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ANALYSIS OF DYADIC STATIONARY PROCESSES USING THE GENERALIZED WALSH FUNCTIONS

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Abstract. This paper deals with continuous-parameter dyadic stationary processes. A necessary and sufficient condition for such a process to assume its spectral representation in terms of the generalized Walsh functions is given. The representation plays an important role in the analysis of such a process: we discuss laws of large numbers, sampling theorem, and the relationship between the dyadic stationary processes with spectral densities and linear dyadic processes.

The existence of a spectral representation shows the possibility of an analysis of dyadic stationary processes similar to that of ordinary stationary processes.

1. Introduction. In communication theory and systems engineering, analysis of signals contaminated by random noise is very important. The general problem is difficult; so it is usually preferable to restrict attention to certain classes of signals.

Recently much attention has been paid to a class of signals that are called dyadic stationary processes [7], [17], [19], [20], [22], [23]. Most discussions, however, have confined themselves to the discrete-parameter case. This is perhaps due to the difficulty arising from the discontinuity of the Walsh functions, in terms of which the processes are represented.

Analysis of the dyadic stationary processes is most readily carried out using the Walsh functions rather than the exponential functions commonly used in the ordinary stationary processes. Many researchers, among others Walsh [28], Fine [8]-[10], Chrestenson [6], Mogenthaler [18], Paley [21], Selfridge [25] and Yano [29], have contributed to Walsh-Fourier analysis, while dyadic calculus involving dyadic derivatives was developed by Gibbs [11], [12], Butzer and Wagner [1]-[3], and Wagner [27].

In this paper we consider continuous-parameter dyadic stationary processes. For their spectral representations, the theory of the generalized Walsh functions is a necessity. In 2 we shall briefly state the definitions and some properties of the generalized Walsh functions which are used frequently afterwards. In 3 we introduce a continuous-parameter dyadic stationary process and give the spectral representations of its covariance function and of the process itself. Laws of large numbers in the sense of both weak and strong convergences are given in 4. A sampling theorem is stated in 5. In 6 we introduce a continuous-parameter linear dyadic process and characterize it as a dyadic stationary process with a spectral density, and vice versa.

2. Preliminaries.

In this section we consider the generalized Walsh functions and their simple properties for later use.

2.1. DYADIC GROUP. Let $\mathscr{F}$ be the field of formal power series

$$\boldsymbol{x} = \sum_{n\geqq M} x_n\zeta^n \quad (x_n \in \{0, 1\})$$

in which $M = M(x)$ is an integer; the addition and the multiplication are defined as follows: for $\boldsymbol{x} = \sum_{n\geqq M} x_n\zeta^n$, $\boldsymbol{y} = \sum_{n\geqq N} y_n\zeta^n \in \mathscr{F}$,

$$\boldsymbol{x} + \boldsymbol{y} = \sum_{n\geqq L} |x_n - y_n|\zeta^n\ , \quad \boldsymbol{xy} = \sum_{n\geqq K} z_n\zeta^n\ ,$$

where $L = \min\{M, N\}$, $K = M + N$, and $z_n = \sum_{M\leqq k\leqq -N+n} x_k y_{n-k}$ ([9]). Define a neighbourhood of zero element $\mathbf{0}$ as the set of $\boldsymbol{x}$ with $x_n = 0$ $(n \geqq N)$ for some fixed integer N. Endowed with this topology $\mathscr{F}$ becomes a totally disconnected locally compact Hausdorff space. We denote the additive group of $\mathscr{F}$ by G and call it the *dyadic group*. Let Γ be the dual group or character group of G. To each $\gamma \in \Gamma$ there is a $\boldsymbol{y} \in G$ such that

$$\gamma(\boldsymbol{x}) = \gamma_1(\boldsymbol{xy}) \tag{2.1}$$

holds for all $\boldsymbol{x} \in G$, where $\gamma_1(z) = \exp\{\pi i z_1\}$. If $\boldsymbol{y} \in G$ is given then $\gamma(\boldsymbol{x})$ defined by (2.1) is a character. The one-to-one correspondence thus established between G and Γ is easily seen to be an isomorphism. Then, on account of (2.1), we may identify Γ with G and, we denote (2.1) by $\gamma(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$, which is the value of the character $\boldsymbol{y} \in G$ at $\boldsymbol{x} \in G$. For further discussion, see Fine [9].

