemannian manifold (Ω, g_Ω) is irreducible if and only if $\mathscr{A}$ is irreducible on $\mathfrak{t}_0$.*

PROOF. Let $\mathfrak{h}'$ be the Lie algebra generated by the set $\mathscr{A}$. Then by (1.3), we have $R(x, y) \in [\mathscr{A}, \mathscr{A}] + \mathscr{A}$ and $R(x, y) \in \mathfrak{h}'$ for all $x, y \in \mathfrak{t}_0$. It is clear that the condition $[\mathscr{A}, \mathfrak{h}'] \subset \mathfrak{h}'$ is satisfied. Therefore, $\mathfrak{h} \subset \mathfrak{h}'$. On the other hand, from the condition $A_{e_i} = 0 (1 \leqq i \leqq r-1)$ (cf. Lemma 2.1 of [16]), it follows that the equality $A_x = \sum_{1 \leqq i<j \leqq r} A_{x_{ij}}$ holds for every $x = (x_{ij}) \in \mathfrak{t}_0$. By $[e_i, x_{ij}] = (1/2\sqrt{n_i}) x_{ij}$ and (1.3), we have $R(e_i, x_{ij}) = (-1/2\sqrt{n_i}) A_{x_{ij}} (1 \leqq i < j \leqq r)$. Therefore, the equality $A_x = \sum_{1 \leqq i<j \leqq r} (-2\sqrt{n_i}) R(e_i, x_{ij})$ holds, and hence, $\mathscr{A} \subset \mathfrak{h}$. From these, it follows that the Lie algebras $\mathfrak{h}'$ and $\mathfrak{h}$ are identical. q.e.d.

We now introduce the following condition (C) on a T-algebra $\mathfrak{A}$ of rank $r (r \geqq 2)$:

(C) *For every pair (i, j) of indices $1 \leqq i \leqq j \leqq r-1$, there exists a series of indices $1 \leqq i_0, i_1, \cdots, i_p \leqq r-1$ satisfying the conditions $i_0 = i$, $i_p = j$ and $n_{i_{\lambda-1} i_\lambda} \neq 0 (1 \leqq \lambda \leqq p)$.*

We next prove the following

LEMMA 2.2. *Let $\mathfrak{m}$ be an $\mathscr{A}$-invariant subspace of $\mathfrak{t}_0$ containing the subspace $\mathfrak{A}_{jj}$ for some index $j (1 \leqq j \leqq r-1)$. If the condition (C) holds, then $\mathfrak{m} = \mathfrak{t}_0$.*

PROOF. By using Lemma 2.2 of [14] and Lemma 2.1 of [16], we can see that the following identities hold:

$$
\begin{aligned}
A_{x_{ij}}(e_j) &= \frac{1}{2\sqrt{n_j}} x_{ij}\,; \quad A_{x_{jk}}(e_j) = \frac{-1}{2\sqrt{n_j}} x_{jk}\,; \\
A_{x_{ij}}(x_{ij}) &= \frac{1}{2} \| x_{ij} \|^2 \left(\frac{1}{\sqrt{n_i}} e_i - \frac{1}{\sqrt{n_j}} e_j \right); \\
A_{x_{jk}}(x_{jk}) &= \frac{1}{2} \| x_{jk} \|^2 \left(\frac{1}{\sqrt{n_j}} e_j - \frac{1}{\sqrt{n_k}} (1 - \delta_{kr}) e_k \right)
\end{aligned}
\tag{2.1}
$$

for $1 \leqq i < j < k \leqq r$. Therefore the condition $n_{ij} \neq 0 (1 \leqq i < j)$ implies $\mathfrak{A}_{ij} \subset \mathfrak{m}$ and $\mathfrak{A}_{ii} \subset \mathfrak{m}$. Moreover, the condition $n_{jk} \neq 0 (j < k < r)$ implies $\mathfrak{A}_{jk} \subset \mathfrak{m}$ and $\mathfrak{A}_{kk} \subset \mathfrak{m}$, and also, the condition $n_{jr} \neq 0$ implies $\mathfrak{A}_{jr} \subset \mathfrak{m}$. Hence, by using the condition (C), we have $\mathfrak{m} = \mathfrak{t}_0$. q.e.d.

Similarly as in the above lemma, we can prove the following.

LEMMA 2.3. *Let $\mathfrak{m}$ be an $\mathscr{A}$-invariant subspace of $\mathfrak{t}_0$ containing the* *subspace* $\mathfrak{A}_{ij}$ *for some pair* (i, j) *of indices* $1 \leqq i < j \leqq r$ *satisfying* $n_{ij} \neq 0$. *If the condition* (C) *holds and* Ω *is not affinely equivalent to a convex cone, then* $\mathfrak{m} = \mathfrak{t}_0$.

