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A REMARK ON MINIMAL FOLIATIONS OF CODIMENSION TWO

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0. Introduction. A foliation $\mathscr{F}$ of a closed Riemannian manifold W is minimal if the leaves are minimal submanifolds of W. A foliation is taut if there is a metric on W for which the foliation is minimal.

Sullivan [S], Rummler [R] and Haefliger [H] found geometrical and topological characterizations of these foliations. A codimension one oriented foliation is taut if and only if every compact leaf is cut out by a closed transversal (Sullivan). For general codimension there is a necessary and sufficient condition for $\mathscr{F}$ to be taut that depends only on the holonomy pseudo group of the foliation (Haefliger). If the leaves of $\mathscr{F}$ are all compact then $\mathscr{F}$ is taut if and only if $\mathscr{F}$ is stable (Rummler).

Recently, Oshikiri [O], proved that for $\mathscr{F}$ of codimension one and W with non-negative Ricci curvature tensor, $\mathscr{F}$ minimal implies that $\mathscr{F}$ and $\mathscr{F}^{\perp}$ are totally geodesic, where $\mathscr{F}^{\perp}$ denotes the normal flow to $\mathscr{F}$. In particular, $\mathscr{F}$ is defined by a closed form.

In this paper we generalize this theorem for the case of codimension two. Precisely, we prove the following:

THEOREM. *Let W^{n+2} be an oriented closed $(n+2)$-dimensional Riemannian manifold and $\mathscr{F}_1$ a minimal, codimension two C^∞ foliation of W. Suppose the normal distribution of $\mathscr{F}_1$, say $\mathscr{F}_2$, is C^∞ and integrable and that both $\mathscr{F}_1$ and $\mathscr{F}_2$ are orientable.*

(1) *If* $\mathrm{Ricc}(W) > 0$ *then* $\varepsilon(\mathscr{F}_2) \neq 0$.

(2) If $\mathrm{Ricc}(W) \geqq 0$ *then either* $\mathscr{F}_1$ *is totally geodesic or* $\varepsilon(\mathscr{F}_2) \neq 0$. *(Both can occur simultaneously.)*

(3) *If W has non-negative sectional curvature then either $\varepsilon(\mathscr{F}_2) \neq 0$ or $\mathscr{F}_1$ and $\mathscr{F}_2$ are totally geodesic. (Both can occur simultaneously.) Here $\varepsilon(\mathscr{F}_2)$ denotes the Euler class of $\mathscr{F}_2$ and* $\mathrm{Ricc}(W)$ *is the Ricci curvature tensor of W.*

REMARKS.

(a) For the case of non-negative sectional curvature the theorem

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is a complete generalization of Oshikiri's result for codimension two. (Notice that the Euler class of a one dimensional orientable foliation is always zero.)

(b) For the case of positive Ricci curvature the theorem provides a topological obstruction to the integrability of the normal bundle of a minimal foliation. Let us illustrate that with one example.

Let $S^3 \subset \boldsymbol{R}^4$ be the standard unit 3-sphere of constant curvature. Set $W = S^3 \times S^3$ with the Riemannian product metric. It is easy to see that $\operatorname{Ricc}(W) > 0$. There are orientable codimension two foliations on W such that the normal bundle is also an orientable foliation. The product of two Reeb foliations of S^3 is such an example. This foliation is not minimal.

There are also minimal foliations of codimension two on W. For instance, consider the fibration $\pi = H \circ \pi_1 \colon S^3 \times S^3 \to S^2$, where: $\pi_1 \colon S^3 \times S^3 \to S^3$, $\pi_1(x, y) = x$; $H \colon S^3 \to S^2$ is the Hopf fibration. The fibration $\pi \colon S^3 \times S^3 \to S^2$ defines a totally geodesic (hence minimal) foliation $\mathscr{F}$ of W where each leaf is a totally geodesic $S^3 \times S^1 \subset S^3 \times S^3$. The normal bundle of this foliation, say $\mathscr{F}^\perp$, is not integrable because $\varepsilon(\mathscr{F}^\perp) \in H^2(W, \boldsymbol{R}) = 0$.