2.2. CORRESPONDENCE BETWEEN THE FUNCTIONS ON G AND R_+, AND THE MEASURES ON THEM. Let $R_+ = [0, \infty)$ and D_+ be the set of dyadic rationals on $R_+ \setminus \{0\}$. Define a mapping λ from G onto R_+ by

$$\lambda(\boldsymbol{x}) = \lambda(\sum x_n\zeta^n) = \sum x_n 2^{-n} \quad (\boldsymbol{x} \in G)\ .$$

The image $\lambda(\boldsymbol{x})$ represents a dyadic expansion of a positive number. The dyadic rational assumes two representations, i.e., a finite and an infinite expansions; hence the mapping λ is not injective. Let $\mathscr{E} = \{\boldsymbol{x}: \boldsymbol{x} = \sum x_n\zeta^n, x_n = 1\ (n \geqq M)$ for some $M\}$. We shall always take the finite expansion for a dyadic rational in D_+. Then the restriction of λ on $\mathscr{E}^c$

is one-to-one and onto R_+, where $\mathscr{E}^c$ denotes the complement of $\mathscr{E}$. Its inverse μ satisfies $\lambda(\mu(x)) = x$ $(x \in R_+)$ and $\mu(\lambda(\boldsymbol{x})) = \boldsymbol{x}$ $(\boldsymbol{x} \in \mathscr{E}^c)$. Let us define the addition $x \oplus y$ of $x \in R_+$ and $y \in R_+$ by the relation $x \oplus y = \lambda(\mu(x) + \mu(y))$. For each real-valued function f on R_+ define the corresponding function ϕ on G by

$$\begin{aligned}\phi(\boldsymbol{x}) &= f(\lambda(\boldsymbol{x})) \quad (\boldsymbol{x} \in \mathscr{E}^c) \\ &= \limsup_{y \to x} \phi(\boldsymbol{y}) \quad (\boldsymbol{x} \in \mathscr{E}) ,\end{aligned}$$

where the approach is over those $\boldsymbol{y} \in \mathscr{E}^c$. For brevity we shall call ϕ the *G-extension* of f and write the relation as $\phi \sim f$. If f is continuous so is ϕ, but not conversely.

A function f on R_+ is said to be *W-continuous at* $t \in R_+$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t \oplus h) - f(t)| < \varepsilon$ whenever $h < \delta$ $(h \in R_+)$. If f is W-continuous at every $t \in R_+$, then we call it simply *W-continuous.* A continuous function is W-continuous. A function f on R_+ is said to be *W-positive definite*, if it satisfies

$$\sum_{i,j}^{n} c_i \bar{c}_j f(t_i \oplus t_j) \geqq 0$$

for any positive integer n, any sequence of complex numbers $\{c_i\}$ and any sequence $\{t_i\} \subset R_+$ with $\mu(t_i) + \mu(t_j) \in \mathscr{E}^c$ $(i, j = 1, \cdots, n)$. The following result states that the "W-properties" are natural candidates to replace ordinary ones on the real lines [18].

LEMMA 2.1. (a) *If f is W-continuous, and if $f(t-) = \lim_{s \uparrow t} f(s)$ exists and is finite at any $t \in D_+$, then its G-extension ϕ is continuous on G.*

(b) *If f satisfies the conditions in* (a), *and f is also W-positive definite, then its G-extension ϕ is positive definite.*

(c) *If ϕ is continuous on G, then the function $f(t) = \phi(\mu(t))$ is W-continuous and $f(t-)$ exists and is finite at every dyadic rational $t \in D_+$.*

(d) *If ϕ is positive definite, then its counterpart f defined in* (c) is *W-positive definite.*

A measure α defined on $\mathscr{B}(G)$, consisting of all Borel subsets of G, is decomposed uniquely into a *usual* measure, vanishing on all Borel subsets of $\mathscr{E}$, and an *unusual* measure, vanishing on all Borel subsets of $\mathscr{E}^c$. There is a one-to-one correspondence between usual measures on G and measures on R_+ such that

(2.2) $$\alpha(A) = m(\lambda(A \cap \mathscr{E}^c)) \quad (A \in \mathscr{B}(G))$$

or

$$m(A) = \alpha(\mu(A)) \quad (A \in \mathscr{B}(R_+)) , \tag{2.3}$$

where m is a measure on $\mathscr{B}(R_+)$ consisting of all Borel subsets of R_+. We denote the correspondence as $\alpha \sim m$. The following result is obvious [10].

LEMMA 2.2. (a) *If $\phi \sim f$ and $\alpha \sim m$, then*

$$\int_G \phi(\boldsymbol{x})\alpha(d\boldsymbol{x}) = \int_0^\infty f(x)m(dx) .$$

(b) *In particular,* if $\phi \sim f$, *then the Haar measure $d\boldsymbol{x}$ on G is adjustable such that*

$$\int_G \phi(\boldsymbol{x})d\boldsymbol{x} = \int_0^\infty f(x)dx ,$$

where dx is the Lebesgue measure on R_+.