PROOF. By (2.1), $(1/\sqrt{n_i})e_i - (1/\sqrt{n_j})(1 - \delta_{jr})e_j \in \mathfrak{m}$. Therefore, if $j = r$, then the subspace $\mathfrak{A}_{ii}$ is contained in $\mathfrak{m}$. In this case, by Lemma 2.2, we have $\mathfrak{m} = \mathfrak{t}_0$. Now, we can assume that $j \leqq r - 1$. If $n_{kr} = 0$ for every index $k(1 \leqq k \leqq r - 1)$, then the domain Ω is affinely equivalent to a convex cone ([18], [16]). Hence, there exists an index $k(1 \leqq k \leqq r - 1)$ satisfying the condition $n_{kr} \neq 0$. If $k = j$, then the condition $n_{ij}n_{jr} \neq 0$ implies $n_{ir} \neq 0$ (cf. [18] or (1.2) in [16]). Therefore, we can assume that $k \neq j$. We now want to show that the subspace $\mathfrak{A}_{kk}$ is contained in $\mathfrak{m}$. By the condition (C), there exist different indices $1 \leqq i_0, i_1, \cdots, i_p \leqq r - 1$ such that $i_0 = j$, $i_p = k$ and $n_{i_{\lambda-1}i_\lambda} \neq 0$ $(1 \leqq \lambda \leqq p)$. Using (1.3), (2.1) and the condition $(1/\sqrt{n_i})e_i - (1\sqrt{n_j})e_j \in \mathfrak{m}$, we have inductively that $\mathfrak{A}_{i'_{\lambda-1}i'_\lambda} \subset \mathfrak{m}$ for every $\lambda(1 \leqq \lambda \leqq p)$, where $(i'_{\lambda-1}, i'_\lambda)$ means $(i_{\lambda-1}, i_\lambda)$ or $(i_\lambda, i_{\lambda-1})$ in accordance with $i_{\lambda-1} < i_\lambda$ or $i_\lambda < i_{\lambda-1}$. Therefore, from (2.1) and the condition $\mathfrak{A}_{i'_{p-1}k'} \subset \mathfrak{m}$, it follows that $(1/\sqrt{n_{i_{p-1}}})e_{i_{p-1}} - (1/\sqrt{n_k})e_k$ is contained in $\mathfrak{m}$. Again by using (2.1), we have

$$A_{x_{kr}}\left(\frac{1}{\sqrt{n_{i_{p-1}}}}e_{i_{p-1}} - \frac{1}{\sqrt{n_k}}e_k\right) = \frac{1}{2n_k}x_{kr} \in \mathfrak{m}$$

for every $x_{kr} \in \mathfrak{A}_{kr}$, and hence, $\mathfrak{A}_{kr} \subset \mathfrak{m}$. From this, it follows that the condition $A_{x_{kr}}(x_{kr}) = (1/2\sqrt{n_k})\|x_{kr}\|^2 e_k \in \mathfrak{m}$ holds. Therefore, $\mathfrak{A}_{kk} \subset \mathfrak{m}$. By Lemma 2.2, we get $\mathfrak{m} = \mathfrak{t}_0$. q.e.d.

By (1.3), we can easily verify that the following formulas hold:

$$\langle \alpha(a_{ij}, x), b_{ij} \rangle = \frac{1}{2}\left\{\frac{1}{\sqrt{n_j}}(1 - \delta_{jr})\langle x_{jj}, e_j \rangle - \frac{1}{\sqrt{n_i}}\langle x_{ii}, e_i \rangle\right\}\langle a_{ij}, b_{ij} \rangle \tag{2.2}$$

and

$$\langle \alpha(a_{ij}, x), e_i \rangle = \frac{1}{2\sqrt{n_i}}\langle a_{ij}, x_{ij} \rangle \tag{2.3}$$

for all $x \in \mathfrak{t}_0$ and $a_{ij}, b_{ij} \in \mathfrak{A}_{ij}(1 \leqq i < j \leqq r)$;

$$\langle \alpha(a_{ij}, x), e_j \rangle = \frac{-1}{2\sqrt{n_j}}\langle a_{ij}, x_{ij} \rangle \tag{2.4}$$

for all $x \in \mathfrak{t}_0$ and $a_{ij} \in \mathfrak{A}_{ij}(1 \leqq i < j \leqq r - 1)$.

