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The Fejér-Riesz inequality for Siegel domains

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Introduction. The classical Fejér-Riesz inequality ([2]) was extended from the unit disc of the complex plane $\boldsymbol{C}$ to balls and polydiscs of $\boldsymbol{C}^n$ ([4], [9], and [10]). For unbounded domains, Hille and Tamarkin derived an analogous inequality. Let $f \in H^p(\boldsymbol{R}_+^2)$, $1 \leqq p < \infty$, where $\boldsymbol{R}_+^2$ denotes the upper half-plane $\{z \in \boldsymbol{C} \mid \operatorname{Im} z > 0\}$. Then the following holds for every $x \in \boldsymbol{R}$ ([5, Theorem 4.1]):

$$\int_{\boldsymbol{R}_+} |f(x+iy)|^p dy \leqq 2^{-1} \sup_{y>0} \int_{\boldsymbol{R}} |f(x+iy)|^p dx \,, \tag{1}$$

where $\boldsymbol{R}_+$ denotes the positive numbers. Kawata [6] and Krylov [8] showed that the main results of the Hille-Tamarkin's H^p theory are valid for all $p > 0$. The inequality (1) is also seen to hold in this case. Our purpose is to deal with this inequality in a setting of higher dimensions and a wider class of functions. We shall obtain an inequality of the same sort for functions u such that $u \geqq 0$ and $\log u$ are plurisubharmonic on certain Siegel domains in $\boldsymbol{C}^n \times \boldsymbol{C}^m$. The principal result is Theorem 1 in Section 2. Section 3 is concerned with Hardy space results.

1. Preliminaries. Let u be a real-valued function on $\boldsymbol{R}_+^2$. If $u \geqq 0$ and $\log u$ is subharmonic we shall call u a log. subharmonic function. Such functions are called functions of class PL and then basic properties are found in [11]. We shall denote by $LH^p(\boldsymbol{R}_+^2)$, $0 < p < \infty$, the class of log. subharmonic functions u satisfying the condition

$$M(u, p; \boldsymbol{R}_+^2) := \sup_{y>0} \int_{\boldsymbol{R}} u(x+iy)^p dx < \infty \,. \tag{2}$$

Let Ω be an open cone in $\boldsymbol{R}^n$ which is the interior of the convex hull of n linearly independent half-lines starting from the origin. We shall call Ω an n-polygonal cone. The tube domain with base Ω is defined by $T_\Omega = \{X + iY \in \boldsymbol{C}^n \mid X \in \boldsymbol{R}^n,\ Y \in \Omega\}$. Let u be a real-valued function defined on T_Ω and $u \geqq 0$. If $\log u$ is plurisubharmonic we shall call u a log.

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plurisubharmonic function. We define the class $LH^p(T_\Omega)$, $0 < p < \infty$, as the family of log. plurisubharmonic functions u satisfying the condition

$$M(u, p; T_\Omega) = \sup_{Y \in \Omega} \int_{R^n} u(X + iY)^p dX < \infty ,$$

where $dX = dx_1 \cdots dx_n$, the volume element in $\boldsymbol{R}^n$. The Hardy space $H^p(T_\Omega)$ consists of holomorphic functions f on T_Ω such that $M(|f|, p; T_\Omega) < \infty$ ([14]). The Siegel domain of type II we shall throughout consider is the domain in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ defined by an n-polygonal cone $\Omega \subset \boldsymbol{R}^n$ and an Ω-hermitian form Φ: $\boldsymbol{C}^m \times \boldsymbol{C}^m \to \boldsymbol{C}^n$, i.e., $D = D(\Omega, \Phi) = \{(Z, W) \in \boldsymbol{C}^n \times \boldsymbol{C}^m | \operatorname{Im} Z - \Phi(W, W) \in \Omega\}$. If $n = 1$, $\Omega = \boldsymbol{R}_+$, and $\Phi(W, W) = \sum_{j=1}^m |w_j|^2$ for $W = (w_1, \cdots, w_m)$, the associated domain, D_0, is biholomorphic with the unit ball of $\boldsymbol{C}^{m+1}$. Let u be a log. plurisubharmonic function on D. Then $u(X + i(Y + \Phi(W, W)), W)$ is an upper semi-continuous function of $(X + iY, W) \in T_\Omega \times \boldsymbol{C}^m$. We define the class $LH^p(D)$, $0 < p < \infty$, as the totality of log. plurisubharmonic functions u on D satisfying