We assume, in the sequel, that the Haar measure on G is always adjusted as above.

2.3. DEFINITION OF THE GENERALIZED WALSH FUNCTIONS. Now we shall define the *generalized Walsh functions* following Fine [9]. For any $t \in R_+$, put

$$W(x, t) = \langle \mu(x), \mu(t) \rangle \quad (x \in R_+) ,$$

and call $\{W(x, t) : x \in R_+, t \in R_+\}$ the generalized Walsh functions. If $t \in R_+$ is an integer, then $W(x, t)$ reduces to the ordinary Walsh function. Because of this the functions so defined deserve the name. In the sequel we simply call them "the Walsh functions" instead of "the generalized Walsh functions".

The Walsh functions possess the following properties:

$$|W(x, t)| = 1 \quad (x \in R_+, t \in R_+) ; \tag{2.4}$$

$$\begin{aligned} \int_0^1 W(x, t)dt &= 1 \quad (0 \leqq x < 1) , \\ &= 0 \quad (1 \leqq x) ; \end{aligned} \tag{2.5}$$

$$W(x, t) = W(t, [x])W(x, [t]) \quad (x, t \in R_+) , \tag{2.6}$$

where $[x]$ denotes the integer part of x;

$$\begin{aligned} \mathrm{W}(x, t)W(y, t) &= W(x \oplus y, t) \quad (\mu(x) + \mu(y) \in \mathscr{E}^c) , \\ &= W(x \oplus y - , t) \quad (\mu(x) + \mu(y) \in \mathscr{E}) ; \end{aligned} \tag{2.7}$$

$$W(\lambda(\boldsymbol{x}) - , t) = \langle \boldsymbol{x}, \mu(t) \rangle \quad (\boldsymbol{x} \in \mathscr{E}) . \tag{2.8}$$

By (2.6) the symmetric relation

$$W(x, t) = W(t, x) \quad (t, x \in R_+)$$

holds, so that relations similar to (2.7) and (2.8) on the second variable are valid. The Walsh functions are W-continuous, since at every (dyadic rational) discontinuity $x \in D_+$, $x \oplus h > x$ for sufficiently small $h \in R_+$, and the Walsh functions are continuous on the right. The Walsh function is considered to be the eigen function corresponding to the eigenvalue $t \in R_+$, of the equation

$$Df(x) - tf(x) = 0 ,$$

where D is the 'derivative' operator [1]-[3], [11], [12].

LEMMA 2.3.

$$\lim_{n\to\infty} 2^{-n} \int_{I(n)} \langle \boldsymbol{x}, \boldsymbol{t} \rangle d\boldsymbol{t} = 1 \quad (\boldsymbol{x} = \boldsymbol{0}) , \qquad (2.9)$$
$$= 0 \quad (\boldsymbol{x} \neq \boldsymbol{0}) ,$$

where $I(n) = \{\boldsymbol{t}\colon \lambda(\boldsymbol{t}) < 2^n\}$.

We shall omitt the proof because it will be reduced, on account of the property of the set $\mathscr{E}$ and the correspondence between the characters on G and the Walsh functions on R_+, to the special case of Lemma 2.4 below.

LEMMA 2.4. (a) *If k is a positive integer, then*

$$\lim_{k\to\infty} k^{-1}D(x; k) = 1 \quad (x = 0) \qquad (2.10)$$
$$= 0 \quad (0 < x < 1) ,$$

where $D(x; k) = \sum_{i=0}^{k-1} W(x, i)$ *is the Dirichlet kernel.*

(b)
$$\lim_{R\to\infty} R^{-1} \int_0^R W(x, t)dt = 1 \quad (x = 0) \qquad (2.11)$$
$$= 0 \quad (x \neq 0) .$$

PROOF. (a) For every integer $k > 0$, it is clear that $k^{-1}D(0, k) = 1$. For any fixed $x > 0$ there is an integer n such that $2^{-n} \leqq x < 2^{-n+1}$. For any small $\varepsilon > 0$ there is an integer $K > 0$ for which $2^n K^{-1} < \varepsilon$. Any integer $k > 0$ can be uniquely expanded in the form

$$k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_\nu} , \quad n_1 > n_2 > \cdots > n_\nu \geqq 0 .$$

If $k \geqq K$, and we agree to set $D(x, 0) = 0$, then

$$D(x; k) = D(x; 2^m)D(2^m; p) + W(2^m, p)D(x; q) ,$$

where $i = \max\{j\colon n_j \geqq n\}$, $m = n_i$, $k = p2^m + q$, and $0 \leqq q < 2^m$ (cf. Fine

[8]). Since $D(x; 2^m) = 0$ and $q < 2^n$, we have $|D(x; k)| < 2^n$. Hence, $k^{-1}D(x; k) \leqq 2^n K^{-1} < \varepsilon$.