By making use of the above formulas, we can prove the following

LEMMA 2.4. *Let* $\mathfrak{m}$ *be a non-zero* $\mathscr{A}$*-invariant subspace of* $\mathfrak{t}_0$. *If the condition* (C) *holds, then for an arbitrary index* $i(1 \leqq i \leqq r-1)$, *there exists an element* $x \in \mathfrak{m}$ *such that the* $\mathfrak{A}_{ii}$*-component of* x *is non-zero.*

PROOF. Let us take a non-zero element $y \in \mathfrak{m}$ and fix it. We first suppose that there exists an index $j(1 \leqq j \leqq r-1, j \neq i)$ satisfying $y_{jj} \neq 0$. If an index l is smaller than j and satisfies $n_{lj} \neq 0$, then we take a non-zero element $a_{lj} \in \mathfrak{A}_{lj}$ and put $z = A_{a_{lj}}(y) \in \mathfrak{m}$. By the formula (2.2), we have

$$z_{lj} = \frac{1}{2}\left(\frac{1}{\sqrt{n_j}}\langle y_{jj}, e_j\rangle - \frac{1}{\sqrt{n_l}}\langle y_{ll}, e_l\rangle\right)a_{lj}\ .$$

The condition $y_{jj} \neq 0$ implies $y_{ll} \neq 0$ or $z_{lj} \neq 0$. In the case where $z_{lj} \neq 0$, we put $w = A_{z_{lj}}(z) \in \mathfrak{m}$. Then, by (2.3), $w_{ll} \neq 0$. If an index k satisfies the conditions $n_{jk} \neq 0$ and $j < k \leqq r-1$, then by using (2.2) and (2.4) we have similarly an element $w \in \mathfrak{m}$ such that $w_{kk} \neq 0$. Now, we take different indices $1 \leqq i_0, i_1, \cdots, i_p \leqq r-1$ satisfying the conditions $i_0 = j$, $i_p = i$ and $n_{i_{\lambda-1}i_\lambda} \neq 0 (1 \leqq \lambda \leqq p)$. Then, by the above arguments, we have elements $x^{(1)}, x^{(2)}, \cdots, x^{(p)}$ in $\mathfrak{m}$ such that $x^{(\lambda)}_{i_\lambda i_\lambda} \neq 0 (1 \leqq \lambda \leqq p)$. Hence, putting $x = x^{(p)}$ we get $x \in \mathfrak{m}$ such that $x_{ii} \neq 0$. We next suppose that there exist two indices j and $k(1 \leqq j < k \leqq r)$ satisfying the condition $y_{jk} \neq 0$. Putting $z = A_{y_{jk}}(y)$ and using (2.3), we have $z_{jj} = (1/2\sqrt{n_j})\|y_{jk}\|^2 e_j \neq 0$. By using the same argument as in the first case, we get a desired element $x \in \mathfrak{m}$. q.e.d.

3. Irreducible domains. In this section, by making use of the lemmas obtained in the previous section, we prove the main result of this paper.

The homogeneous convex domain $\Omega(n)$ in $\boldsymbol{R}^n (n \geqq 2)$ defined by

$$\Omega(n) = \{(y^1, y^2, \cdots, y^n) \in \boldsymbol{R}^n;\ y^1 > (y^2)^2 + (y^3)^2 + \cdots + (y^n)^2\} \tag{3.1}$$

is called the *elementary domain* of dimension n.

We now prove the following

THEOREM 3.1. *Let* Ω *be an affine homogeneous convex domain which is not affinely equivalent to a homogeneous convex cone. Then the following conditions are equivalent*:

(1) Ω *is an irreducible convex domain*;

(2) *The condition* (C) *stated in* §2 *holds for the T-algebra* $\mathfrak{A}$ *satisfying* $\Omega = \Omega(\mathfrak{A})$;