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1. Notations. Let $x \in W^{n+2}$ and $U \subset W^{n+2}$ an open neighborhood of x. Let $\{e_1, \cdots, e_{n+2}\}$ be a local orthonormal frame defined on U. The coframe, connection and curvature forms are given by

$$\Theta_I(e_J) = \delta_{IJ} \qquad \delta_{IJ} = 0 \quad \text{if} \quad I \neq J \quad \delta_{II} = 1$$

$$\omega_{IJ}(u) = \langle \nabla_u(e_I), e_J \rangle \,, \qquad \Omega_{IJ} = d\omega_{IJ} - \sum_{K=1}^{n+2} \omega_{IK} \wedge \omega_{KJ}$$

where $1 \leqq I, J \leqq n+2$ and ∇, $\langle\ ,\ \rangle$, denote respectively the Riemannian connection and the scalar product of M.

The Cartan structure equations are:

$$d\Theta_I = \sum_{K=1}^{n+2} \omega_{IK} \wedge \Theta_K \,, \qquad d\omega_{IJ} = \sum_{K=1}^{n+2} \omega_{IK} \wedge \omega_{KJ} + \Omega_{IJ} \,.$$

This is the notation used for instance in [Ch].

2. Some computational lemmas. Let W^{n+2} be an oriented closed Riemannian manifold and $\mathscr{F}_1$ a foliation of codimension 2 satisfying the following conditions:

(a) $\mathscr{F}_1$ is orientable, transversely orientable and has C^∞ differentiability class.

(b) The normal distribution $\mathscr{F}_2 = \mathscr{F}_1^\perp$ is integrable and C^∞.

(c) For $i = 1, 2$, the tangent spaces $\mathscr{F}_i(x)$ at the point x of the leaf $\mathscr{F}_i$ passing through x satisfy $\mathscr{F}_1(x) \oplus \mathscr{F}_2(x) = T_xW$ and $u \in \mathscr{F}_1(x)$, $v \in \mathscr{F}_2(x) \Rightarrow \langle u, v\rangle = 0$.

Throughout this paragraph we shall denote by $\mathscr{F}_i$ both the foliation and the distributions tangent to them.

As a consequence of (a), $\mathscr{F}_2$ is also orientable and transversely orientable.

DEFINITION 2.1. A local orthonormal frame $\{e_1, e_2, \cdots, e_{n+2}\}$ is said to be adapted if the following conditions (i) and (ii) are satisfied:

(i) $e_1(x), \cdots, e_n(x) \in \mathscr{F}_1(x)$, $e_{n+1}(x), e_{n+2}(x) \in \mathscr{F}_2(x)$ for all x.

(ii) $\{e_1, e_2, \cdots, e_{n+2}\}$, $\{e_1, \cdots, e_n\}$ and $\{e_{n+1}, e_{n+2}\}$ are compatible with the orientation of W, $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively.

Let $\{e_1, e_2, \cdots, e_{n+2}\}$ be an adapted local orthonormal frame defined on an open set $U \subset W$. Let ψ be the following $(n+1)$-differential form defined on U:

$$\psi = \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \Theta_{\sigma(2)} \wedge \cdots \wedge \Theta_{\sigma(n-2)} \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \Theta_{\tau(n+2)}$$

where S_n is the group of permutations of the set $\{1, 2, \cdots, n\}$ and $S_2^\perp$ is the group of permutations of the set $\{n+1, n+2\}$. $\operatorname{sgn}(\sigma)$, $\operatorname{sgn}(\tau)$ stand for the signs of the permutations σ and τ.

Let $\bar{E} = \{\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n+2}\}$ be another adapted local orthonormal frame defined on a neighborhood $\bar{U} \subset W$ and $\bar{\Theta}_i$, $\bar{\omega}_{ij}$ be the respective coframe and connection forms associated to $\bar{E}$. Let

$$\bar{\psi} = \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \bar{\Theta}_{\sigma(1)} \wedge \bar{\Theta}_{\sigma(2)} \wedge \cdots \wedge \bar{\Theta}_{\sigma(n-1)} \wedge \bar{\omega}_{\sigma(n)\tau(n+1)} \wedge \bar{\Theta}_{\tau(n+2)} .$$

The following lemma shows that ψ is a global form.