(b) Now

$$R^{-1}\int_0^R W(x, t)dt = R^{-1}\Big(\int_0^{[R]} + \int_{[R]}^R\Big) W(x, t)dt$$
$$= R^{-1}D(x; [R])J(x, t) + R^{-1}W(x, [R])J(x, R - [R]) ,$$

where $J(x, t) = \int_0^t W(x, u)du$. The second term tends to zero as $R \to \infty$, since both $W(x, [R])$ and $J(x, R - [R])$ are bounded. The result follows from (a) and the value of $J(x, 1)$, i.e., $J(x, 1) = 1,\ 0 \leqq x < 1;\ = 0,\ 1 \leqq x$ (cf. [8], [25]).

LEMMA 2.5. *If $\phi(x)$ on G is representable in the form*

$$\phi(\boldsymbol{t}) = \int_G \langle \boldsymbol{x}, \boldsymbol{t}\rangle \alpha(d\boldsymbol{x}) , \tag{2.12}$$

where α is a regular measure on G with finite total variation, then

$$\lim_{n\to\infty} 2^{-n}\int_{I(n)} \phi(\boldsymbol{t})\langle \boldsymbol{x}, \boldsymbol{t}\rangle d\boldsymbol{t} = \alpha(\{\boldsymbol{x}\}) . \tag{2.13}$$

PROOF. Interchanging the integral signs gives

$$2^{-n}\int_{I(n)} \phi(\boldsymbol{t})\langle \boldsymbol{x}, \boldsymbol{t}\rangle d\boldsymbol{t} = \int_G \alpha(d\boldsymbol{y}) 2^{-n}\int_{I(n)} \langle \boldsymbol{y} + \boldsymbol{x}, \boldsymbol{t}\rangle d\boldsymbol{t} ,$$

in which the inner integral multiplied by 2^{-n} is bounded by 1. By Lemma 2.3 the integral on the left therefore converges to $\alpha(\{\boldsymbol{x}\})$.

3. Dyadic stationary processes and their spectral representations.

3.1. DEFINITION OF DYADIC STATIONARY PROCESSES. A second-order stochastic process $\{X(t), t \in R_+\}$ is called *a dyadic stationary process* (*DSP*) in the wide sense, if it has a constant mean and its covariance function

$$r(t, s) = E[X(t) - EX(t)]\overline{[X(s) - EX(s)]} \quad (t, s \in R_+)$$

depends only on the "difference" $\mu(t) + \mu(s)$. Note that the dependence is not on $t \oplus s$ but on $\mu(t) + \mu(s)$. For simplicity we assume throughout that $EX(t) = 0$ $(t \in R_+)$. We shall define a function ρ by

$$\rho(t \oplus s) = r(t, s) \quad (\mu(t) + \mu(s) \in \mathscr{E}^c) . \tag{3.1}$$

It is well-defined on R_+, since the mapping

$$\{(t, s): \mu(t) + \mu(s) \in \mathscr{E}^c\} \ni (t, s) \to t \oplus s \in R_+$$

is onto and the difference $\mu(t) + \mu(s)$ depends on $t \oplus s$ if it belongs to

$\mathscr{E}^c$. Then it is easy to see that the function ρ is W-positive definite, and

$$\rho(t) = \overline{\rho(t)}\ , \quad |\rho(t)| \leqq \rho(0)\ , \tag{3.2}$$

and

$$E|X(t) - X(s)|^2 = 2[\rho(0) - \rho(t \oplus s)] \quad (\mu(t) + \mu(s) \in \mathscr{E}^c)\ . \tag{3.3}$$

If a DSP is such that

$$E|X(t \oplus h) - X(t)|^2 \to 0 \quad (t, h \in R_+)\ , \tag{3.4}$$

as $h \to 0$, then it is said to be *W-mean-continuous at t*. If it is W-mean-continuous at every $t \in R_+$, then we simply call it *W-mean-continuous.*

LEMMA 3.1. *Let ρ be the function defined by* (3.1). *If a DSP $\{X(t), t \in R_+\}$ is measurable* (*in the mean*), *then $\rho(t)$ is continuous from the right at $t = 0$.*

The proof can be carried out similarly to that for the analogous property of the covariance function of an ordinary stationary process.