$$M(u, p; D) = \sup_{Y \in \Omega} \int_{R^n \times C^m} u(X + i(Y + \Phi(W, W)), W)^p dX dW < \infty ,$$

where dW means the volume element in $\boldsymbol{R}^{2m} = \boldsymbol{C}^m$. The Hardy space $H^p(D)$ is the class of holomorphic functions f on D such that $M(|f|, p; D) < \infty$ ([7]). If $f_j \in H^p(D)$, $j = 1, \cdots, l$, then $\sum_{j=1}^l |f_j| \in LH^p(D)$. If $u \in LH^p(D)$ and v is any plurisubharmonic function bounded above, then $ue^v \in LH^p(D)$. Thus $LH^p(D)$ contains discontinuous functions.

We shall frequently use the following basic result. This is found in [14, (4.9) in Chapter *II*] and is valid without the assumption of continuity of u.

LEMMA A. *Let $u(x + iy)$ be a subharmonic function on the half-plane $\boldsymbol{R}_+^2$ and $u \geqq 0$. If u satisfies the condition* (2) *with $p \geqq 1$, then $u(x + iy) \to 0$ as $x^2 + y^2 \to \infty$, provided $y \geqq \rho$ for a constant $\rho > 0$.*

2. The Fejér-Riesz inequality for the domain D. We begin by proving some lemmas. Arguments concerning $\boldsymbol{R}_+^2$ were suggested by the methods of [1] and [2]. However, they must be substantially reformulated to work for the unbouded domain.

LEMMA 1. *Let $f(x + iy)$ be a holomorphic function on $\boldsymbol{R}_+^2$. Then the following inequality holds for $0 < r < R$, $0 < T$:*

$$(3) \qquad 2\int_r^R |f(iy)|^2 dy \leqq \int_{-T}^T |f(x + ir)|^2 dx + \int_{-T}^T |f(x + iR)|^2 dx + \int_r^R |f(-T + iy)|^2 dy + \int_r^R |f(T + iy)|^2 dy .$$

PROOF. Let E and E' be the rectangles in $\boldsymbol{R}^2_+$ with vertices $-T+ir$, ir, iR, $-T+iR$, and ir, $T+ir$, $T+iR$, iR, respectively. Then the Cauchy integral theorem applied to the holomorphic function $f(z)^2$ with respect to E and E' implies

$$\begin{aligned} i\int_r^R f(iy)^2 dy &= -\int_{-T}^0 f(x+ir)^2 dx + i\int_r^R f(-T+iy)^2 dy + \int_{-T}^0 f(x+iR)^2 dx \\ &= \int_0^T f(x+ir)^2 dx + i\int_r^R f(T+iy)^2 dy - \int_0^T f(x+iR)^2 dx \,. \end{aligned}$$

It follows that

$$\begin{aligned} 2\left|\int_r^R f(iy)^2 dy\right| &\leqq \int_{-T}^T |f(x+ir)|^2 dx + \int_{-T}^T |f(x+iR)|^2 dx \\ &\quad + \int_r^R |f(-T+iy)|^2 dy + \int_r^R |f(T+iy)|^2 dy \,. \end{aligned}$$

If $f(z)$ is real-valued on the imaginary axis in $\boldsymbol{R}^2_+$, this becomes the inequality (3). In the general case, let $g(z) = 2^{-1}(f(z) + \overline{f(-\bar{z})})$, $h(z) = (2i)^{-1}(f(z) - \overline{f(-\bar{z})})$, $z \in \boldsymbol{R}^2_+$. Then $g(z)$ add $h(z)$ are holomorphic on $\boldsymbol{R}^2_+$ and real-valued on the imaginary axis, so satisfy the inequality (3). Note that $|f(iy)|^2 = g(iy)^2 + h(iy)^2$, $y \in \boldsymbol{R}$, and $|g(z)|^2 + |h(z)|^2 = 2^{-1}(|f(z)|^2 + |f(-\bar{z})|^2)$, $z \in \boldsymbol{R}^2_+$. It is easily verified that the inequality (3) is valid for $f(z)$. The proof is completed.