(3) (Ω, g_Ω) *is an irreducible Riemannian manifold.*

PROOF. The implications $(1)\rightarrow(2)$ and $(3)\rightarrow(1)$ have been proved in [17] and [16], respectively. So, it remains for us to prove that the implication $(2)\rightarrow(3)$ holds. We now suppose that the rank of the T-algebra $\mathfrak{A}$ is equal to two. If $n_{12}=0$, then Ω is affinely equivalent to the cone of all positive real numbers. Therefore, the number n_{12} must be positive and Ω is affinely equivalent to the elementary domain $\Omega(n_{12}+1)$ (cf. [18]). The elementary domain $\Omega(n_{12}+1)$ is a hyperbolic space form of the sectional curvature $-1/(2n_{12}+4)$ ([16]), and hence, (Ω, g_Ω) is irreducible. So, we can assume that the rank r of the T-algebra $\mathfrak{A}$ is greater than two. Now, let $\mathfrak{m}$ be an arbitrary non-zero $\mathscr{A}$-invariant subspace of $\mathfrak{t}_0$. We first prove that $\mathfrak{m}$ coincides with $\mathfrak{t}_0$ in the case where the condition $n_{ij}n_{jr}\neq 0$ holds for some indices i and $j(1\leqq i<j\leqq r-1)$. By Lemma 2.4, there exists $x\in\mathfrak{m}$ satisfying $x_{ii}\neq 0$. Let us take arbitrary elements $a_{ir}\in\mathfrak{A}_{ir}$ and $a_{jr}\in\mathfrak{A}_{jr}$, and let us denote by A_i and A_j the linear operators $A_{a_{ir}}$ and $A_{a_{jr}}$ on $\mathfrak{t}_0$, respectively. Then by using Lemma 2.2 of [14] and Lemma 2.1 of [16], we get the following equality:

$$
\begin{aligned}
(3.2)\qquad A_i(x) &= \frac{1}{2\sqrt{n_i}}\langle x_{ir}, a_{ir}\rangle e_i - \frac{1}{2\sqrt{n_i}}\langle x_{ii}, e_i\rangle a_{ir} \\
&\quad + \frac{1}{2}\sum_{1\leqq k<i}(x_{kr}a_{ir}^* - x_{ki}a_{ir}) \\
&\quad + \frac{1}{2}\sum_{i<k<r}(a_{ir}x_{kr}^* - x_{ik}^*a_{ir})\ .
\end{aligned}
$$

Similarly by using the condition (1.3) of [16] and the formulas (51), (52) in p. 392 of [18], we have the following formulas:

$$
\begin{aligned}
A_j(A_i(x)) &= \frac{-1}{4\sqrt{n_j}}\langle x_{ij}^*a_{ir}, a_{jr}\rangle e_j - \frac{1}{4\sqrt{n_i}}\langle x_{ii}, e_i\rangle a_{ir}a_{jr}^* \\
&\quad - \frac{1}{4}\sum_{1\leqq k<i}(x_{ki}a_{ir})a_{jr}^* - \frac{1}{4}\sum_{i<k<j}(x_{ik}^*a_{ir})a_{jr}^* \\
&\quad - \frac{1}{4}\sum_{j<k<r}a_{jr}(a_{ir}^*x_{ik}) - \frac{1}{4}(a_{ir}x_{jr}^*)a_{jr}\ ; \\
A_i(A_j(A_i(x))) &= \frac{1}{8\sqrt{n_i}}\langle x_{ii}, e_i\rangle(a_{jr}a_{ir}^*)a_{ir} - \frac{1}{8\sqrt{n_i}}\langle a_{ir}a_{jr}^*, a_{ir}x_{jr}^*\rangle e_i\ ;
\end{aligned}
$$

$$
(3.3)\qquad A_j(A_i(A_j(A_i(x)))) = \frac{1}{16\sqrt{n_in_j}}\langle x_{ii}, e_i\rangle\langle a_{ir}a_{jr}^*, a_{ir}a_{jr}^*\rangle e_j\ .
$$

Here, we take non-zero elements $a_{ij}\in\mathfrak{A}_{ij}$ and $a_{jr}\in\mathfrak{A}_{jr}$, and put $a_{ir}=a_{ij}a_{jr}$. Then $a_{ir}a_{jr}^*\neq 0$. Hence, the conditions $x_{ii}\neq 0$ and (3.3) imply that $\mathfrak{A}_{jj} \subset \mathfrak{m}$. Therefore, by Lemma 2.2, we have $\mathfrak{m} = \mathfrak{t}_0$. We next prove that $\mathfrak{m}$ coincides with $\mathfrak{t}_0$ in the case where the condition $n_{ij}n_{jr} = 0$ holds for every pair (i, j) of indices $1 \leqq i < j \leqq r-1$. Since Ω is not affinely equivalent to a homogeneous convex cone and the condition (C) holds, there exists a pair (i, j) of indices $1 \leqq i < j \leqq r-1$ such that the conditions $n_{ij}n_{ir} \neq 0$ and $n_{jr} = 0$ are satisfied. By Lemma 2.4, there exists an element $x \in \mathfrak{m}$ satisfying $x_{ii} \neq 0$. By (3.2), we have

$$A_i(x) = \frac{1}{2\sqrt{n_i}}\langle x_{ir}, a_{ir}\rangle e_i - \frac{1}{2\sqrt{n_i}}\langle x_{ii}, e_i\rangle a_{ir} \in \mathfrak{m} .$$