LEMMA 3.2. *The measurable DSP $\{X(t), t \in R_+\}$ is W-mean-continuous if and only if*

$$E|X(t \oplus h) - X(t)|^2 \to 0\ , \tag{3.5}$$

as $h \to 0$ with $\mu(t) + \mu(h) \in \mathscr{E}$.

PROOF. The "if" part is also obvious since by Lemma 3.1 and (3.3), (3.5) holds, as $h \to 0$ with $\mu(t) + \mu(h) \in \mathscr{E}^c$.

EXAMPLE 1. Let Y be a random variable with zero mean and unit variance. For a fixed $x \in R_+$, put

$$X(t) = W(x, t)Y \quad (t \in R_+)\ . \tag{3.6}$$

Then the process $\{X(t), t \in R_+\}$ so defined is a W-mean-continuous DSP since

$$EX(t) = 0\ ;$$
$$EX(t)\overline{X(s)} = W(x, t)W(x, s) = \langle \mu(x), \mu(t) + \mu(s) \rangle \quad (t, s \in R_+)\ ,$$

and

$$E|X(t \oplus h) - X(t)|^2 = 2\{1 - W(x, t \oplus h)W(x, t)\} = 2\{1 - W(x, h)\}\ .$$

3.2. SPECTRAL REPRESENTATION OF THE COVARIANCE FUNCTION OF A DSP. First we shall introduce the fundamental assumptions. Let $\{X(t), t \in R_+\}$ be a measurable DSP. Assumptions:

(A) It satisfies (3.5).

(B) The limit $EX(s)\overline{X(0)}$ as $s\uparrow t$ exists and is finite at any $t\in D_+$.
(C) For every dyadic rational $x\in D_+$,

$$\lim_{n\to\infty} 2^{-n}\int_0^{2^n}\rho(t)W(x-,t)dt = 0 .$$

THEOREM 3.1. *Let $r(t,s)$ be the covariance function of a measurable DSP. In order that the covariance function is representable as*

$$r(t,s) = \int_0^\infty W(x,t)W(x,s)m(dx) , \tag{3.7}$$

where m is a unique non-negative finite regular measure on R_+, it is necessary and sufficient that it satisfies the assumptions (A), (B), *and* (C).

PROOF. (Sufficiency) It follows from Lemma 2.1 that the G-extension γ of ρ defined by (3.1) is continuous and positive definite, since the conditions (a) and (b) are satisfied by (A) and (B). Hence by Bochner's theorem on positive definite functions on locally compact Abelian groups [24], γ has a representation of the the form

$$\gamma(\boldsymbol{t}) = \int_G \langle \boldsymbol{x}, \boldsymbol{t}\rangle \alpha(d\boldsymbol{x}) ,$$

where α is a unique non-negative finite regular measure on G. By definition,

$$\rho(t) = \gamma(\mu(t)) = \left(\int_{\mathscr{E}^c} + \int_{\mathscr{E}}\right)\langle \boldsymbol{x}, \mu(t)\rangle \alpha(d\boldsymbol{x}) = J_1 + J_2, \text{ say} . \tag{3.8}$$

First we evaluate J_2. Since the Haar measure of $\mathscr{E}$ is zero, for every integer $n>0$

$$\int_{I(n)} \gamma(\boldsymbol{t})\langle \boldsymbol{x}, \boldsymbol{t}\rangle d\boldsymbol{t} = \int_0^{2^n}\rho(t)W(x-,t)dt ,$$

where $\boldsymbol{x}\in\mathscr{E}$ and $\lambda(\boldsymbol{x}) = x$. By (C) the last integral multiplied by 2^{-n} tends to zero as $n\to\infty$. Hence an application of Lemma 2.5 gives $\alpha(\{\boldsymbol{x}\}) = 0$ ($\boldsymbol{x}\in\mathscr{E}$) and thus $J_2 = 0$. Next put $m(A) = \alpha(\mu(A))$ ($A\in\mathscr{B}(R_+)$); then

$$\rho(t) = \int_0^\infty \langle \mu(x), \mu(t)\rangle \alpha(d\mu(x)) = \int_0^\infty W(x,t)m(dx) . \tag{3.9}$$

Thus

$$r(t,s) = \int_0^\infty W(x, t\oplus s)m(dx) = \int_0^\infty W(x,t)W(x,s)m(dx) ,$$

whenever $\mu(t)+\mu(s)\in\mathscr{E}^c$. In the case of $\mu(t)+\mu(s)\in\mathscr{E}$, by (A)

$$r(t, s) = EX(t)\overline{X(s)} = \lim_{h\to 0} EX(t)\overline{X(s \oplus h)}$$
$$= \lim_{h\to 0}\int_0^\infty W(x, t)W(x, s\oplus h)m(dx) = \int_0^\infty W(x, t)W(x, s)m(dx) ,$$

where the approach is over those h with $\mu(s) + \mu(h) \in \mathscr{E}^c$. The last equality is justified by the boundedness and W-continuity of the Walsh functions.