We shall write $P(x, y) = \pi^{-1} y (x^2 + y^2)^{-1}$, the Poisson kernel for $\boldsymbol{R}^2_+$, and $u_\rho(x+iy) = u(x + i(\rho + y))$ for a constant ρ.

LEMMA 2. *Let* $u \in LH^1(\boldsymbol{R}^2_+)$ *and let* $u_{\rho,\varepsilon}(x+iy) = (u_\rho(x+iy) + \varepsilon)^{1/2}$ *for* $\rho, \varepsilon > 0$. *Let*

$$h_{\rho,\varepsilon}(x+iy) = \int_{\boldsymbol{R}} \log u_{\rho,\varepsilon}(t) P(x-t, y) dt \,, \quad x+iy \in \boldsymbol{R}^2_+ \,. \tag{4}$$

Then $h_{\rho,\varepsilon}$ *is a harmonic majorant of* $\log u_{\rho,\varepsilon}$ *on* $\boldsymbol{R}^2_+$.

PROOF. The sum of two log. subharmonic functions is log. subharmonic, so the function $\log u_{\rho,\varepsilon}$ is upper semi-continuous on the closure $\overline{\boldsymbol{R}^2_+}$ of $\boldsymbol{R}^2_+$, and subharmonic on $\boldsymbol{R}^2_+$. Lemma A implies that $\log u_{\rho,\varepsilon}(x+iy) \to 2^{-1}\log\varepsilon$ as $x^2 + y^2 \to \infty$, $y \geqq 0$. Thus $\log u_{\rho,\varepsilon}(t)$ is bounded on $\boldsymbol{R}$ and $h_{\rho,\varepsilon}$ is defined and harmonic on the whole of $\boldsymbol{R}^2_+$. We can choose a sequence of bounded continuous functions $u_k(t)$ on $\boldsymbol{R}$ such that $u_1(t) \geqq u_2(t) \geqq \cdots$ and $u_k(t) \to \log u_{\rho,\varepsilon}(t)$ as $k \to \infty$. Let

$$h_k(x+iy) = \int_{\boldsymbol{R}} u_k(t) P(x-t, y) dt \,, \quad x+iy \in \boldsymbol{R}^2_+ \,, \quad k = 1, 2, \cdots \,.$$

Then h_k are continuous on $\overline{\boldsymbol{R}^2_+}$, harmonic and satisfy $2^{-1}\log\varepsilon \leqq h_k$ on $\boldsymbol{R}^2_+$. Since $\log u_{\rho,\varepsilon}(t) \leqq u_k(t) = h_k(t)$, $t \in \boldsymbol{R}$, the maximum principle for subharmonic functions implies that $\log u_{\rho,\varepsilon} \leqq h_k$ on $\boldsymbol{R}^2_+$. Letting $k \to \infty$ we get the relation $\log u_{\rho,\varepsilon} \leqq h_{\rho,\varepsilon}$. The proof is completed.

LEMMA 3. *Let $u \in LH^1(\boldsymbol{R}^2_+)$ and $\rho > 0$. Then the following inequality holds for every $x \in \boldsymbol{R}$:*

$$\int_{\boldsymbol{R}_+} u_\rho(x+iy)dy \leqq 2^{-1}\int_{\boldsymbol{R}} u_\rho(x)dx\ .$$

PROOF. We may assume that $x = 0$. Let $\varepsilon > 0$ and define the function $h_{\rho,\varepsilon}$ by (4). Let $F(z) = \exp(h_{\rho,\varepsilon}(z) + ig_{\rho,\varepsilon}(z))$, $z \in \boldsymbol{R}^2_+$, where $g_{\rho,\varepsilon}$ is so chosen that $h_{\rho,\varepsilon} + ig_{\rho,\varepsilon}$ is holomorphic. From Lemma 2 we see that $u_\rho(z) + \varepsilon \leqq \exp(2h_{\rho,\varepsilon}(z)) = |F(z)|^2$. Let $0 < r < R$, $0 < T$. The inequality (3) applied to $F(z)$ implies that

$$\begin{aligned}2I(r, R) :&= 2\int_r^R u_\rho(iy)dy < 2\int_r^R |F(iy)|^2 dy\\ &\leqq \int_{-T}^T |F(x+ir)|^2 dx + \int_{-T}^T |F(x+iR)|^2 dx\\ &\quad + \int_r^R |F(-T+iy)|^2 dy + \int_r^R |F(T+iy)|^2 dy\ .\end{aligned}$$