LEMMA 2.2.

$$\psi|_{U\cap\bar{U}} = \bar{\psi}|_{U\cap\bar{U}} .$$

PROOF. Set $e_i = \sum_{j=1}^n a_{ij} e_j$ $(1 \leqq i \leqq n)$ and $\bar{e}_\alpha = \sum_{\beta=n+1}^{n+2} a_{\alpha\beta} e_\beta$ $(n+1 \leqq \alpha \leqq n+2)$. Then we have $\bar{\theta}_i = \sum_{j=1}^n a_{ij}\theta_j$, $\bar{\theta}_\alpha = \sum_{\beta=n+1}^{n+2} a_{\alpha\beta}\theta_\beta$, $\bar{\omega}_{i\alpha} = \sum_{j=1}^n \sum_{\beta=n+1}^{n+2} a_{ij} a_{\alpha\beta} \omega_{j\beta}$, for $1 \leqq i \leqq n$, $n+1 \leqq \alpha \leqq n+2$. Thus

$$\bar{\psi}|_{U\cap\bar{U}} = \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Big(\sum_{j_1=1}^n a_{\sigma(1)j_1} \Theta_{j_1}\Big) \wedge \Big(\sum_{j_2=1}^n a_{\sigma(2)j_2} \Theta_{j_2}\Big) \wedge \cdots \wedge$$

$$\wedge\Big(\sum_{j_{n-1}=1}^{n} a_{\sigma(n-1)j_{n-1}}\Theta_{j_{n-1}}\Big)\wedge\Big(\sum_{j_n=1}^{n}\sum_{\beta_1=n+1}^{n+2} a_{\sigma(n)j_n}a_{\tau(n+1)\beta_1}\omega_{j_n\beta_1}\Big)$$

$$\wedge\Big(\sum_{\beta_2=n+1}^{n+2} a_{\tau(n+2)\beta_2}\Theta_{\beta_2}\Big)\Big|_{U\cap\bar U}$$

$$= \sum_{\sigma\in S_n}\sum_{\tau\in S_2^\perp}\sum_{j_1,j_2,\cdots,j_n=1}^{n}\sum_{\beta_1,\beta_2=n+1}^{n+2}\cdot\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\, a_{\sigma(1)j_1}\cdot a_{\sigma(2)j_2}\cdot\ \cdots$$

$$\cdot a_{\sigma(n)j_n}\cdot a_{\tau(n+1)\beta_1}\cdot a_{\tau(n+2)\beta_2}$$

$$\cdot\Theta_{j_1}\wedge\Theta_{j_2}\wedge\ \cdots\ \wedge\Theta_{j_{n-1}}\wedge\omega_{j_n\beta_1}\wedge\Theta_{\beta_2}|_{U\cap\bar U}\ .$$

The fact that $\Theta_I\wedge\Theta_I = 0$ and the symmetry of S_n gives us immediately:

$$\psi|_{U\cap\bar U} = \sum_{\sigma\in S_n}\sum_{\tau\in S_2^\perp}\sum_{\eta\in S_n}\sum_{\mu\in S_2^\perp}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\, a_{\sigma(1)\eta(1)}\cdot a_{\sigma(2)\eta(2)}\ \cdots$$

$$\cdot a_{\sigma(n)\eta(n)}\cdot a_{\tau(n+1)\mu(n+1)}\cdot a_{\tau(n+2)\mu(n+2)}$$

$$\cdot\Theta_{\eta(1)}\wedge\ \cdots\ \wedge\Theta_{\eta(n-1)}\wedge\omega_{\eta(n)\mu(n+1)}\wedge\Theta_{\mu(n+2)}|_{U\cap\bar U}\ .$$

But $\bar E$ is an adapted frame. Then $\det(a_{ij}) = 1$ $(1\leqq i, j\leqq u)$, $\det(a_{\alpha\beta}) = 1$ $(n+1\leqq\alpha, \beta\leqq n+2)$ and $\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)a_{\sigma(1)\eta(1)}\cdots a_{\sigma(n)\eta(n)} = \operatorname{sgn}(\eta)\cdot\det(a_{ij}) = \operatorname{sgn}(\eta)$.