(Necessity) The assumption (A) is obvious, since

$$E|X(t\oplus h) - X(t)|^2 = \int_0^\infty |W(x, t\oplus h) - W(x, t)|^2 m(dx)$$

and the Walsh functions are W-continuous.

Since $\lim_{s\uparrow t} W(x, s)$ exists at every $t \in R_+$,

$$EX(s)\overline{X(0)} = \int_0^\infty W(x, s)m(dx) \to \int_0^\infty W(x, t-)m(dx)$$

as $s \uparrow t$, and hence (B) is proved. Next as for (C), let $\rho(t\oplus s) = r(t, s)$ $(\mu(t) + \mu(s) \in \mathscr{E}^c)$, and define the G-extensions $\gamma \sim \rho$ and $\alpha \sim m$ respectively; then

$$\gamma(\mu(t)) = \int_{\mathscr{E}^c} \langle \boldsymbol{x}, \mu(t)\rangle \alpha(d\boldsymbol{x})$$

so that for $x \in D_+$,

$$(3.10)\qquad 2^{-n}\int_0^{2^n} \rho(t)W(x-, t)dt = 2^{-n}\int_0^{2^n}\gamma(\mu(t))\langle \boldsymbol{x}, \mu(t)\rangle dt$$
$$= \int_{\mathscr{E}^c}\alpha(d\boldsymbol{y})2^{-n}\int_0^{2^n}\langle \boldsymbol{y} + \boldsymbol{x}, \mu(t)\rangle dt \quad (\boldsymbol{x}\in\mathscr{E}, \lambda(\boldsymbol{x}) = x) .$$

Note that $\boldsymbol{y} \neq \boldsymbol{x}$ in the last integral. Hence by Lemma 2.3 the inner integral multiplied by 2^{-n} converges to zero boundedly as $n \to \infty$.

Now (3.7) can be rewritten as

$$(3.11)\qquad r(t, s) = \int_0^\infty \langle \mu(x), \mu(t) + \mu(s)\rangle m(dx) ,$$

which depends only on the difference $\mu(t) + \mu(s)$. We shall call m the *spectral measure* (*distribution*) of the *DSP*. It is obvious that $m(R_+) = E|X(t)|^2$ $(t \in R_+)$. If the spectral measure is absolutely continuous with respect to Lebesgue measure, then its Radon-Nykodym derivative dm/dt is called the *spectral density* of the *DSP*.

Let m be a non-negative finite regular measure on R_+. Then there exists a *DSP* which has m as its spectral measure. We shall illustrate this by an example.

EXAMPLE 2. Let Y be a non-negative valued random variable with a probability measure $\Pr\{Y \in B\} = m(B)/m(R_+)$ $(B \in \mathscr{B}(R_+))$. Let Z be a random variable independent of Y and $EZ = 0$, $E|Z|^2 = m(R_+)$. Define $X(t) = ZW(Y, t)$ $(t \in R_+)$. Then $EX(t) = 0$ and

$$EX(t)\overline{X(s)} = E|Z|^2 W(Y, t) W(Y, s) = \int_0^\infty W(y, t) W(y, s) m(dy) .$$

REMARKS. (a) We shall introduce an equivalent definition of DSP's. A second-order stochastic process is called a DSP, if it has constant mean and its covariance function satisfies

$$r(t, s) = r(t \oplus u, s \oplus u) \tag{3.12}$$

whenever $\mu(t) + \mu(u) \in \mathscr{E}^c$ and $\mu(s) + \mu(u) \in \mathscr{E}^c$. Then

$$r(t, s) = r(t \oplus s, 0) \quad (\mu(t) + \mu(s) \in \mathscr{E}^c) ,$$

and $r(t, 0)$ plays the role of ρ defined by (3.1). The restrictions $\mu(t) + \mu(u) \in \mathscr{E}^c$ and $\mu(s) + \mu(u) \in \mathscr{E}^c$ are essential. If this is not the case, even a simpler process as in Example 1 cannot be a DSP, since $W(x, t \oplus u) = W(x, t) W(x, u)$ is not an identity (see (2.7)), and (3.12) does not hold for some pairs $(t, u) \in R_+^2$. For ordinary stationary processes, on the other hand, such a situation does not occur, because the exponential functions, which are the counterpart of the Walsh functions, are continuous everywhere.