Using inequalities

$$|F(z)|^2 \leqq \left(\int_{\boldsymbol{R}} u_{\rho,\varepsilon}(t)P(x-t, y)dt\right)^2 \leqq \int_{\boldsymbol{R}} u_{\rho,\varepsilon}(t)^2 P(x-t, y)dt\ ,$$

we have

$$\begin{aligned}2I(r, R) &< \int_{-T}^T dx\int_{\boldsymbol{R}} u_{\rho,\varepsilon}(t)^2 P(x-t, r)dt + \int_{-T}^T\left(\int_{\boldsymbol{R}} u_{\rho,\varepsilon}(t)P(x-t, R)dt\right)^2 dx\\ &\quad + \int_r^R dy\int_{\boldsymbol{R}} u_{\rho,\varepsilon}(t)^2 P(-T-t, y)dt + \int_r^R dy\int_{\boldsymbol{R}} u_{\rho,\varepsilon}(t)^2 P(T-t, y)dt\ .\end{aligned}$$

Letting $\varepsilon \to 0$ we see that

$$\begin{aligned}2I(r, R) &\leqq \int_{-T}^T dx\int_{\boldsymbol{R}} u_\rho(t)P(x-t, r)dt + \int_{-T}^T\left(\int_{\boldsymbol{R}} u_\rho(t)^{1/2}P(x-t, R)dt\right)^2 dx\\ &\quad + \int_r^R dy\int_{\boldsymbol{R}} u_\rho(t)P(-T-t, y)dt + \int_r^R dy\int_{\boldsymbol{R}} u_\rho(t)P(T-t, y)dt\\ &=: I_1(r, T) + I_2(R, T) + I_3(r, R, T) + I_4(r, R, T)\ .\end{aligned}$$

Clearly, $I_1(r, T) \leqq \int_{\boldsymbol{R}} u_\rho(t)dt$. We treat I_j, $j = 3, 4$. Let

$$v(\mp T, y) = \int_{\boldsymbol{R}} u_\rho(t)P(\mp T - t, y)dt\ .$$

Since $u_\rho(t) \to 0$ as $|t| \to \infty$ by Lemma A, we can take $K > 0$ such that $u_\rho(t) < R^{-2}$ for $|t| > K$. Using the inequality $y(a^2 + y^2)^{-1} < a^{-1}$ for $a, y > 0$ and taking an arbitrary $T > K$, we can see that

$$v(\mp T, y) < R^{-2} + \int_{|t| \leqq K} u_\rho(t) P(\mp T - t, y) dt < R^{-2} + (T - K)^{-1} \int_{\boldsymbol{R}} u_\rho(t) dt .$$

It follows that

$$I_j(r, R, T) < R^{-1} + (T - K)^{-1} R \int_{\boldsymbol{R}} u_\rho(t) dt , \quad T > K , \quad j = 3, 4 .$$

To estimate the integral $I_2(R, T)$, let

$$G(x, y) = \int_{\boldsymbol{R}} u_\rho(t)^{1/2} P(x - t, y) dt , \quad x + iy \in \boldsymbol{R}^2_+ .$$

Since $u_\rho(t)^{1/2} \in L^2(\boldsymbol{R})$, we have $G(x, y) \leqq Cv(x)$, $y > 0$, where C is a constant and $v(x)$ is the Hardy-Littlewood maximal function of $u_\rho(x)^{1/2}$. Note that $v(x) \in L^2(\boldsymbol{R})$ and $G(x, R)^2 \to 0$ as $R \to \infty$. Now in the inequality

$$2I(r, R) \leqq \int_{\boldsymbol{R}} u_\rho(x) dx + \int_{\boldsymbol{R}} G(x, R)^2 dx + 2R^{-1} + 2(T - K)^{-1} R \int_{\boldsymbol{R}} u_\rho(x) dx , \quad T > K ,$$

letting first $T \to \infty$ and then $R \to \infty$, $r \to 0$, we have

$$2 \int_{\boldsymbol{R}_+} u_\rho(iy) dy \leqq \int_{\boldsymbol{R}} u_\rho(x) dx ,$$

which completes the proof.