Similarly

$$\sum_{\tau\in S_2^\perp}\operatorname{sgn}(\tau)a_{\tau(n+1)\mu(n+1)}\cdot a_{\tau(n+2)\mu(n+2)} = \operatorname{sgn}(\mu)\ .$$

Thus

$$\bar\psi_{U\cap\bar U} = \sum_{\eta\in S_n}\sum_{\mu\in S_2^\perp}\operatorname{sgn}(\eta)\operatorname{sgn}(\mu)\,\Theta_{\eta(1)}\wedge\ \cdots\ \wedge\Theta_{\eta(n-1)}\wedge\omega_{\eta(n)\mu(n+1)}\wedge\Theta_{\mu(n+2)}$$

$$= \psi|_{U\cap\bar U}\ . \qquad\square$$

From now until the end of this paragraph let us suppose that $n\geqq 2$. Using the same notations as before we define the forms ϕ_1 and ϕ_2 and Ω in $\bigwedge^{n+2}(W, R)$ by

$$\phi_1 = \sum_{\sigma\in S_n}\sum_{\tau\in S_2^\perp}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\Big(\sum_{\beta=n+1}^{n+2}\omega_{\sigma(1)\beta}\wedge\omega_{\beta\sigma(2)}\Big)$$

$$\wedge\Theta_{\sigma(3)}\wedge\ \cdots\ \wedge\Theta_{\sigma(n)}\wedge\Theta_{\tau(n+1))}\wedge\Theta_{\tau(n+2)}\ ,$$

$$\phi_2 = \sum_{\sigma\in S_n}\sum_{\tau\in S_2^\perp}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\theta_{\sigma(1)}\wedge\ \cdots\ \wedge\theta_{\sigma(n)}\wedge\Big(\sum_{k=1}^{n}\omega_{\tau(n+1)k}\wedge\omega_{k\tau(n+2)}\Big)\ ,$$

$$\Omega = \sum_{\sigma\in S_n}\sum_{\tau\in S_2^\perp}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\Theta_{\sigma(1)}\wedge\ \cdots\ \wedge\Theta_{\sigma(n-1)}\wedge\Omega_{\sigma(n)\tau(n+1)}\wedge\Theta_{\tau(n+2)}\ ,$$

for $n\geqq 2$.

REMARK. ϕ_1, and ϕ_2 and Ω are global forms in the sense that they do not depend on the choice of the particular adapted local frame. The proof of that fact is a straightforward computation similar to that of Lemma 2.2.

LEMMA 2.3. *If $n \geqq 2$, then*

$$d\psi = (-1)^n[((n-1)/2)\phi_1 + (1/n)\phi_2] + (-1)^{n+1}\Omega .$$

PROOF. Let

$$d\psi = A + B + C , \tag{1}$$

where

$$A = \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \sum_{j=1}^{n-1} (-1)^{j+1} \theta_{\sigma(1)} \wedge \cdots \wedge \theta_{\sigma(j-1)} \wedge d\Theta_{\sigma(j)} \wedge \Theta_{\sigma(j+1)} \wedge \cdots$$
$$\wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \Theta_{\tau(n+2)} ,$$
$$B = (-1)^{n+1} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge d\omega_{\sigma(n)\tau(n+1)} \wedge \Theta_{\tau(n+2)} ,$$
$$C = (-1)^{n+2} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge d\Theta_{\tau(n+2)} .$$

Permuting 1 and j on A, $1 \neq j$, we get:

$$A = \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \sum_{j=1}^{n-1} (-1)^{j+1} \cdot (-1)^{j+1} d\theta_{\sigma(1)} \wedge \theta_{\sigma(2)} \wedge \cdots$$
$$\wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \Theta_{\tau(n+2)} .$$

But $d\Theta_{\sigma(1)} = \sum_{K=1}^{n+2} \omega_{\sigma(1)_K} \wedge \Theta_K$. Thus

$$A = A_1 + A_2 \tag{2}$$

where

$$\begin{aligned} A_1 &= (n-1) \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, \omega_{\sigma(1)\sigma(n)} \wedge \Theta_{\sigma(n)} \wedge \theta_{\sigma(2)} \wedge \theta_{\sigma(3)} \wedge \cdots \\ &\qquad \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \Theta_{\tau(n+2)} , \qquad \text{and} \\ A_2 &= (n-1) \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \omega_{\sigma(1)\tau(n+1)} \wedge \Theta_{\tau(n+1)} \wedge \Theta_{\sigma(2)} \\ &\qquad \wedge \Theta_{\sigma(3)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \Theta_{\tau(n+2)} . \end{aligned} \tag{3}$$