(b) In the discrete-parameter case Nagai [19] proved the spectral representation theorems of a DSP (counter part of Theorems 3.1 and 3.2), under the same condition as the ordinary (time-invariant) stationary processes. It is, however, shown by Endow [7] that it also needs to annex a 'smoothness' condition to the covariance function at jump points of the Walsh functions.

3.3. SPECTRAL REPRESENTATION OF A *DSP*.

THEOREM 3.2. *Let $\{X(t), t \in R_+\}$ be a DSP. If its covariance function has a representation in the form of* (3.7), *then*

$$X(t) = \int_0^\infty W(x, t) \zeta(dx) \quad \text{(with prob. 1)} , \tag{3.13}$$

where ζ is an orthogonal random measure with $E\zeta(A) = 0$ and $E\zeta(A)\overline{\zeta(B)} = m(A \cap B)$ $(A, B \in \mathscr{B}(R_+))$. Conversely, if a DSP has a representation in the form of (3.13), *then its covariance function is representable in the form of* (3.7).

PROOF. The first part of the statement is a direct corollary of a more general result (cf. [13, p. 201]). The converse part is also clear, since

$$EX(t) = \int_0^\infty W(x, t)E\zeta(dx) = 0$$

and

$$r(t, s) = EX(t)\overline{X(s)} = \int_0^\infty\int_0^\infty W(x, t)W(y, s)E\zeta(dx)\overline{\zeta(dy)}$$
$$= \int_0^\infty X(x, t)W(x, s)m(dx) .$$

Habib and Cambanius [14] defined a *DSP* as a special class of Walsh-harmonizable processes; when the random measure ζ of a Walsh-harmonizable process is orthogonal, or, equivalently, when its spectral measure is supported by the diagonal of R_+^2, then it is called a dyadic stationary process. They also stated that its covariance function has a form of

$$r(t, s) = \int_0^\infty W(t \oplus s, x)m(dx) .$$

It, however, must be modified as (3.7) since the equation

$$W(t, x)W(s, x) = W(t \oplus s, x)$$

is not an identity (see (2.7)).

4. Laws of large numbers. Let $\{X(t), t \in R_+\}$ be a measurable *DSP*. We adopt the assumptions (A), (B) and (C) in 3; hence it assumes the spectral representations (3.7) and (3.13).

The stochastic integral

$$\int_a^b X(t)dt \quad (0 \leqq a < b < \infty)$$

exists in quadratic mean, since

$$\int_a^b \sqrt{(E|X(t)|^2)}\,dt = \int_a^b \sqrt{\left(E\left|\int_0^\infty W(x, t)\zeta(dx)\right|^2\right)}dt$$
$$= \int_a^b \sqrt{\left(\int_0^\infty m(dx)\right)}dt = \sqrt{m(R_+)}(b - a) < \infty .$$

For every $R > 0$,

$$E\left|R^{-1}\int_0^R X(t)dt - \zeta(\{0\})\right|^2 = E\left|\int_{x\neq 0}\zeta(dx)R^{-1}\int_0^R W(x, t)dt\right|^2$$
$$= \int_{x\neq 0}\left|R^{-1}\int_0^R W(x, t)dt\right|^2 m(dx) .$$

By Lemma 2.4 the inner integral multiplied by R^{-1} converges boundedly to zero for $x \neq 0$. Hence we obtain the following result.

THEOREM 4.1. *Let the measurable DSP* $\{X(t),\ t \in R_+\}$ *satisfy the assumptions* (A), (B), *and* (C). *Then the weak law of large numbers holds*:

$$\operatorname*{l.i.m.}_{R\to\infty} R^{-1}\int_0^R X(t)dt = \zeta(\{0\}) .$$

COROLLARY 4.1. *Suppose that the conditions in Theorem* 4.1 *are satisfied. Then a DSP is ergodic if and only if* $m(\{0\}) = 0$.

COROLLARY 4.2. *Suppose that the conditions in Theorem* 4.1 *are satisfied. Then a DSP is ergodic if and only if*

$$\lim_{n\to\infty} 2^{-n}\int_0^{2^n} \rho(t)dt = 0 ,$$

where ρ *is the function defined by* (3.1).

PROOF. Using (3.9), we obtain

$$2^{-n}\int_0^{2^n}\rho(t)dt = \int_0^\infty m(dx)2^{-n}\int_0^{2^n} W(x, t)dt = \int_0^\infty 2^{-n}D(x;\ 2^n)m(dx) . \tag{4.1}$$

It follows from Lemma 2.4 that the last integral in (4.1) converges to $m(\{0\})$ as $n \to \infty$, and hence the conclusion follows from Corollary 4.1.