LEMMA 4. *Let T_Ω be a tube with base Ω which is an n-polygonal cone in $\boldsymbol{R}^n$. Let $u \in LH^1(T_\Omega)$ and let $u_\rho(X + iY) = u(X + i(\rho + Y))$ where $\rho = (\rho_1, \cdots, \rho_n) \in \Omega$. Then for any $X \in \boldsymbol{R}^n$*

$$\int_\Omega u_\rho(X + iY) dY \leqq 2^{-n} \int_{\boldsymbol{R}^n} u_\rho(X) dX . \tag{5}$$

PROOF. To begin with, we suppose Ω is the first octant in $\boldsymbol{R}^n$, i.e., $\Omega = \{Y = (y_1, \cdots, y_n) \in \boldsymbol{R}^n \,|\, y_1, \cdots, y_n > 0\}$. Clearly we may consider T_Ω as $\boldsymbol{R}^2_+ \times \cdots \times \boldsymbol{R}^2_+$, the Cartesian product of n half-planes. Let $\Omega' = \{Y' = (y_2, \cdots, y_n) \,|\, y_2, \cdots, y_n > 0\}$. Then we can write $T_\Omega = \boldsymbol{R}^2_+ \times T_{\Omega'}$ and $X + iY = (x_1 + iy_1, X' + iY')$ for $X + iY \in T_\Omega$. We shall show that if $z_1 \in \boldsymbol{R}^2_+$ is fixed then $u(z_1, Z')$ belongs to $LH^1(T_{\Omega'})$ as a function of $Z' \in T_{\Omega'}$. It is clear that $u(z_1, Z')$ is log. plurisubharmonic on $T_{\Omega'}$. Take $r > 0$ such that $\Delta = \{w \in \boldsymbol{C} \,|\, |w - z_1| \leqq r\} \subset \boldsymbol{R}^2_+$ and let $\delta = \operatorname{Im} z_1 - r$. Since $u(w, Z')$ is subharmonic as a function of $w = x_1 + iy_1$ for an arbitrary $Z' = X' + iY' \in T_{\Omega'}$, we have

$$u(z_1, Z') \leqq (\pi r^2)^{-1}\int_{\Delta} u(x_1 + iy_1, Z')dx_1 dy_1$$
$$\leqq (\pi r^2)^{-1}\int_{\delta}^{2r+\delta} dy_1 \int_{R} u(x_1 + iy_1, Z')dx_1 .$$

Integrating with respect to $dX' = dx_2 \cdots dx_n$ we get

$$\int_{R^{n-1}} u(z_1, Z')dX' \leqq (\pi r^2)^{-1}\int_{\delta}^{2r+\delta} dy_1 \int_{R^n} u(x_1 + iy_1, Z')dX$$
$$\leqq (\pi r^2)^{-1} 2rM(u, 1; T_{\Omega}) .$$

Similarly, it is seen that if $Z' \in T_{\Omega'}$ is fixed then $u(z_1, Z')$ belongs to $LH^1(\boldsymbol{R}^2_+)$ as a function of $z_1 \in \boldsymbol{R}^2_+$ ([14, p. 116]). The inequality (5) is proved in Lemma 3 for $n = 1$. Now we assume that it is valid for $n - 1$. Writing $\rho' = (\rho_2, \cdots, \rho_n) \in \Omega'$, we obtain

$$\int_{\Omega} u_{\rho}(X + iY)dY = \int_{R_+} dy_1 \int_{\Omega'} u(x_1 + i(\rho_1 + y_1), X' + i(\rho' + Y'))dY'$$
$$\leqq 2^{-(n-1)}\int_{R^{n-1}} dX' \int_{R_+} u(x_1 + i(\rho_1 + y_1), X' + i\rho')dy_1$$
$$\leqq 2^{-n}\int_{R^n} u_{\rho}(X)dX .$$

If Ω is an n-polygonal cone we can proceed as in [14, p. 118]. Take n linearly independent vectors generating Ω and let A be the matrix with these vectors as its columns. Then the linear map $\tilde{X} \to A\tilde{X}$, $\tilde{X} \in \boldsymbol{R}^n$, transforms the first octant Λ onto Ω and can be extended to $\boldsymbol{C}^n$ by $A(\tilde{Z}) = A\tilde{X} + iA\tilde{Y}$, $\tilde{Z} = \tilde{X} + i\tilde{Y} \in \boldsymbol{C}^n$. The function $u \circ A$ belongs to $LH^1(T_{\Lambda})$, so we have

$$\int_{\Omega} u(X + i(\rho + Y))dY = |\det A| \int_{\Lambda} (u \circ A)(\tilde{X} + i(\tilde{\rho} + \tilde{Y}))d\tilde{Y}$$
$$\leqq 2^{-n}|\det A| \int_{R^n} (u \circ A)(\tilde{X} + i\tilde{\rho})d\tilde{X}$$
$$= 2^{-n}\int_{R^n} u(X + i\rho)dX ,$$

which completes the proof.

