Tôhoku Math. Journ.
36 (1983), 591–609.

RATIONAL MAPPINGS OF DEL PEZZO SURFACES, AND SINGULAR COMPACTIFICATIONS OF TWO-DIMENSIONAL AFFINE VARIETIES*

DAVID BINDSCHADLER, LAWRENCE BRENTON AND DANIEL DRUCKER

(Received December 5, 1983)

Introduction. Let F be an algebraically closed field and let $(X, \mathcal{O}_X)$ be a two-dimensional, rational, normal compact Gorenstein space over F. If the anticanonical divisor of X is ample, then X is called a (possibly singular) *Del Pezzo surface*. For $F = \boldsymbol{C}$, these surfaces were studied systematically by Du Val [12] in his investigation of the relation between rational double points and subgroups of reflection groups of regular polyhedra (cf. also [8], [13], [20]). They have recently attracted new interest as singular fibres of versal deformations of elliptic singularities ([9], [16], [21], [26]). We have studied certain of these spaces as examples of singular complex surfaces of the homology, cohomology, or homotopy type of $\boldsymbol{CP}^2$ ([2], [7]).

If X is a Del Pezzo surface, the *degree* of X is the integer $d = K \cdot K$, where K is the canonical divisor. If X is singular, each singularity is a double point of type A_k, D_k, or E_k, and the Dynkin diagram Γ corresponding to the singular set is the Coxeter graph of a subgroup of one of the reflection groups A_1, $A_2 + A_1$, A_4, D_5, or E_k, $6 \leqq k \leqq 8$ ([12]). The number n of vertices of Γ is always less than or equal to $9 - d$; if $n = 9 - d$, X will be called *maximally degenerate*. In this case $H^i(X, \boldsymbol{Q}) \cong H^i(\boldsymbol{P}^2, \boldsymbol{Q})$ $\forall i$, with $H^1(X, \boldsymbol{Z}) \cong 0$, $H^2(X, \boldsymbol{Z}) \cong \boldsymbol{Z}$, and $H^3(X, \boldsymbol{Z})$ a finite group of order $\sqrt{(\det(\Gamma))/d}$, where $\det(\Gamma)$ is the determinant of the associated Cartan matrix. Except when X is the singular quadric hypersurface $\boldsymbol{Q}_0^2 \subset \boldsymbol{P}^3(F)$, the Chern class of K generates $H^2(X, \boldsymbol{Z})$ and so the degree also gives the cohomology ring structure. In the maximally degenerate case (but not in general), the singularity type determines the surface up to a deformation through fibres of the same singularity type ([12], [20]).

Now let X be any Del Pezzo surface. Since X is rational, there exists a birational mapping f of X onto $\boldsymbol{P}^2(F)$. Factoring f into a

* Some of the results of Part I of this paper were announced in [3], where a preliminary version of the present paper was referred to under the title: Graph theoretic techniques in algebraic geometry III.

sequence of monoidal point transformations and their inverses, we obtain a commuting diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ X \arrow[rr, "f"] & & P^2 \end{tikzcd}$$

where $\tilde{X}$ is non-singular and ρ and π are regular (cf. Nagata [19]). The meromorphic map f will be called *minimal* if $\tilde{X}$ can be taken to be the minimal non-singular model of X. In that case, ρ and π are unique and exhibit the construction of X as the result of blowing up $9-d$ points in $\boldsymbol{P}^2$, then collapsing the union of $n = 10 - d - b_2(X)$ non-singular curves of grade -2 to one or more singular points. The purpose of this paper is to give the complete classification of such maps f in the maximally degenerate case. Our techniques rely on the existence of certain "global extensions" $\tilde{\Gamma}$ of Dynkin diagrams Γ, having the property that for each vertex w_i of $\tilde{\Gamma} - \Gamma$, the subgraph of $\tilde{\Gamma}$ induced by w_i and the vertices of the components of Γ which meet w_i, is an extended Dynkin diagram of type $\bar{A}_k$, $\bar{D}_k$, or $\bar{E}_k$.

In part II we will use our results to study compactifications of affine varieties V of $\boldsymbol{CP}^2$. For $V = \boldsymbol{C}^2$, $\boldsymbol{C}\times\boldsymbol{C}^*$, and $(\boldsymbol{C}^*)^2$, *non-singular* compactifications have been studied in such recent papers as [17], [24], [25], and [27]. Our techniques allow us to treat certain singular compactifications as well. Indeed, we will find all "minimal projective compactifications" (defined precisely below) $h: V \rightarrow X$ of affine varieties $V \subset \boldsymbol{CP}^2$, where X is a minimal singular Gorenstein surface with vanishing geometric genus. In particular, this gives a classification of all compactifications of this type for affine surfaces of the form $(\boldsymbol{C} - \{P_1, \cdots, P_r\}) \times (\boldsymbol{C} - \{Q_1, \cdots, Q_s\})$.

I. Rational mappings of maximally degenerate Del Pezzo surfaces. In this section we will obtain the list of all minimal birational mappings $f: X \rightarrow \boldsymbol{P}^2(F)$, modulo automorphisms of $\boldsymbol{P}^2(F)$, where F is any algebraically closed field of characteristic different from 2, and where X is a maximally degenerate Del Pezzo surface over F. Let $X \neq \boldsymbol{P}^2, \boldsymbol{Q}_0^2$ be such a surface, and let $\pi: \tilde{X} \rightarrow X$ be the minimal resolution of singularities. Then $\tilde{X}$ contains only finitely many negatively embedded irreducible curves, and each such curve is non-singular rational with self-intersection -2 or -1. Let $C_1, \cdots, C_n$ be the curves of grade -2 and let $D_1, \cdots, D_m$ be the curves of grade -1. Then $n \leqq 8$ and the dual intersection graph of the C_i's is the Dynkin diagram Γ corresponding to the singular points of X. The a priori possibilities for Γ are those Coxeter graphs that correspond to generators of subgroups of certain distinguished finite reflection groups on $\boldsymbol{R}^n$, $3 \leqq n \leqq 8$, as listed below .

$$\begin{aligned} n = 8:\quad & E_8, D_8, E_7 + A_1, A_8, E_6 + A_2, A_7 + A_1, D_5 + A_3, \\ & \qquad D_6 + 2A_1, 2D_4, 2A_4, A_5 + A_2 + A_1, 2A_3 + 2A_1, 4A_2, \\ & \qquad D_4 + 4A_1, 8A_1 \\ n = 7:\quad & E_7, A_7, D_6 + A_1, A_5 + A_2, 2A_3 + A_1, D_4 + 3A_1, 7A_1 \\ n = 6:\quad & E_6, 3A_2, A_5 + A_1 \\ n = 5:\quad & D_5, A_3 + 2A_1 \\ n = 4:\quad & A_4 \\ n = 3:\quad & A_2 + A_1 \end{aligned}$$

(Cf. [8], [12], [13]. Several other characterizations of these graphs are given in [3].)