Using the symmetry with respect to the group S_n and the laws of commutativity of the wedge product, we get

$$A_2 = (n-1)(-1)^{n+1} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \omega_{\sigma(1)\tau(n+1)} \wedge \omega_{\sigma(2)\tau(n+1)}$$
$$\wedge \Theta_{\sigma(3)} \wedge \cdots \wedge \Theta_{\sigma(n)} \wedge \Theta_{\tau(n+1)} \wedge \Theta_{\tau(n+2)} .$$

Or, equivalently:

$$A_2 = (n-1)(-1)^n \cdot (1/2)\phi_1 \tag{4}$$

Then, using (2) and (4) we get

$$A = A_1 + (n-1)(-1)^n \cdot (1/2)\phi_1 . \tag{5}$$

On the other hand

$$B = B_1 + B_2 \tag{6}$$

where

$$B_1 = (-1)^{n+1}\Omega \tag{7}$$

and

$$B_2 = (-1)^{n+1} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \Theta_{\sigma(2)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \left(\sum_{K=1}^{n+2} \omega_{\sigma(n)K} \wedge \omega_{K\tau(n+1)} \right) \wedge \Theta_{\tau(n+2)} . \tag{8}$$

From (8) we obtain:

$$B_2 = B_{21} + B_{22} \tag{9}$$

and

$$B_{21} = (-1)^{n+1} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp}{}' \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \Theta_{\sigma(2)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \left(\sum_{j=1}^{n-1} \omega_{\sigma(n)\sigma(j)} \wedge \omega_{\sigma(j)\tau(n+1)} \right) \wedge \Theta_{\tau(n+2)} , \tag{10}$$

$$B_{22} = (-1)^{n+1} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \Theta_{\sigma(2)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+2)} \wedge \omega_{\tau(n+2)\tau(n+1)} \wedge \Theta_{\tau(n+2)} . \tag{11}$$

Using again the symmetry of S_n and the laws of commutativity of the wedge product "$\wedge$", the equality (10) becomes:

$$B_{21} = (-1)^{n+1}(n-1) \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \Theta_{\sigma(2)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\sigma(1)} \wedge \omega_{\sigma(1)\tau(n+1)} \wedge \Theta_{\tau(n+2)} . \tag{12}$$

For C, we have

$$C = C_1 + C_2 \tag{13}$$

i.e. where

$$C_1 = (-1)^{n+2} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\tau(1)} \wedge \Theta_{\sigma(2)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \omega_{\tau(n+2)\sigma(n)} \wedge \Theta_{\sigma(n)} \tag{14}$$

and

$$(15) \qquad C_2 = (-1)^{n+2} \sum_{\sigma \in S_n} \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \Theta_{\sigma(1)} \wedge \cdots \wedge \Theta_{\sigma(n-1)} \wedge \omega_{\sigma(n)\tau(n+1)} \wedge \omega_{\tau(n+2)\tau(n+1)} \wedge \Theta_{\tau(n+1)} .$$

It is easy to see, from (14) that

$$(16) \qquad C_1 = (-1)^{n+2} \cdot (1/n)\phi_2 .$$

From (3), (12), (11) and (15), we get

$$(17) \qquad A_1 = -B_{21} ,$$

$$(18) \qquad B_{22} = -C_2 .$$

Using (1), (5), (6), (7), (9), (13), (16), we finally obtain from (17) and (18):

$$d\psi = (-1)^n[((n-1)/2)\phi_1 + (1/n)\phi_2] + (-1)^{n+1}\Omega . \qquad \square$$

Let M_x be the leaf of $\mathscr{F}_1$ passing through a point $x \in W$. M_x is an immersed manifold on W and its metric is that induced by the metric of W. Let $\{e_1, e_2, \cdots, e_n, e_{n+1}, e_{n+2}\}$ be an adapted frame defined in a neighbourhood of $x \in W$. Define $A_x = (h_{ij}^\alpha(x))$ $1 \leqq i, j \leqq n$, $n+1 \leqq \alpha \leqq n+2$ by $h_{ij}^\alpha = \langle \nabla_{e_i} e_\alpha, e_j \rangle = -\omega_{j\alpha}(e_i)$. With these notations it is easy to see that

$$\phi_1 = -(2!)^2(n-2)! \sum_{1=i<j}^{n} \sum_{\alpha=n+1}^{n+2} (h_{ii}^\alpha h_{jj}^\alpha - h_{ji}^\alpha h_{ij}^\alpha)\nu$$