THEOREM 1. *Let $D = D(\Omega, \Phi)$ be a Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ with an n-polygonal cone Ω. Let $u \in LH^p(D)$, $0 < p < \infty$. Then for any $X \in \boldsymbol{R}^n$*

$$\int_{\Omega \times C^m} u(X + i(Y + \Phi(W, W)), W)^p dY dW \leqq 2^{-n}M(u, p; D) .$$

PROOF. It suffices to prove for $p = 1$. For $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n) \in \Omega$ and

$W \in \boldsymbol{C}^m$ put $v(Z; \varepsilon, W) = u(Z + i(\varepsilon + \Phi(W, W)), W)$, $Z = X + iY \in T_\Omega$. Then it is seen from the same argument as in [12] that $v(Z; \varepsilon, W)$ belongs to $LH^1(T_\Omega)$ as a function of Z. It follows from (5) that for any $\rho \in \Omega$

$$\int_\Omega u(X + i(\rho + \varepsilon + Y + \Phi(W, W)), W)dY$$
$$\leqq 2^{-n}\int_{\boldsymbol{R}^n} u(X + i(\rho + \varepsilon + \Phi(W, W)), W)dX .$$

Integration with respect to dW and arbitrariness of $\rho + \varepsilon$ imply the desired inequality. The proof of Theorem 1 is completed.

REMARK. It should be noted that the condition we imposed on the cone Ω is not restrictive when $n = 1$ and 2. Thus all the results hold for general Siegel domains for these cases.

Recall that D_0 is biholomorphic with the unit ball of $\boldsymbol{C}^{m+1}$. We write $|W|^2 = \sum_{j=1}^m |w_j|^2$ for $W \in \boldsymbol{C}^m$.

COROLLARY 1. (i) *Let* $u \in LH^p(T_\Omega)$. *Then for any* $X \in \boldsymbol{R}^n$

$$\int_\Omega u(X + iY)^p dY \leqq 2^{-n}M(u, p; T_\Omega) .$$

(ii) *Let* $u \in LH^p(D_0)$. *Then for any* $x \in \boldsymbol{R}$

$$\int_{\boldsymbol{R}_+ \times \boldsymbol{C}^m} u(x + i(y + |W|^2), W)^p dy dW \leqq 2^{-1}M(u, p; D_0) .$$

REMARK. The Poisson kernel for the unit disc U provides an example that the Fejér-Riesz inequality does not necessarily hold for harmonic functions on U ([2, p. 311]). Similarly, the Poisson kernel $P(x, y)$ shows that (i) is not necessarily valid for harmonic function on $\boldsymbol{R}^2_+$.

The following result is related to [13, Theorem C] and the first half is known for $|f|^p$, $f \in H^p(\boldsymbol{R}^2_+)$ ([6]). We write $r \leqq s$ if and only if $s - r \in \{0\} \cup \Omega$ for $r, s \in \Omega$, and $|Y|^2 = \sum_{j=1}^n y^2$ for $Y \in \boldsymbol{R}^n$.

THEOREM 2. *Let* $u \in LH^p(D)$, $0 < p < \infty$, *where* $D = D(\Omega, \Phi)$ *with an* n*-polygonal cone* Ω. Let

$$\psi(Y) = \int_{\boldsymbol{R}^n \times \boldsymbol{C}^m} u(X + i(Y + \Phi(W, W)), W)^p dX dW , \quad Y \in \Omega .$$

Then $\psi(Y)$ *is a decreasing function of* Y. *If* $Y \geqq Y_0$ *for some* $Y_0 \in \Omega$ *and* $|Y| \to \infty$, *then* $\psi(Y) \to 0$.