If $n < 8$ then blowing up $8 - n$ additional points in sufficiently general position on the curves D_i produces a new surface $\tilde{\tilde{X}}$ of the same type and from which the properties of $\tilde{X}$ can easily be recovered. Thus we can assume $n = 8$. Let $\mathscr{E}$ be the vector subspace of $H^2(\tilde{X}, \boldsymbol{R})$ orthogonal to the canonical divisor K. Then the Chern classes γ_i of the divisors C_i form a basis for $\mathscr{E}$ and a base for the root system of the complex Lie algebra $\mathfrak{g}$ corresponding to Γ. If δ_j denotes the orthogonal projection of the Chern class of D_j into $\mathscr{E}$, then the vectors δ_j represent certain of the "directrices" of the system ([11]). Taking the negative of the cup product pairing as an inner product on $\mathscr{E}$, it is easy to verify that

$$\langle \delta_i, \delta_j \rangle = \begin{cases} 2 & \text{if} \quad i = j \\ 1 & \text{if} \quad D_i \cap D_j = \varnothing \\ 0 & \text{if} \quad D_i \quad \text{meets} \quad D_j,\ i \neq j\ , \end{cases}$$

while $\langle \gamma_i, \delta_j \rangle = -(C_i \cdot D_j) \leqq 0\ \forall i, j$. These observations motivate the following lemma.

LEMMA 1. *Let $B = \{v_1, \cdots, v_8\}$ be a base in $\boldsymbol{R}^8$ for a root subsystem of E_8, with $\|v_i\|^2 = 2\ \forall i$. Then there exists a unique (up to an orthogonal transformation of $\boldsymbol{R}^8$ taking B onto itself) collection $B' = \{w_1, \cdots, w_m\}$ of vectors of squared length 2 satisfying*

(1) *$\langle v_i, w_j \rangle$ is a non-positive integer $\forall i, j$,*
(2) *$\langle w_i, w_j \rangle$ is a non-negative integer $\forall i, j$, and*
(3) *B' is maximal with respect to (1) and (2).*

The Lemma is proved by means of a graphical construction. For any vector w in $\boldsymbol{R}^8$ that satisfies condition (1) and has squared length 2, we now show that the weighted intersection multigraph Γ_w of the set $\tilde{B} = \{v_1, \cdots, v_8, w\}$ is the union of an extended Dynkin diagram of type $\bar{A}$, $\bar{D}$, or $\bar{E}$ with a (possibly empty) collection of Dynkin diagrams of types A, D, and E. (A "multigraph" is a graph with multiple edges allowed. The term "quasi-graph" allows unanchored edges, pictured ●—; cf. [7]. In this paper, the extended Dynkin diagram $\bar{A}_1$ is the only multigraph we encounter that has a multiple edge.) Let $\tilde{B}_w = \{w\} \cup \{v \in B\colon v \text{ is linked to } w\}$, where "$v$ is linked to w" means that there is a sequence $v_0, \cdots, v_k$ $(k \geqq 1)$ of vectors in $\tilde{B}$ such that $v_0 = v$, $v_k = w$, and $\langle v_{i-1}, v_i\rangle \neq 0$ for $i = 1, \cdots, k$. Then $\tilde{B}_w$ is indecomposable (not the union of two nonempty orthogonal subsets), $\tilde{B}_w \perp (\tilde{B} - \tilde{B}_w)$, and $\tilde{B} = \tilde{B}_w \cup (\tilde{B} - \tilde{B}_w)$. Since $\tilde{B}$ is dependent and $\tilde{B} - \tilde{B}_w$ is independent, $\tilde{B}_w$ must be dependent. Γ_w is the union of the multigraphs for $\tilde{B}_w$ and $\tilde{B} - \tilde{B}_w$, so the fact that Γ_w has the form described above follows from standard arguments of the type used to classify Dynkin diagrams. (See [10] for a suitably general classification result.)

Conversely, if $\hat{\Gamma}$ is a 9-vertex multigraph containing Γ with an extended Dynkin diagram of type $\bar{A}$, $\bar{D}$, or $\bar{E}$ as one component, and if the other components (if any) are Dynkin diagrams of types A, D, and E, then $\hat{\Gamma}$ uniquely determines a vector w such that $\hat{\Gamma} = \Gamma_w$. In fact, $w = \sum_{i,j} (-c_{ij})\sqrt{N_i}v_j$, where (c_{ij}) is the inverse of the Cartan matrix of Γ and N_i is the number of edges joining v_i to the single vertex of $\hat{\Gamma} - \Gamma$.

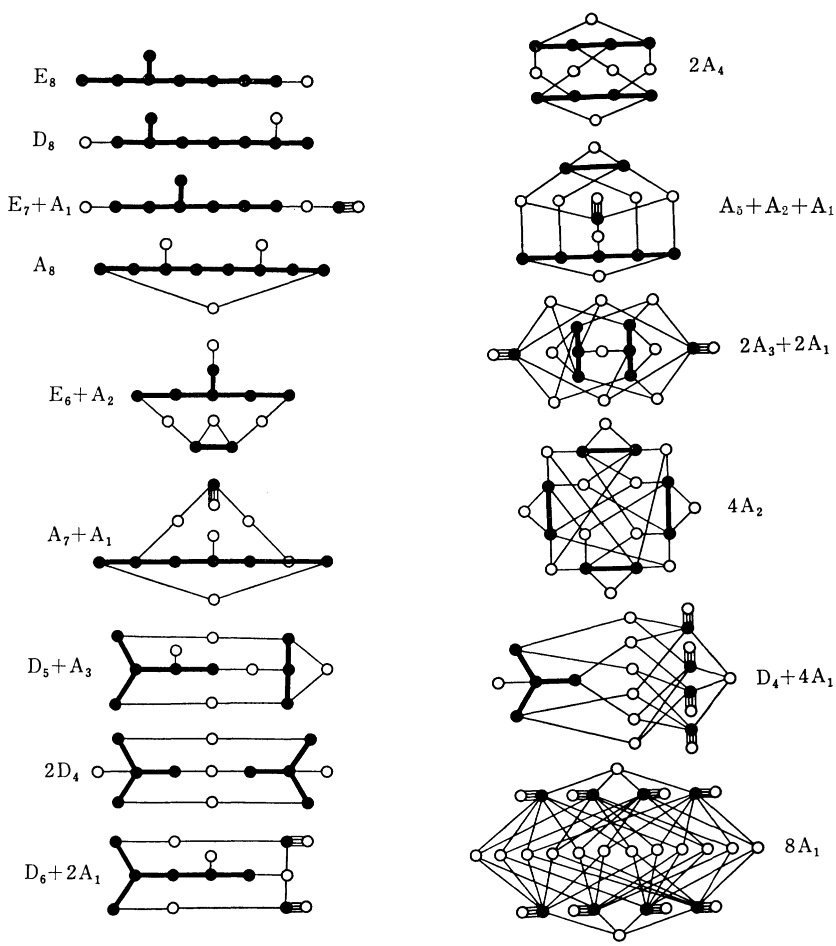
To find the desired maximal collection B', select the graph Γ corresponding to B from the list of all 15 graphs for $n = 8$ given earlier. Adjoin vertices $w_1, w_2, \cdots$ to Γ in such a way that for each j the multigraph of $\tilde{B}_{w_j}$ is an extended Dynkin diagram of type $\bar{A}$, $\bar{D}$, or $\bar{E}$, and $\langle w_i, w_j\rangle$ is a non-negative integer for every $i < j$. When this process can no longer be continued, one obtains a maximal collection $B' = \{w_1, \cdots, w_m\}$ together with a multigraph $\tilde{\Gamma} = \bigcup_{i=1}^m \Gamma_{w_i}$.