where ν is the volume element of W. Or, equivalently:

$$\phi_1 = (2!)(n-2)! \sum_{\alpha=n+1}^{n+2} [\operatorname{tr} A_\alpha^2 - (\operatorname{tr} A_\alpha)^2]\nu$$

where

$$A_\alpha = (h_{ij}^\alpha) , \qquad A_\alpha^2 = A_\alpha \circ A_\alpha ,$$

$$\operatorname{tr} A_\alpha = \sum_{i=1}^{n} h_{ii}^\alpha , \qquad \operatorname{tr} A_\alpha^2 = \sum_{i,k=1}^{n} h_{ik}^\alpha h_{ki}^\alpha .$$

Because A_α is symmetric, we have

$$\operatorname{tr} A_\alpha^2 = \sum_{i,j=1}^{n} (h_{ij}^\alpha)^2 .$$

If $\mathscr{F}_1$ is a minimal foliation, then

$$(19) \qquad \begin{aligned} &\sum_{i=1}^{n} h_{ii}^\alpha = \operatorname{tr} A_\alpha = 0 \quad n+1 \leqq \alpha \leqq n+2 , \qquad \text{and} \\ &\phi_1 = (2!)(n-2)!\, S\nu \end{aligned}$$

where $S = \sum_{i,j=1}^{n} \sum_{\alpha=n+1}^{n+2} (h_{ij}^\alpha)^2$ is the square of the length of the second fundamental form of M_x.

Denote by $\varepsilon(\mathscr{F}_2)$ the Euler class of the tangent bundle to $\mathscr{F}_2$. Using the notations as before, we can write (see [M]).

$$\varepsilon(\mathscr{F}_2) = -(1/2!) \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\tau)\Big[\Big(\sum_{k=1}^{n} \omega_{\tau(n+1)k} \wedge \omega_{k\tau(n+2)}\Big) + \Omega_{\tau(n+1)\tau(n+2)}\Big] .$$

On the other hand,

$$\phi_2 = n! \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\tau)\nu_1 \wedge \Big(\sum_{k=1}^{n} \omega_{\tau(n+1)k} \wedge \omega_{k\tau(n+2)}\Big)$$

where $\nu_1 = \Theta_1 \wedge \cdots \wedge \Theta_n$ is the volume element of $\mathscr{F}_1$. Then

$$\phi_2 = -2n!\, \varepsilon(\mathscr{F}_2) \wedge \nu_1 - n! \sum_{\tau \in S_2^\perp} \operatorname{sgn}(\tau)\Omega_{\tau(n+1)\tau(n+2)} \wedge \nu_1 .$$

Let $c_{n+1,n+2}$ denote the sectional curvature of W in the direction of the plane determined by e_{n+1} and e_{n+2}. According to our notations, we have

$$\Omega_{n+1,n+2}(e_{n+1}, e_{n+2}) = -c_{n+1,n+2} .$$

Thus

$$\phi_2 = -2n!\, \varepsilon(\mathscr{F}_2) \wedge \nu_1 + 2!\, n!\, c_{n+1,n+2}\nu . \tag{20}$$

$c_{i\alpha}$ is the sectional curvature in the direction of the plane determined by e_i and e_α, $1 \leqq i \leqq n$, $n+1 \leqq \alpha \leqq n+2$ and

$$\Omega = -(n-1)! \sum_{i,\alpha} c_{i\alpha}\nu . \tag{21}$$

The following Lemma is an easy consequence of (19), (20) and (21), Lemma 2.3 and the Stokes theorem.

LEMMA 2.4. *If $\mathscr{F}_2$ is a minimal foliation, then*

$$\int_W S_\nu - 2\varepsilon(\mathscr{F}_2) \wedge \nu_1 + \sum_{\alpha=n+1}^{n+2} \operatorname{Ricc}(e_\alpha)\nu = 0 ,$$

where $\operatorname{Ricc}(e_\alpha) = \sum_{K=1, K\neq\alpha}^{n+2} c_{K\alpha}$ *is the Ricci curvature of W in the direction of e_α.*

REMARK. Lemma 2.4 remains true even if the normal distribution to $\mathscr{F}_1$, say $\mathscr{F}_2$, is not integrable. Observe as well that $\sum_{\alpha=n+1}^{n+2} \operatorname{Ricc}(e_\alpha)$ does not depend on the choice of the particular adapted frame.