PROOF. It is sufficient to prove for $p = 1$. First we prove the assertions by induction on n assuming that Ω is the first octant in $\boldsymbol{R}^n$ and $u \in LH^1(T_\Omega)$. We denote by $\psi^{(n)}(Y)$ the integral of $u(X + iY)$ with respect to dX over $\boldsymbol{R}^n$. Let $u \in LH^1(\boldsymbol{R}^2_+)$. Suppose u is continuous and let $v = u^{1/2}$. Then v is subharmonic and $M(v, 2; \boldsymbol{R}^2_+) < \infty$. It is proved implicitly in [13, Theorem C] that in this case $\psi^{(1)}(y)$ is a decreasing function of $y > 0$. When u is only upper semi-continuons, let $G_\rho = \{x+iy \in \boldsymbol{R}^2_+ \mid y > \rho\}$, $\rho > 0$, and let $u_r(x+iy)$ be the function defined to be the mean value of u over the disc of radius r, $r < \rho$, centered at the point $x + iy$. u_r is a continuous subharmonic function on G_ρ aud $\{u_r\}$ tends to u decreasingly as $r \to 0$. It is seen from Fubini's theorem that $M(u_r, 1; G_\rho) \leqq M(u, 1; \boldsymbol{R}^2_+)$, hence $\psi_r^{(1)}(y)$, the integral of $u_r(x + iy)$, is a decreasing function of $y > \rho$. Taking limit as $r \to 0$, we can get the same conclusion for u. Let $h(x + iy)$ be the Poisson integral of $v_\rho(t)$ with a constant $\rho > 0$. Then from the same reasoning as in Lemma 2 we can see that $v_\rho(x + iy)$ is majorized by $h(x + iy)$ on $\boldsymbol{R}^2_+$. The maximal function of v_ρ belongs to $L^2(\boldsymbol{R})$, so $\psi^{(1)}(\rho + y)$ tends to 0 as $y \to \infty$. Next supposing $\psi^{n-1}(Y')$ is a decreasing function, we can easily see that $\psi^{(n)}(s) \leqq \psi^{(n)}(r)$ if $r \leqq s$ in Ω. If $|Y| = |(y_1, Y')| \to \infty$, $Y \geqq Y_0$, we may suppose $y_1 \to \infty$ increasingly. Let $t_1 = y_1 - \varepsilon > 0$, $\varepsilon > 0$. From $Y \geqq (t_1, Y'_0)$ we have

$$\psi^{(n)}(Y) \leqq \int_{\boldsymbol{R}^n} u(x_1 + it_1, X' + iY'_0)dX = \int_{\boldsymbol{R}^{n-1}} dX' \int_{\boldsymbol{R}} u(x_1 + it_1, X' + iY'_0)dx_1 .$$

Here, the inner integral tends to 0 decreasingly as $t_1 \to \infty$ for every $X' \in \boldsymbol{R}^{n-1}$, so $\psi^{(n)}(Y) \to 0$ as $y_1 \to \infty$. Let Ω be an n-polygonal cone and A be the matrix employed in the proof of Lemma 4. Then we can write

$$\psi^{(n)}(Y) = |\det A| \int_{\boldsymbol{R}^n} (u \circ A)(\tilde{X} + i\tilde{Y})d\tilde{X} ,$$

where $Y = A\tilde{Y}$, $\tilde{Y} \in \Lambda$. Since $r \leqq s$ in Ω if and only if $\tilde{r} \leqq \tilde{s}$ in Λ, $\psi^{(n)}(Y)$ is seen to be a decreasing function. The second assertion follows from the fact that $|Y| \to \infty$ if and only if $|\tilde{Y}| \to \infty$. Now let $u \in LH^1(D)$ and $r \leqq s$, $r, s \in \Omega$. Take $\varepsilon \in \Omega$ so that $r = \varepsilon + \rho$, $s = \varepsilon + \sigma$ for some $\rho, \sigma \in \Omega$. Then $v(Z; \varepsilon, W) \in LH^1(T_\Omega)$ for any $W \in \boldsymbol{C}^m$, so we have

$$\int_{\boldsymbol{R}^n} u(X + i(\varepsilon + \sigma + \Phi(W, W)), W)dX \tag{6}$$
$$\leqq \int_{\boldsymbol{R}^n} u(X + i(\varepsilon + \rho + \Phi(W, W)), W)dX .$$