To prove uniqueness, consider each of the 15 graphs in turn and construct all possible maximal collections together with the corresponding multigraphs. For each Γ, it is found that all the multigraphs obtained from maximal collections are isomorphic. From this it follows easily that the maximal collections correspond under suitably chosen orthogonal transformations of $\boldsymbol{R}^8$. Note that $\tilde{\Gamma}$ depends only on Γ, not on the particular choice of B or B'.

The multigraph $\tilde{\Gamma}$ constructed in the proof from a Dynkin diagram Γ of a base B will be called the *maximal special global extension* of Γ (cf. [3]). The maximal special global extensions of the 15 admissible 8-

point Dynkin diagrams are shown in Table I. Solid dots represent the vertices v_i of Γ, and circles represent the vertices w_j of $\tilde{\Gamma} - \Gamma$. Note that the full intersection matrix of $\{v_1, \cdots, v_8, w_1, \cdots, w_m\}$ can be recovered from $\tilde{\Gamma}$ by using the fact that $\langle w_i, w_j \rangle$ equals 1 or 0 according to whether or not Γ has a component that is joined to both w_i and w_j in $\tilde{\Gamma}$.

TABLE I: MAXIMAL SPECIAL GLOBAL EXTENSIONS OF ADMISSIBLE 8-POINT DYNKIN DIAGRAMS.



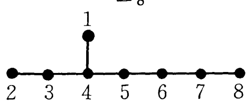
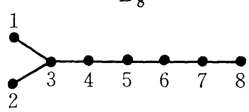
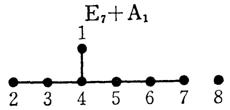
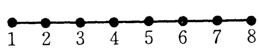
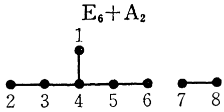
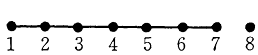
From this list of maximal extensions we will obtain the classification of minimal birational mappings $f: X \to \boldsymbol{P}^2(F)$ of maximally degenerate Del Pezzo surfaces X. As noted in the introduction, any such f uniquely determines the map ρ in the diagram

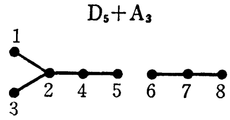
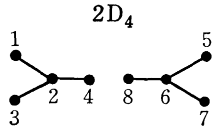
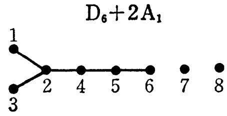
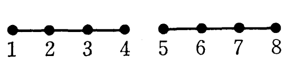
$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ X \arrow[rr, "f"] & & P^2 \end{tikzcd}$$

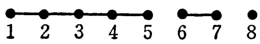
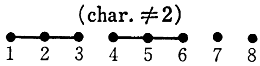
and X determines π. Thus all minimal birational mappings of X can be found by finding all maps $\rho: \tilde{X} \to P^2$ inverse to blowing up points. For each such ρ, the dual intersection graph Γ_ρ of the exceptional curves of ρ is a subgraph of the maximal special global extension $\tilde{\Gamma}$ of the Dynkin diagram Γ which characterizes X. (This follows from the remarks preceding Lemma 1 and the fact that D_i cannot meet D_j if both are exceptional curves of ρ of grade -1.) All rational maps ρ, then, can be found by determining which sequences of eight vertices in the multigraphs of Table I can be blown down to non-singular points. The remaining vertices must correspond to irreducible curves on $P^2(F)$, and a check to see if such curves actually occur on $P^2(F)$ confirms that the expected exceptional curves of ρ exist on $\tilde{X}$. Once all maps for $n = 8$ are found, the search for $n < 8$ can be completed by observing which sequences have intermediate stages where the dual intersection graph of the curves of self-intersection -2 form a Dynkin diagram on the appropriate number of vertices. For the sake of completeness we will also include the case $\mathrm{char}(F) = 2$, where the singular space X may not exist (cf. Artin [1]), although the "non-singular model" $\tilde{X}$ can be obtained from $P^2(F)$ as in the general case by blowing up points as described by ρ^{-1}.

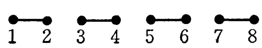
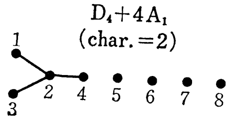
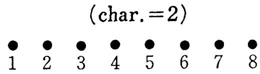
In Table II we list all maps $\rho: \tilde{X} \to P^2(F)$ derived in this fashion. Every entry in the table occurs over every algebraically closed field F, except that the graph $2A_3 + 2A_1$ does not occur in the case $\mathrm{char}(F) = 2$, while the graphs $7A_1$, $D_4 + 4A_1$, and $8A_1$ occur *only* in this case. (In particular, these three graphs do not occur over C; cf. [12], [20], [3]). First we give the Dynkin diagram Γ of curves on $\tilde{X}$ with self-intersection -2. (Recall that Γ determines $\tilde{X}$ uniquely up to an algebraic deformation in the maximally degenerate case.) The second column is the list of curves with self-intersection -1 that are blown down by ρ, described by identifying which curves of self-intersection -2 they meet (e.g., 178 is the curve which intersects the curves numbered 1, 7, and 8, while 88 is the curve which intersects curve 8 twice). The third column is the list of those curves in the Dynkin diagram that are blown down. Finally we give the variety, normalized by an appropriate linear fractional transformation of projective space, which is the image on $P^2(F)$ of the remaining -2 curves.