If $x_{ir} = 0$, then $\mathfrak{A}_{ir} \subset \mathfrak{m}$. Hence, by Lemma 2.3, $\mathfrak{m} = \mathfrak{t}_0$. If $x_{ir} \neq 0$, then we take an arbitrary element $a_{ij} \in \mathfrak{A}_{ij}$. By Lemma 2.2 of [14] and Lemma 2.1 of [16], we have

$$A_{a_{ij}}(A_i(x)) = -\frac{1}{4n_i}\langle x_{ir}, a_{ir}\rangle a_{ij} \in \mathfrak{m} .$$

Hence, $\mathfrak{A}_{ij} \subset \mathfrak{m}$, and again by Lemma 2.3, we get $\mathfrak{m} = \mathfrak{t}_0$. Therefore, $\mathscr{A}$ is irreducible on $\mathfrak{t}_0$, and by Lemma 2.1, the Riemannian manifold (Ω, g_Ω) is irreducible. q.e.d.

Finally in this section, we remark that in the case of homogeneous convex cones, the condition (C) was treated in [1].

4. **Tube domains.** For an affine homogeneous convex domain Ω in $\boldsymbol{R}^n$, the domain $D(\Omega) = \{x + \sqrt{-1}y \in \boldsymbol{C}^n;\ y \in \Omega\}$ is called the *tube domain* over Ω. In this section, we study Riemannian geometric relations between Ω and $D(\Omega)$.

The group of all affine transformations:

$$x + \sqrt{-1}y \in D(\Omega) \rightarrow (Ax + a) + \sqrt{-1}Ay \in D(\Omega) \quad (A \in G(\Omega),\ a \in \boldsymbol{R}^n)$$

acts on $D(\Omega)$ transitively. Therefore, $D(\Omega)$ is holomorphically equivalent to a homogeneous bounded domain in $\boldsymbol{C}^n$, and there exists the Bergman kernel function $k\colon D(\Omega) \times D(\Omega) \rightarrow \boldsymbol{C}$ of $D(\Omega)$. By using properties of the Bergman kernel function (cf. Lemma 6.1 of [12]) and the characteristic function φ_Ω of Ω (cf. [18]), we can see that there exists a positive number c satisfying

$$k(z, z) = c(\varphi_\Omega(y))^2 \tag{4.1}$$

for all $z = x + \sqrt{-1}y \in D(\Omega)$. We now denote by $g_{D(\Omega)}$ the Bergman metric of $D(\Omega)$, that is,

$$g_{D(\Omega)}(z) = 2\sum_{1 \leqq i, j \leqq n} \frac{\partial^2 \log k(z, z)}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j \tag{4.2}$$

for $z=(z^1, z^2, \cdots, z^n) \in D(\Omega)$ (cf. p. 73 of [12]).

The following proposition is well known but for the sake of completeness, we give a proof here (cf. [6]).

PROPOSITION 4.1. *For an affine homogeneous convex domain Ω in $\boldsymbol{R}^n$, the Riemannian manifold (Ω, g_Ω) is a totally geodesic submanifold of the tube domain $(D(\Omega), g_{D(\Omega)})$ by a natural imbedding $\sigma: y \in \Omega \to \sqrt{-1}y \in D(\Omega)$.*

PROOF. By (4.1) and (4.2) we have

$$g_{D(\Omega)}(z) = \sum_{1 \leqq i,j \leqq n} \frac{\partial^2 \log \varphi_\Omega(y)}{\partial y^i \partial y^j} dz^i d\bar{z}^j \tag{4.3}$$

for all $z = x + \sqrt{-1}y \in D(\Omega)$, $z^i = x^i + \sqrt{-1}y^i (1 \leqq i \leqq n)$. Therefore, it can be easily verified that $\sigma^* g_{D(\Omega)} = g_\Omega$ holds on Ω. We next consider the following mapping

$$\tau: x + \sqrt{-1}y \in D(\Omega) \to -x + \sqrt{-1}y \in D(\Omega) .$$

Using (4.3), we can see that τ is isometric with respect to the Bergman metric. Moreover, the set of all fixed points of τ coincides with $\sigma(\Omega)$. Therefore, (Ω, g_Ω) is a totally geodesic submanifold of $(D(\Omega), g_{D(\Omega)})$ (cf. §8 in Chap. VII of [5]). q.e.d.

We now give a typical example of tube domains.