It follows that $\psi(s) \leqq \psi(r)$. Finally take $\varepsilon \in \Omega$ such that $Y_0 = \varepsilon + Y^*_0$ for some $Y^*_0 \in \Omega$. Then $Y = \varepsilon + Y^*$, $Y^* \geqq Y^*_0$, and $|Y^*| \to \infty$. Therefore

$$\int_{\boldsymbol{R}^n} u(X + i(\varepsilon + Y^* + \Phi(W, W)), W)dX$$
$$\leqq \int_{\boldsymbol{R}^n} u(X + i(Y_0 + \Phi(W, W)), W)dX ,$$

the left-hand side tending to 0 as $|Y| \to \infty$. The dominated convergence theorem shows that $\psi(Y) \to 0$ as $|Y| \to \infty$. The proof is completed.

COROLLARY 2. *Let $u \in LH^p(D)$. Then $u(Z + i\Phi(W, W), W)$ belongs to $LH^p(T_\Omega)$ as a function of $Z \in T_\Omega$ for almost every $W \in \boldsymbol{C}^m$.*

PROOF. Let $p = 1$. We can take a sequence $\{\varepsilon^{(j)}\} \subset \Omega$ such that $\varepsilon^{(1)} \geqq \varepsilon^{(2)} \geqq \cdots$, $\varepsilon^{(j)} \to 0$ and $\psi(\varepsilon^{(j)}) \to M(u, 1; D)$ as $j \to \infty$. For $W \in \boldsymbol{C}^m$ let

$$g_j(W) = \int_{\boldsymbol{R}^n} u(X + i(\varepsilon^{(j)} + \Phi(W, W)), W)dX , \quad j = 1, 2, \cdots ,$$

$$g(W) = \sup_{Y \in \Omega} \int_{\boldsymbol{R}^n} u(X + i(Y + \Phi(W, W)), W)dX .$$

Then from the inequality (6) and the choice of $\{\varepsilon^{(j)}\}$ it follows that $g_j(W) \to g(W)$ increasingly as $j \to \infty$ for every $W \in \boldsymbol{C}^m$. We can see that $g(W) < \infty$ for a.e. W from

$$\int_{\boldsymbol{C}^m} g(W)dW = \lim_{j \to \infty} \psi(\varepsilon^{(j)}) = M(u, 1; D) < \infty .$$

3. The case of holomorphic functions. If $f \in H^p(\boldsymbol{R}^2_+)$, $0 < p < \infty$, the boundary value $f^*(x)$ exists for a.e. $x \in \boldsymbol{R}$. Here $f^* \in L^p(\boldsymbol{R})$ and $f(x + iy) \to f^*(x)$ as $y \to 0$ in the sense of L^p-convergence. As a consequence of Corollary 1 and Theorem 2 we have the inequality (1).

PROPOSITION 1. *Let $f \in H^p(\boldsymbol{R}^2_+)$, $0 < p < \infty$. Then for any $x \in \boldsymbol{R}$*

$$\int_{\boldsymbol{R}_+} |f(x + iy)|^p dy \leqq 2^{-1} \int_{\boldsymbol{R}} |f^*(x)|^p dx .$$

Let D be a Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ and $f \in H^p(D)$, $0 < p < \infty$. Then the boundary value f^* exists almost everywhere, i.e., $f^*(X + i\Phi(W, W), W) = \lim_{Y \to 0} f(X + i(Y + \Phi(W, W)), W)$ for a.e. $(X, W) \in \boldsymbol{R}^n \times \boldsymbol{C}^m$, and $f^* \in L^p(\boldsymbol{R}^n \times \boldsymbol{C}^m)$ ([12]).

PROPOSITION 2. *Let $D = D(\Omega, \Phi)$, where Ω is an n-polygonal cone in $\boldsymbol{R}^n$. Let $f \in H^p(D)$, $0 < p < \infty$. Then for any $X \in \boldsymbol{R}^n$*

$$\int_{\Omega \times \boldsymbol{C}^m} |f(X + i(Y + \Phi(W, W)), W)|^p dY dW$$
$$\leqq 2^{-n} \int_{\boldsymbol{R}^n \times \boldsymbol{C}^m} |f^*(X + i\Phi(W, W), W)|^p dX dW .$$

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