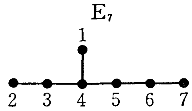
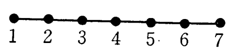
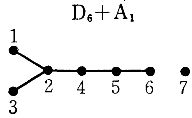
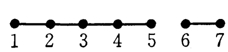
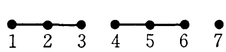
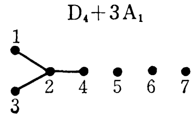
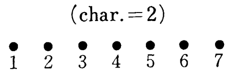
TABLE II: MINIMAL BIRATIONAL MAPPINGS OF MAXIMALLY DEGENERATE DEL PEZZO SURFACES ONTO P^2

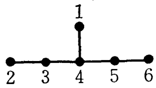
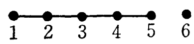
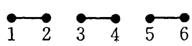
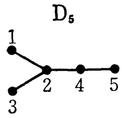
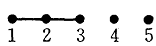
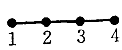
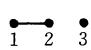
E_8 	8	2345678	y
D_8 	2	2345678	$x^2 - yz$
	2,7	234578	yz
	2,7	123567	yz
	2,7	134567	$y(x^2 - yz)$
E_7+A_1 	78,2	234568	yz
	78,2	123467	yz
	78,2	124567	$y(x^2 - yz)$
	78,88	234567	$y(x^3 + pxyz - yz^2)$
A_8 	18	2345678	$x^3 + pxyz - yz^2$ where $p \neq 0$
	3,6	123678	yz
	18,3	235678	$x(x^2 - yz)$
	18,3	134567	$x(x^2 - yz)$
	18,3,6	13467	xyz
	18,3,6	12456	xyz
E_6+A_2 	27,1	145678	yz
	27,78	234568	$y(x^3 + pxyz - yz^2)$
	27,68,1	12356	xyz
	27,68,1	12458	xyz
	27,68,1	23458	$xy(x^2 - yz)$
	27,68,78	12346	$y(y - z)(x^2 - yz)$
A_7+A_1 	28,4	124567	$y(x^2 - yz)$
	28,17	345678	$x(x^2 - yz)$
	28,88	234567	$(x^2 - yz)(2x^3 - 3x^2y + y^3 - x^2z - 2xyz + 3y^2z + yz^2)$
	28,17,4	34678	xyz
	28,17,4	14568	xyz
	28,17,4	23567	$xy(x^2 - yz)$
	28,68,4	12467	$yz(x^2 - yz)$
	28,68,17	13456	$x(x^2 - yz)(x^2 - y^2 - yz)$
	28,68,88	12456	$xy(x^3 - 2x^2y + xy^2 + pyz^2)$ where $p \neq 0$
	28,68,17,4	1346	$xyz(y-z)$
	28,68,17,4	2356	$xyz(x^2 - yz)$

D_5+A_3 	16,4	234678	$y(x^2 - yz)$
	16,38	234567	$(x^2 - yz)(x^2 - xy - yz)$
	16,68	124578	$y(x^3 + pxyz - yz^2)$ where $p \neq 0$
	16,38,4	14578	xyz
	16,57,4	12378	$yz(x^2 - yz)$
	16,57,4	23478	$yz(x^2 - yz)$
	16,38,57	24568	$x(x - z)(x^2 - yz)$
	16,57,68	12358	$y(y - z)(x^2 - yz)$
	16,57,68	23458	$y(y - z)(x^2 - yz)$
	16,38,57,4	1258	$xyz(y - z)$
	16,38,57,4	1347	$xyz(y - z)$
	16,38,57,4	2468	$xyz(y - z)$
	16,38,57,68	1345	$yz(x - y)(x - z)$
	16,38,57,68	1247	$xy(x - z)(x^2 - yz)$
$2D_4$ 	15,37	124678	$(x^2 - yz)(x^2 - pyz)$ where $p(p-1) \neq 0$
	15,37,2	24568	$yz(x^2 - yz)$
	15,37,48,2	1467	$xyz(y - z)$
	15,37,48,2	2578	$xyz(y - z)$
D_6+2A_1 	17,38	124568	$(x^2 - yz)(x^2 - y^2 - yz)$
	17,38,5	12568	$yz(y - z)$
	17,38,5	24578	$yz(y - z)$
	678,17,5	12358	$yz(x^2 - yz)$
	678,17,5	23458	$yz(x^2 - yz)$
	678,17,38	13456	$x(x^2 - yz)(x^2 - y^2 - yz)$
	678,17,77	12356	$xy(x^3 - 2x^2y + xy^2 + pyz^2)$ where $p \neq 0$
	678,17,77	12458	$xy(x^3 - 2x^2y + xy^2 + pyz^2)$ where $p \neq 0$
	678,17,77	23456	$y(x^2 - yz)(2x^3 - 3x^2y + y^3 - x^2z - 2xyz + 3y^2z + yz^2)$
	678,17,38,5	1345	$xyz(y - z)$
	678,17,38,5	1246	$xyz(x^2 - yz)$
$2A_4$ 	16,58	123478	$x(x^2 - yz)$
	16,14	234678	$y(x^3 + pxyz - yz^2)$ where $p \neq 0$
	16,47,58	12357	$x(x - z)(x^2 - yz)$
	16,35,47	12578	$y(y - z)(x^2 - yz)$
	16,35,58	13478	$y(y - z)(x^2 - yz)$
	16,35,14	23678	$z(x^2 - yz)(x^2 - xy - yz)$
	16,35,28,47	1458	$yz(x - y)(x - z)$
	16,35,47,58	1248	$yz(x - y)(x - z)$
	16,35,47,58	2368	$xy(x - z)(x^2 - yz)$
	16,35,28,58	3467	$yz(x - y)(x^2 - yz)$
	16,35,28,47,58	137	$xyz(x - z)(y - z)$