EXAMPLE 1([6]). Let Ω be the elementary domain in $\boldsymbol{R}^n (n \geqq 2)$ defined by (3.1). We consider the following holomorphic imbedding

$$\Phi: D(\Omega) \to \boldsymbol{C}^n,\ \Phi(z) = \left(z^1 + \frac{\sqrt{-1}}{2} \sum_{2 \leqq k \leqq n} (z^k)^2, \frac{1}{\sqrt{2}} z^2, \cdots, \frac{1}{\sqrt{2}} z^n\right)$$

for all $z = (z^1, z^2, \cdots, z^n) \in D(\Omega)$. Then the image $\Phi(D(\Omega))$ coincides with the domain $\{z \in \boldsymbol{C}^n; y^1 > \sum_{2 \leqq k \leqq n} |z^k|^2\}$, which is holomorphically equivalent to the open unit ball in $\boldsymbol{C}^n$. It should be noted that the domain $D(\Omega)$ can not be realized as a tube domain over a convex cone (cf. [3], [10]).

By using the above proposition we can prove the following.

THEOREM 4.2. *An affine homogeneous convex domain Ω is Riemannian symmetric with respect to the canonical metric if and only if the tube domain $D(\Omega)$ is Hermitian symmetric with respect to the Bergman metric.*

PROOF. In order to prove the assertion, we can assume that Ω is an irreducible domain (cf. Proposition 1.1 of [16]). Let (Ω, g_Ω) be Riemannian symmetric. Then by Theorem 4.2 of [16], Ω is affinely equivalent either to an irreducible homogeneous self-dual cone or to an elementary domain.

The assertion in the first case was proved by Rothaus [11]. In the second case, by Example 1, $D(\Omega)$ is holomorphically equivalent to an open unit ball. Therefore, $(D(\Omega), g_{D(\Omega)})$ is Hermitian symmetric. The converse assertion follows from Proposition 4.1. q.e.d.

We next consider the irreducibility of a convex domain and the tube domain over it. For this purpose, we employ the notion of Siegel domains from [2], [6] and [10].

Let X and Y be real vector spaces and V_0 be a homogeneous convex cone in X. Then a symmetric bilinear mapping $F: Y \times Y \to X$ is called a *homogeneous V_0-symmetric form* if the following three conditions are satisfied: (1) $F(y, y) \in \bar{V}_0$ (the topological closure of V_0 in X) for every $y \in Y$; (2) $F(y, y) = 0$ implies $y = 0$; (3) The subgroup of $G(V_0)$ defined by $\{A \in G(V_0);$ there exists $B \in \mathrm{GL}(Y)$ such that $AF(y, y) = F(By, By)$ for all $y \in Y\}$ is transitive on V_0.

For a homogeneous V_0-symmetric form $F: Y \times Y \to X$, the *real Siegel domain*

$$\Omega(V_0, F) = \{(x, y) \in X \times Y;\ x - F(y, y) \in V_0\} \tag{4.4}$$

is an affine homogeneous convex domain. Let X^C (resp. Y^C) be the complexification of X (resp. Y). Then from the affine homogeneous convex domain $\Omega(V_0, F)$, we can construct a homogeneous Siegel domain in $X^C \times Y^C$ as follows: Let $F^C: Y^C \times Y^C \to X^C$ be the Hermitian extension of F, that is,

$$\begin{aligned} F^C(y_1 + \sqrt{-1}y_2, y_3 + \sqrt{-1}y_4) = \{&F(y_1, y_3) + F(y_2, y_4)\} \\ &+ \sqrt{-1}\{F(y_2, y_3) - F(y_1, y_4)\}\,. \end{aligned}$$

Then F^C is a V_0-Hermitian form on Y^C and the domain

$$D(V_0, F^C) = \{(z, u) \in X^C \times V^C;\ \mathrm{Im}\, z - F^C(u, u) \in V_0\} \tag{4.5}$$

is a homogeneous Siegel domain of type II (cf. [10]).

LEMMA 4.3. *For an affine homogeneous convex domain $\Omega = \Omega(V_0, F)$, the tube domain $D(\Omega)$ is holomorphically equivalent to the Siegel domain $D(V_0, F^C)$.*

PROOF. Following Gindikin [2] we consider the holomorphic imbedding

$$\Phi: (z, u) \in D(\Omega) \to \left(z + \frac{\sqrt{-1}}{2}F^C(u, \bar{u}), \frac{1}{\sqrt{2}}u\right) \in X^C \times Y^C\,.$$

Then by (4.4) and (4.5) we can verify that Φ maps $D(\Omega)$ onto $D(V_0, F^C)$. q.e.d.

We next give an example of a non-symmetric tube domain by making use of the above lemma.