3. Proof of the theorem. Suppose $n \geqq 2$. Let us observe first that the minimality of $\mathscr{F}_1$ and the integrability of its normal bundle imply $d\nu_1 = 0$ (see [R]). Then ν_1 is a cycle and $\nu_1 \in H^n(W, \boldsymbol{R})$, where $H^n(W, \boldsymbol{R})$ is the n-th de Rham cohomology group of W.

The Euler class $\varepsilon(\mathscr{F}_2)$ is also a cycle and $\varepsilon(\mathscr{F}_2) \in H^2(W, \boldsymbol{R})$.

We first prove (i). When $\mathrm{Ricc}(W) > 0$, suppose that $\varepsilon(\mathscr{F}_2) = 0$ as an element of $H^2(W, \boldsymbol{R})$, i.e., $\varepsilon(\mathscr{F}_2)$ is an exact form. Consider the cup product $\wedge$ in the cohomology ring $H^*(W, \boldsymbol{R})$

$$\wedge : H^2(W, \boldsymbol{R}) \times H^n(W, \boldsymbol{R}) \to H^{n+2}(W, \boldsymbol{R}) .$$

Since $\varepsilon(\mathscr{F}_2) = 0$, we have $\varepsilon(\mathscr{F}) \wedge \nu_1 = 0 \in H^{n+2}(W, \boldsymbol{R})$. Now, by the de Rham theorem

$$\int_W \varepsilon(\mathscr{F}) \wedge \nu_1 = 0 , \quad \text{which contradicts Lemma 2.4 .}$$

This completes the proof of part (1) of the theorem.

Suppose now $\mathrm{Ricc}(W) \geqq 0$ and $\varepsilon(\mathscr{F}_2) = 0$. Then

$$\int_W \varepsilon(\mathscr{F}_2) \wedge \nu_1 = 0 = \int_W \Big(S + \sum_{\alpha=n+1}^{n+2} \mathrm{Ricc}\,(e_\alpha)\Big)\nu ,$$

by Lemma 2.4. Thus $S \equiv 0$ and $\mathrm{Ricc}\,(e_\alpha) \equiv 0$ $e_\alpha \perp \mathscr{F}_1$.

If $S \equiv 0$ then $\mathscr{F}_1$ is totally geodesic and this completes the proof of part (2) of the theorem.

Suppose now $\varepsilon(\mathscr{F}_2) = 0$ and W has non-negative sectional curvature in the direction of every 2-plane and at every point of W. Then, in particular $\mathrm{Ricc}(W) \geqq 0$ and, by part (2) we see that $\mathscr{F}_1$ is totally geodesic.

Moreover, the following proposition is a part of a theorem proved by Abe [*Ab*].

PROPOSITION. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two orthogonal foliations of complementary dimensions over a complete Riemannian manifold W with non-negative sectional curvatures. Suppose $\mathscr{F}_1$ is totally geodesic. Then $\mathscr{F}_2$ is totally geodesic.*

This completes the proof of part (3) of the theorem for the case $n \geqq 2$.

Let us now suppose that $n = 1$. $\mathscr{F}_1$ is now a minimal one dimensional foliation. In other words, $\mathscr{F}_1$ is totally geodesic.

Let $\{e_1, e_2, e_3\}$ be an adapted local frame and set

$$\psi = \omega_{21} \wedge \Theta_3 + \Theta_2 \wedge \omega_{31} .$$

It is easy to see that ψ is globally defined (see [*A*], [*BLR*]). Exterior differentiation of ψ and the Stokes theorem give

$$2\int_W \varepsilon(\mathscr{F}_2) \wedge \Theta_1 = \int_W (\mathrm{Ricc}\,(e_2) + \mathrm{Ricc}\,(e_3))\nu .$$

If $\mathscr{F}_1$ is totally geodesic and $\mathscr{F}_2 = \mathscr{F}_1^{\perp}$ is a foliation then $d\Theta_1 = 0$.

The same argument used for the case $n \geqq 2$ shows that if $\operatorname{Ricc}(W) > 0$ then $\varepsilon(\mathscr{F}_2) \neq 0$.

This concludes part (1).

Part (2) is a consequence of the fact that $\mathscr{F}_1$ is totally geodesic. We can prove Part (3) by repeating the argument used in the case $n \geqq 2$. $\square$

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