	168,15	234567	$y(x^3 + pxyz - yz^2)$ where $p \neq 0$
$A_5+A_2+A_1$	168,67	123457	$(x^2-yz)(2x^3-3x^2y+ y^3 - x^2z - 2xyz + 3y^2z + yz^2)$
	27,38,15	45678	xyz
	27,38,15	13467	$xy(x^2 - yz)$
	27,38,46	12458	$yz(x^2 - yz)$
	168,27,15	34578	$y(y - z)(x^2 - yz)$
	168,27,38	13457	$z(x^2 - yz)(x^2 - xy - yz)$
	168,27,67	23458	$z(x^2 - yz)(x^2 - y^2 - yz)$
	168,38,15	23567	$z(x^2 - yz)(x^2-xy - yz)$
	168,38,88	34567	$yz(x^3 - 3xyz + y^2z + yz^2)$
	168,46,67	12457	$xy(x^3 - 2x^2y + xy^2 + pyz^2)$ where $p \neq 0$
	168,46,67	23478	$xy(x^3 - 2x^2y + xy^2 + pyz^2)$ where $p \neq 0$
	168,578,88	23456	$x(x^2 - yz)(3x^3 + xy^2 - x^2z - 3xyz + yz^2)$
	168,27,38,15	2456	$xy(x - z)(x^2 - yz)$
	168,27,38,46	1457	$yz(x - z)(x^2 - yz)$
	168,27,46,15	3478	$yz(x - z)(x^2 - yz)$
	168,27,46,15	2358	$yz(x - z)(x^2 - yz)$
	168,27,46,67	2458	$yz(y - z)(x^2 - yz)$
	168,578,27,15	2346	$z(x - y)(x^2 - yz)(x^2 - xy - yz)$
	168,578,27,38	2456	$z(x - y)(x^2 - yz)(x^2 - xy - yz)$
	168,578,27,67	1345	$xz(x^2 - yz)(x^2 - y^2 - yz)$
	168,578,38,88	1245	$xyz(x^3 - 3xyz + y^2z + yz^2)$
	168,578,38,88	2356	$xyz(xy^2 + x^2z - 3xyz + yz^2)$
	168,27,38,46,15	357	$xyz(x - z)(y - z)$
	168,578,27,38,15	346	$yz(x - y)(x - z)(x^2 - yz)$
	168,578,27,38,46	135	$yz(x - y)(x - z)(x^2 - yz)$
	168,578,27,46,67	235	$xyz(y - z)(x^2 - yz)$
	168,578,27,38,46,15	24	$xyz(x - y)(x - z)(y - z)$
	148,278,13	34567	$z(x^2 - yz)(x^2 - y^2 - yz)$
$2A_3+2A_1$ (char. $\neq 2$)	148,167,13	23568	$xy(x^3 - 2x^2y + xy^2 + pyz^2)$ where $p \neq 0$
	148,278,88	23456	$y(x^2 - yz)(8x^3 - 4x^2y + 4y^2z + yz^2 - 8xyz)$
	148,167,347	23568	$(x^2 - yz)(x^2 - xy - yz)(x^2 + 4xz - yz)$
	148,167,13,25	3578	$yz(y - z)(x^2 - yz)$
	148,278,13,25	3567	$yz(y - z)(x^2 - yz)$
	148,167,25,46	2378	$yz(2x - y - z)(x^2 - yz)$
	148,278,368,88	3457	$xyz(x^3 - 2x^2y + xy^2 + 8xyz - 4yz^2)$
	148,278,368,25	1357	$yz(x^2 - yz)(xy + xz - 2yz)$
	148,278,357,25	1356	$yz(x^2 - yz)(xy+xz - 2yz)$

	148,278,578,88	2356	$yz(x^2-yz)(x^2y-x^2z+4xyz-y^2z+yz^2)$
	148,278,167,578	2346	$(x+2z)(x^2-yz)(x^2-xy-yz) \times(4x^2-4xy-y^2-8yz)$
	148,278,368,13,25	467	$xyz(2x-y-z)(x^2-yz)$
	148,167,368,13,25	247	$yz(x-z)(y-z)(x^2-yz)$
	148,278,167,13,25	346	$xyz(y-z)(x^2-yz)$
	148,278,368,167,13	245	$z(x-y)(x-z)(x^2-yz)(x^2-xy-yz)$
	148,278,368,578,88	135	$xyz(y-z)(x^2z+8xy^2-4y^2z+4yz^2-8xyz)$
	148,278,167,578, 347,25	36	$yz(x-z)(y-z)(x+y)(x^2-yz)$
	148,278,368,167,578, 347	25	$(x-y)(x-z)(x+y)(x+z)(x^2-yz) \times(x^2+yz)$
	148,278,368,167,347, 13,25	5	$xyz(x-y)(x-z)(y-z)(x-y-z)$
$4A_2$ 	138,145,12	25678	$yz(x^3-3xyz+y^2z+yz^2)$
	138,145,236,12	3578	$z(x-y-z)(x^2-yz)(x^2-xy-yz)$
	138,145,167,12	2357	$xyz(xy^2+x^2z-3xyz+yz^2)$
	138,145,167,236,357	248	$z(x-y)((p+1)x-y-pz)(x^2-yz) \times(x^2-xy+(p-1)yz)$ where $p^2-p+1=0$
	138,145,167,236,247, 258,357,468		$xyz(x-z)(y-pz)(x-y-z)((p+1) \times x-y)((p+1)x-y-z)$ where $p^2+p+1=0$
D_4+4A_1 (char. $=2$) 	156,457,178	23468	$(x^2-yz)(x^2-y^2-yz)(x^2+pyz)$ where $p(p-1)\neq 0$
	156,457,178,2	2368	$yz(y-z)(x^2-yz)$
	156,457,358,2	2678	$yz(y-z)(x^2-yz)$
	156,457,358,55	1278	$xy(x-y)(px^3+(p+1)x^2y+xy^2+(p+1)yz^2)$ where $p(p-1)\neq 0$
	5678,156,457,55	2348	$yz(x^2-yz)(x^2y+x^2z+py^2z+yz^2)$ where $p-1\neq 0$
	5678,156,457,367	1348	$(x-z)(x^2-yz)(x^2+pyz)(x^2+py^2+(p+1)yz)$ where $p(p+1)\neq 0$
	156,457,178,358,2	346	$yz(x-z)(y-z)(x^2-yz)$
	5678,156,457,358,55	134	$xyz(y-z)(x^2y+px^2z+y^2z+yz^2)$ where $p(p-1)\neq 0$
	5678,156,457,178,358, 468	23	$(x-y)(x-z)((p+1)x-z)(x-(p+1)y)(x^2-yz)(x^2+pxy+yz)$ where $p(p-1)\neq 0$
	156,457,178,358,468, 367,2	2	$xyz(x-y)(x-z)(y-z)(x-y-z)$
$8A_1$ (char. $=2$) 	1234,1256,1368,2367	4578	$(x^2-yz)(x^2-pyz)((q^2-p)x^2+p(p-1)y^2+(q^2-p^2)yz)((q^2-p)x^2+p(q^2-1)yz+q^2(p-1)z^2)$ where $p(p-1)(q^2-1)(q^2-p)\neq 0$