EXAMPLE 2. Let us consider the affine homogeneous convex domain

$$\Omega = \{y = (y_{ij}) \in H^+(3, \boldsymbol{C});\ y_{33} = 1\}$$

in the affine subspace $\{y = (y_{ij}) \in H(3, \boldsymbol{C});\ y_{33} = 1\}$ of $H(3, \boldsymbol{C})$, where $H^+(p, \boldsymbol{C})$ is the cone of all positive definite elements in the real vector space $H(p, \boldsymbol{C})$ of all complex Hermitian matrices of degree p. Then Ω is realized as a real Siegel demain over the cone $H^+(2, \boldsymbol{C})$ by the following symmetric bilinear mapping

$$F: \boldsymbol{C}^2 \times \boldsymbol{C}^2 \to H(2, \boldsymbol{C}),\ F(y_1, y_2) = \frac{1}{2}({}^t y_1 \bar{y}_2 + {}^t y_2 \bar{y}_1)\ .$$

Using Theorem 4.2 and Lemma 4.3, we can easily show that the tube domain $D(\Omega)$ is holomorphically equivalent to the non-symmetric Siegel domain given by (2.7.b) in p. 43 of [4].

Let $D(V, F)$ be a homogeneous Siegel domain associated with a homogeneous convex cone V and a V-Hermitian form F. Then it was proved by Kaneyuki [3] that $D(V, F)$ is irreducible with respect to the Bergman metric if and only if the cone V is irreducible. Making use of this and the results obtained above, we can prove the following

THEOREM 4.4. *An affine homogeneous convex domain Ω in $\boldsymbol{R}^n (n \geqq 2)$ is irreducible if and only if the tube domain $D(\Omega)$ is irreducible with respect to the Bergman metric.*

PROOF. As was noted above, the assertion was proved by Kaneyuki [3] in the case of homogeneous convex cones. So, we may assume that Ω is not affinely equivalent to a homogeneous convex cone. Let $\mathfrak{A}$ be a T-algebra of rank $r (r \geqq 2)$ such that Ω is affinely equivalent to the domain $\Omega(\mathfrak{A})$ defined by (1.2). Then by Theorem 3.1, Ω is irreducible if and only if the condition (C) holds for the T-algebra $\mathfrak{A}$. On the other hand, the convex domain $\Omega(\mathfrak{A})$ is realized as a real Siegel domain as follows: Let $\mathfrak{A}_0$ be the subspace of $\mathfrak{A}$ defined by $\mathfrak{A}_0 = \sum_{1 \leqq i, j \leqq r-1} \mathfrak{A}_{ij}$, and let $X(\mathfrak{A}_0)$ be the subspace of all Hermitian elements in $\mathfrak{A}_0$. Furthermore, we put $Y = \sum_{1 \leqq i \leqq r-1} \mathfrak{A}_{ir}$ and define a symmetric bilinear mapping

$$F: Y \times Y \to X(\mathfrak{A}_0) \quad \text{by} \quad F(a, b) = \frac{1}{2}(ab^* + ba^*)$$

for every $a, b \in Y$. Then by using results in Chap. III of [18], we can see that the set

$$V_0 = \pi(\{tt^*;\ t \in T(\mathfrak{A}) \cap (\mathfrak{A}_0 + \boldsymbol{R}e)\}) = \pi(V(\mathfrak{A}) \cap (X(\mathfrak{A}_0) + \boldsymbol{R}e))$$

is a homogeneous convex cone in $X(\mathfrak{A}_0)$ and F is a homogeneous V_0-symmetric form, where π is the projection of $X(\mathfrak{A})$ onto $X(\mathfrak{A}_0)$. Moreover, the real Siegel domain $\Omega(V_0, F)$ is affinely equivalent to $\Omega(\mathfrak{A})$ (For the notation, see also (1.1), (1.2) and (4.4).). Hence, by Lemma 4.3, the tube domain $D(\Omega)$ is holomorphically equivalent to the homogeneous Siegel domain $D(V_0, F^C)$. On the other hand, according to Asano [1], the convex cone V_0 is irreducible if and only if the condition (C) holds for the T-algebra $\mathfrak{A}$. Therefore, from the result of [3] stated above, it follows that the Siegel domain $D(V_0, F^C)$ is irreducible with respect to the Bergman metric if and only if Ω is an irreducible convex domain. q.e.d.

Combining Lemma 4.3 and Theorem 4.4 with the result of [3] used in the above proof, we have the following

COROLLARY 4.5. *Let Ω be an affine homogeneous convex domain which is affinely equivalent to a real Siegel domain $\Omega(V_0, F)$ associated with a homogeneous convex cone V_0 and a homogeneous V_0-symmetric form F. Then Ω is irreducible if and only if V_0 is irreducible.*

5. Einstein convex domains. It is known that the Bergman metric of an arbitrary homogeneous bounded domain in $\boldsymbol{C}^n$ is Einstein (cf. e.g., [5]). In this section, we determine all affine homogeneous convex domains whose canonical metrics are Einstein.