	1234,1256,1278,1368	457	$yz(y-z)(x^2-yz)((p+q)x^2y+q(p+1)x^2z+p(q+1)y^2z+p(p+q)yz^2)$ where $p(p-1)(q-1)(p+q)(p^2+q)(pq^2+p^2+pq+q)\neq 0$
	1234,1256,1278,1368, 1357,2358,2367	4	$xyz(x-y-z)((p+q+1)xy+pqxz+pqyz+pqz^2)(qx^2+qxy+qyz+p(p+q+1)yz)(pxy+py^2+q(p+q+1)xz+pyz)$ where $pq(p+q+1)\neq 0$
	1234,1256,1278,1368, 1357,1458,1467,11		$xyz(x-y)(x-z)(y-z)(x-y-z)\times(x^2y-xy^2-qx^2z+py^2z+qxz^2-pyz^2)$ where $pq(p-1)(q-1)(p-q)\times(p+q+1)\neq 0$
E_7 	7	234567	y
A_7 	2	234567	x^2-yz
	2,6	23467	yz
D_6+A_1 	67,3	23457	yz
	67,3	12356	yz
	67,3	12456	$y(x^2-yz)$
A_5+A_2 	17,3	34567	yz
	17,56	23457	$x(x^2-yz)$
	17,56,3	2357	xyz
	17,56,3	1245	xyz
$2A_3+A_1$ 	14,57	12356	$y(x^2-yz)$
	14,57,36	1267	xyz
	14,57,27	2356	$yz(x^2-yz)$
	14,57,36,27	135	$xyz(y-z)$
D_4+3A_1 	15,37,46	1267	$yz(y-z)$
	567,15,37	1246	$yz(x^2-yz)$
	567,15,37,46	134	$xyz(y-z)$
$7A_1$ (char.$=2$) 	123,145,246,167	357	$yz(y-z)(x^2-yz)$
	123,145,246,167, 357,257,356		$xyz(x-y)(x-z)(y-z)\times(x-y-z)$

E_6 	6	23456	y
A_5+A_1 	16	12345	x^2-yz
	16,4	1245	yz
	16,4	2346	yz
$3A_2$ 	16,23	3456	yz
	16,23,45	246	xyz
D_5 	3	2345	y
A_3+2A_1 	15,34	235	yz
A_4 	2	234	y
A_2+A_1 	13	12	y

II. Compactifications of affine surfaces. An *analytic* (or *algebraic*) *compactification* of a two-dimensional affine variety V is an injective holomorphic (regular) mapping h of V into a normal compact complex surface X (the *compactifying surface*) such that $X-h(V)$ is a closed curve Y (the *curve at infinity*) on X. For $V=C^2$ the problem of finding all analytic compactifications with X non-singular and for which Y consists of non-singnlar components meeting transversally was solved in detail by Morrow [17], following important contributions by Remmert and Van de Ven [23], Van de Ven [28], Ramanujam [22], and Kodaira [14]. In [4] and [6] we took up the question of the possible singularities of Y and of X if the conditions of normal meetings and of smoothness are relaxed.

Meanwhile attention turned to compactifications of other affine spaces such as $\boldsymbol{C} \times \boldsymbol{C}^*$ and $(\boldsymbol{C}^*)^2$. The complete list (modulo blowing up extra points at infinity) of rational compactifying surfaces of these varieties was given by Suzuki [25], while for $V = (\boldsymbol{C}^*)^2$, all non-rational compactifying surfaces were listed by Ueda [27]. (See also Simha [24]. For $V = \boldsymbol{C}^2$ and $V = \boldsymbol{C} \times \boldsymbol{C}^*$, all compactifying surfaces are rational (Kodaira [14]).) These ideas have recently been extended by Enoki [15], with especially interesting results in the non-algebraic case.

Now if X is a Del Pezzo surface over $\boldsymbol{C}$ and if $f: X \to \boldsymbol{CP}^2$ is one of the rational mappings listed in Table II, with associated plane curve C, then $(f|_{P^2-C})^{-1}: \boldsymbol{P}^2 - C \to X$ is a compactification of the affine variety $V = \boldsymbol{P}^2 - C$. Conversely, if $h: V \to X$ is any *algebraic* compactification, then h^{-1} extends meromorphically across the curve at infinity in X to provide a birational mapping $f: X \to \boldsymbol{CP}^2$. Thus classifying minimal birational mappings of Del Pezzo surfaces onto $\boldsymbol{CP}^2$ is equivalent to classifying certain compactifications of affine subvarieties of $\boldsymbol{CP}^2$ (cf. [6]).

DEFINITIONS. Let V be an open two-dimensional variety, let S be a compact surface containing V as a Zariski open subset, and let $h: V \to X$ be an algebraic compactification of V. Then h is called a *minimal compactification of V in S* provided (a) h does not extend regularly across any component of the curve $S - V$, and (b) if $\tilde{h}: S \to X$ is the meromorphic extension of h to S, then $\tilde{h}$ admits a resolution of singularities

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ S \arrow[rr, "\tilde{h}"] & & X \end{tikzcd}$$

for which π is the minimal resolution of singularities of X. For any affine variety V, a compactification $h: V \to X$ will be called a *minimal projective compactification* if V admits an embedding into $\boldsymbol{CP}^2$ for which h is a minimal compactification of V in $\boldsymbol{CP}^2$. A normal compact surface X is called *minimal* if X has smallest second Betti number among all normal surfaces dominated by its non-singular model.

LEMMA 2. *Let V be an affine subvariety of $\boldsymbol{CP}^2$, and let $h: V \to X$ be a minimal projective compactification of V such that X is a singular* *minimal Gorenstein surface with vanishing geometric genus. Then X is a maximally degenerate Del Pezzo surface and h^{-1} extends to one of the rational maps classified in Table II. In particular, the only varieties V admitting such compactifications are the complements in $\boldsymbol{P}^2$ of the curves C listed in the fourth column of the Table.*

To prove the lemma one notes that the existence of h implies that X is rational. But a minimal rational surface $X \neq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ has $b_2(X) = 1$ ([19]). In this context, $p_g = 0$ implies that the canonical bundle K_X is negative ([2]), so X is a maximally degenerate Del Pezzo surface as claimed. That h^{-1} extends to a mapping of the type classified is clear from the definitions of minimal in each section.

We will now apply our work to the problem of finding compactifications for a particularly interesting class of affine surfaces, which includes, for instance, the spaces $\boldsymbol{C}^2$, $\boldsymbol{C} \times \boldsymbol{C}^*$, and $(\boldsymbol{C}^*)^2$.

LEMMA 3. *Let $P_1, \cdots, P_r, Q_1, \cdots, Q_s$ be points of $\boldsymbol{C}$, with $r \leqq s$. Put $V = (\boldsymbol{C} - \{P_1, \cdots, P_r\}) \times (\boldsymbol{C} - \{Q_1, \cdots, Q_s\})$, and let $h: V \to S$ be an algebraic compactification with S non-singular. Denote by $Y = \bigcup_{i=1}^n Y_1$ the curve $S - h(V)$, and assume that* (a) *each component Y_i is non-singular,* (b) *Y_i meets Y_j transversally in a single point, if at all, and* (c) *$Y_i \cap Y_j \cap Y_k = \varnothing$ for all distinct vertices i, j, k. Denote the weighted dual intersection graph of Y by Γ_Y, the subgraph of Γ_Y consisting of the union of all elementary cycles* (i.e., *the "two-connected block" of Γ_Y) by Γ'_Y, and the fundamental group of the boundary of a tubular neighborhood of Y in S by $\pi_1(\Gamma_Y)$. (If $H^1(Y, \boldsymbol{R}) = 0$, then $\pi_1(\Gamma_Y)$ is the group on n generators $x_1, \cdots, x_n$ with relations $\prod_{j=1}^n x_j^{Y_j \cdot Y_i} = 1$ $\forall i$ and $(x_i x_j)^{Y_i \cdot Y_j} = (x_j x_i)^{Y_i \cdot Y_j}$ $\forall i, j$; see Mumford* [18].) *Then Γ_Y has the following properties.*

(1) *The number n of vertices of Γ_Y is equal to $r + s + b_2(S)$, where b_2 is the second Betti number.*

(2) *If c denotes the circuit rank (the number of independent cycles of a graph), then $c(\Gamma_Y) = \dim H^1(Y, \boldsymbol{R}) = rs$.*

(3) *If $r = 0$ then $\Gamma'_Y = \varnothing$ and $\pi_1(\Gamma_Y)$ is free on s generators.*

(4) *If $r \geqq 1$ then Γ'_Y contains at least $r + 1$ vertices of order $s + 1$, and each component of the weighted graph $\Gamma_Y - \Gamma'_Y$ has determinant equal to ± 1.*

PROOF. Suppose that $h_1: V \to S_1$ and $h_2: V \to S_2$ are two such compactifications. Then, since the maps are algebraic, $h_2 \circ h_1^{-1}$ extends to a birational mapping $f: S_1 \to S_2$. If properties (a), (b), and (c) hold for both S_1 and S_2, then the singularities of f can be resolved by a diagram

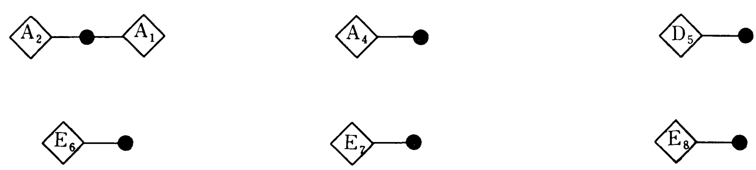
$$\begin{tikzcd} & \tilde{S} \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ S_1 \arrow[rr, "f"] & & S_2 \end{tikzcd}$$

where ρ_1 and ρ_2 are inverse to a sequence of monoidal transformations at points of $S_1 - h_1(V)$ and $S_2 - h_2(V)$, and where conditions (a), (b), and (c) continue to hold for the curve at infinity in each intermediate step between S_1 and $\tilde{S}$ and between S_2 and $\tilde{S}$. Since under these conditions properties (1), (2), (3), and (4) are preserved by monoidal point transformations and their inverses, we conclude that if V admits any such compactification satisfying (1)-(4), then all such compactifications will.

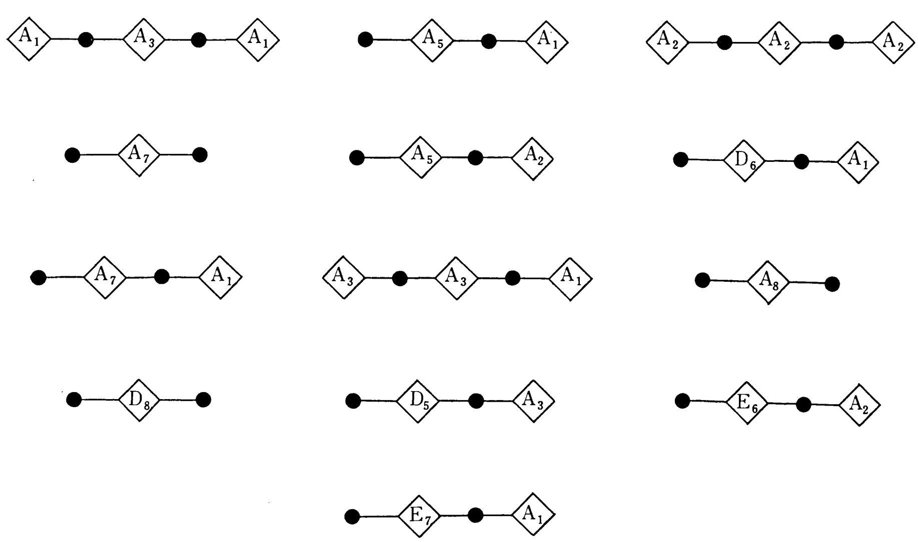
But clearly (a)-(c) and (1)-(4) hold for the standard embedding $(\boldsymbol{C} - \{P_1, \cdots, P_r\}) \times (\boldsymbol{C} - \{Q_1, \cdots, Q_s\}) \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$, for which Γ_Y is the bipartite

TABLE III: MINIMAL SINGULAR PROJECTIVE COMPACTIFICATIONS OF $(\boldsymbol{C} - \{P_1, \cdots, P_r\}) \times (\boldsymbol{C} - \{Q_1, \cdots, Q_s\})$