Let S be the Ricci tensor for the canonical metric of an affine homogeneous convex domain $\Omega = \Omega(\mathfrak{A})$. Then S is given as follows (cf. e.g., [5]):

(5.1) $$S: \mathfrak{t}_0 \times \mathfrak{t}_0 \to \boldsymbol{R}\ ,$$

$$S(x, y) = \text{trace of the linear mapping: } z \in \mathfrak{t}_0 \to R(z, x)y \in \mathfrak{t}_0\ .$$

The canonical metric of Ω is Einstein if and only if there exists a constant number c satisfying the equality $S = c\langle\ ,\ \rangle$ on $\mathfrak{t}_0 \times \mathfrak{t}_0$.

By using Lemmas 1.1 and 2.2 of [14] and the formula (1.3), we can easily see that the following identities hold:

$$[e_i, x] = \frac{-1}{2\sqrt{n_i}}\Big(\sum_{1 \leqq k < i} x_{ki} - \sum_{i < k \leqq r} x_{ik}\Big) \quad (1 \leqq i \leqq r-1)$$

and

$$R(x, e_i)e_j = \frac{1}{4\sqrt{n_i n_j}} x_{ij} \quad (1 \leqq i < j \leqq r-1)$$

for every $x \in \mathfrak{t}_0$. Therefore, by (5.1) we have

$$S(e_i, e_j) = \frac{1}{4\sqrt{n_i n_j}} n_{ij} \quad (1 \leqq i < j \leqq r-1) . \tag{5.2}$$

Several conditions for an affine homogeneous convex domain in order to be affinely equivalent to an elementary domain are known. Using them, we can state the following

THEOREM 5.1. *For an irreducible affine homogeneous convex domain* Ω *in* $\boldsymbol{R}^n (n \geqq 2)$, *the following conditions are equivalent*:

(1) *The canonical metric* g_Ω *of* Ω *is Einstein*;

(2) Ω *is affinely equivalent to the n-dimensional elementary domain*;

(3) (Ω, g_Ω) *is an irreducible Riemannian symmetric space*;

(4) *The sectional curvature of* (Ω, g_Ω) *is a negative constant*;

(5) *The sectional curvature of* (Ω, g_Ω) *is strictly negative*;

(6) (Ω, g_Ω) *admits an infinitesimal non-affine isometry*;

(7) *The tube domain over* Ω *is holomorphically equivalent to the open unit ball in* $\boldsymbol{C}^n$.

PROOF. The implications $(2) \leftrightarrow (3)$ and $(2) \rightarrow (4)$ have been proved in [16], and $(2) \leftrightarrow (5)$ is due to Shima [13]. The proof for $(2) \leftrightarrow (6)$ can be found in [17]. The implication $(4) \rightarrow (5)$ is trivial. For $(2) \rightarrow (7)$, see Example 1 in §4. The implication $(7) \rightarrow (5)$ follows from Proposition 4.1. We prove the implications $(1) \leftrightarrow (2)$ here. Now, we suppose that the condition (1) holds. Then, by the conditions (5.2) and $\langle e_i, e_j \rangle = \delta_{ij}$, we can see that $n_{ij} = 0$ is satisfied for every pair (i, j) of indices $1 \leqq i < j \leqq r-1$. Since Ω is an irreducible convex domain, the condition (C) is satisfied (cf. [1] and Theorem 3.1). Thus, the rank of the T-algebra $\mathfrak{A}$ satisfying the condition $\Omega(\mathfrak{A}) = \Omega$ must be equal to two. If $n_{12} = 0$, then Ω is affinely equivalent to the cone of all positive real numbers and the dimension of Ω is equal to one. Hence, n_{12} must be positive and Ω is affinely equivalent to the elementary domain $\Omega(n)$, where $n = n_{12} + 1$ (cf. [18]). Conversely, we suppose that the condition (2) holds. Then (Ω, g_Ω) is a hyperbolic space form of the sectional curvature $-1/(2n+2)$ ([16]). Therefore, (Ω, g_Ω) is Einstein (cf. e.g., [5]), and hence, the condition (1) holds. q.e.d.

By using the above theorem, we can easily verify the following

COROLLARY 5.2. *Let* Ω *be a reducible affine homogeneous convex domain. Then the canonical metric of* Ω *is Einstein if and only if* Ω *is affinely equivalent either to a direct product of elementary domains of the same dimension or to a direct product of the half-lines of all positive real numbers.*

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