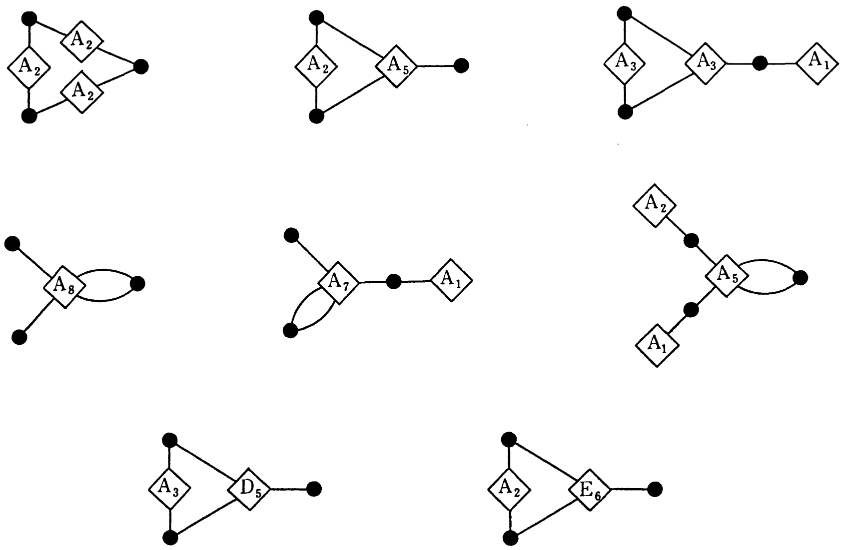
Compactifications of $\boldsymbol{C}^2$



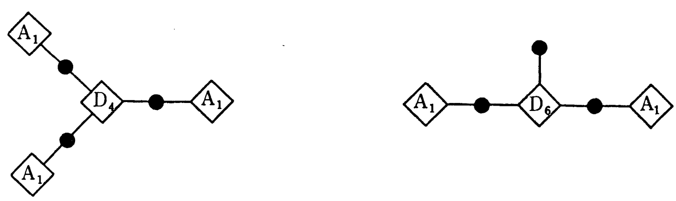
Compactifications of $\boldsymbol{C} \times \boldsymbol{C}^*$



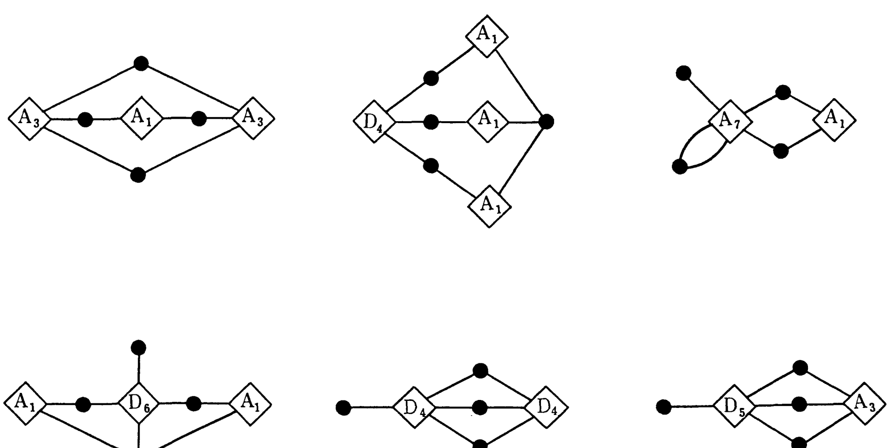
Compactifications of $(C^*)^2$



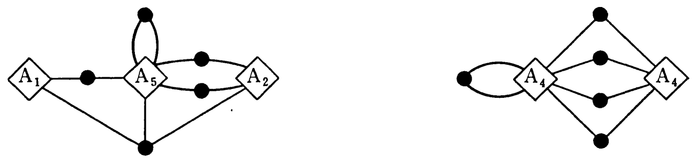
Compactifications of $C \times (C - \{0, 1\})$



Compactifications of $C^* \times (C-\{0, 1\})$



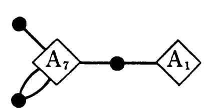
Compactifications $(\boldsymbol{C} - \{0, 1\})^2$



graph $K_{r+1,s+1}$ with all weights zero. This completes the proof.

THEOREM. *Let V be an affine variety of the type $(\boldsymbol{C} - \{P_1, \cdots, P_r\}) \times (\boldsymbol{C} - \{Q_1, \cdots, Q_s\})$. Suppose that V admits a minimal projective compactification $h: V \to X$, where X is a minimal singular Gorenstein surface with $p_g = 0$. Then $r \leqq 2$, $s \leqq 2$, and X is a maximally degenerate Del Pezzo surface. Furthermore, every such compactification is represented in Table III.*

(We identify the compactification by drawing the graph of the curve $Y = X - h(V)$ at infinity. The solid dots represent the irreducible components of Y, while the symbol Γ_k inserted in an edge (or multiple edge or quasi-edge) means that the corresponding curves intersect in the indicated singular point of X. Thus for example the picture below means



that Y has three irreducible components, the first passing simply through the A_7 singularity, the second with two local components passing through A_7, the third passing through both A_7 and A_1, and with no other points of intersection.)

PROOF OF THE THEOREM. By Lemma 2 each compactification of the type under consideration is represented in Table II. Indeed, for each entry of Table II the given map $\rho: X \to \boldsymbol{CP}^2$ represents a minimal projective compactification of a space V of type $(\boldsymbol{C} - \{P_1, \cdots, P_r\}) \times (\boldsymbol{C} - \{Q, \cdots, Q_s\})$ if and only if the variety $\boldsymbol{CP}^2 - C$, for C the projective plane curve indicated in the fourth column, is isomorphic to such a V. When C is one of the collections of lines

$$\begin{array}{ll} y & xy(x-y) \\ xy & xyz(y-z) \\ xyz & xyz(x-y)(z-x)\ , \end{array}$$

this is trivially the case. The Cremona transformations $[x, y, z] \to [xz,$

$yz - x^2, z^2]$ and $[x, y, z] \to [x^2, yz, xy]$ on $\boldsymbol{CP}^2$ give three additional isomorphisms $\boldsymbol{CP}^2 - C \cong V$ as follows:

$$\begin{aligned} y(yz - x^2)\,, &\quad \text{for} \quad V = \boldsymbol{C} \times \boldsymbol{C}^* \\ xy(yz - x^2)\,, &\quad \text{for} \quad V = (\boldsymbol{C}^*)^2 \\ xyz(yz - x^2)\,, &\quad \text{for} \quad V = \boldsymbol{C}^* \times (\boldsymbol{C} - \{0, 1\}) \end{aligned}$$

For each occurrence (a total of 37) in Table II of one of these 9 admissible curves, the "graph at infinity" for the corresponding compactification is easily determined from the graph Γ_Y described in the second column of Table II, by collapsing each component of the Dynkin diagram Γ to a (quasi-) edge. For every other curve C appearing in the fourth column of Table II, it is a straightforward verification that the corresponding graph Γ_Y does not satisfy properties (1)-(4) of Lemma 3 for any choice of r and s, so the only compactifications of this type are the 37 listed.

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MATHEMATICS DEPARTMENT
WAYNE STATE UNIVERSITY
DETROIT, MI 48202
U.S.A.