Next we consider the strong law of large numbers.

THEOREM 4.2. *Let the real-valued measurable DSP* $\{X(t),\ t \in R_+\}$ *satisfy the assumptions* (A), (B), *and* (C). *If*

$$\sum_{k=1}^\infty k^2 m(I(-k)) < \infty , \tag{4.2}$$

then

$$\lim_{R\to\infty} R^{-1}\int_0^R X(t)dt = 0 \quad (\textit{with prob.}\ 1) ,$$

where $I(-k) = [0, 2^{-k})$.

PROOF. We shall only check the convergences of the series

$$\sum_{k=1}^\infty \Pr\{2^{-k}S(2^k) \geqq \varepsilon/(k+1)\} \tag{4.3}$$

and

$$\sum_{k=1}^\infty \sum_{p=0}^{k-1} \sum_{q=0}^{2^{k-p}-1} \Pr\{2^{-k}|S(2^k + (q+1)2^p) - S(2^k + q2^p)| \geqq \varepsilon/(k+1\} , \tag{4.4}$$

where

$$S(a) = \int_0^a X(t)dt \ ,$$

since the proof parallels almost word-for-word the one, due to Verbitskaya [26] in the ordinary stationary case. Now by Tchebychev's inequality the series (4.3) converges if the series

$$\sum_{k=0}^{\infty} 2^{-2k}(k+1)^2 E\left|\int_0^{2^k} X(t)dt\right|^2 \tag{4.5}$$

converges. Rewriting

$$E\left|\int_0^{2^k} X(t)dt\right|^2 = \int_0^{2^k}\int_0^{2^k} r(u, v)dudv = \int_0^{\infty}\left(\int_0^{2^k} W(x, t)dt\right)^2 m(dx)$$
$$= \int_0^{\infty} D^2(x; 2^k)m(dx) = 2^{2k}m(I(-k)) \ ,$$

we see that if (4.2) holds then (4.5) converges. Similarly (4.4) converges if the following series converges:

$$\sum_{k=1}^{\infty}\sum_{p=0}^{k-1}\sum_{q=0}^{2^{k-p}-1} 2^{-2k}(k+1)^2\int_0^{2^p}\int_0^{2^p} r(2^k + q2^p + u, 2^k + q2^p + v)dudv \ . \tag{4.6}$$

Noting that $2^k \oplus q2^p \oplus u = 2^k + q2^p + u$ and $\mu(2^k \oplus q2^p) + \mu(u) \in \mathscr{E}^c$ $(k > p$, $q < 2^{k-p}$, $u < 2^p)$, we have

$$\int_0^{2^p}\int_0^{2^p} r(2^k + q2^p + u, 2^k + q2^p + v)dudv = \int_0^{2^p}\int_0^{2^p} r(u, v)dudv$$
$$= 2^{2p}m(I(-p)) \ .$$

Hence (4.6) reduces to

$$\sum_{k=1}^{\infty}\sum_{p=0}^{k-1} 2^{-2k}(k+1)^2 2^{2p}m(I(-p))\sum_{q=0}^{2^{k-p}-1} 1 = \sum_{k=1}^{\infty}\sum_{p=0}^{k-1} 2^{-k+p}(k+1)^2 m(I(-p))$$
$$= \sum_{p=0}^{\infty} 2^p m(I(-p))\sum_{k=p+1}^{\infty} 2^{-k}(k+1)^2 \ ,$$

which is majorized by the series (4.2) with some multiplicative constant.

5. Sampling theorem. A sampling theorem for a sequency band-limited (non-random) signal was proved by Maqusi [15]. He also proved there one for *DSP's* with spectral densities. Here we shall show a sampling theorem for a sequency band-limited *DSP* with a spectral representation.

A sampling theorem based on Walsh analysis techniques for not necessarily sequency-limited (non-random) signals is constructed by Butzer and Splettstößer [4]. Habib and Cambanis [14] derived a dyadic sampling representation for Walsh-harmonizable random signals.

Let a measurable *DSP* $\{X(t), t \in R_+\}$ satisfy the assumptions (A), (B) and (C). Suppose also that $m(A \cap [N, \infty)) = 0$ $(A \in \mathscr{B}(R_+))$, where m is the spectral measure of $\{X(t), t \in R_+\}$ and N is a positive integer. For fixed $t \in R_+$, we consider the Walsh function $W(x, t)$ as a function of x and expand it into the Walsh-Fourier series on $[0, N)$:

$$W(x, t) \sim \sum c_n W(xN^{-1}, n) ,$$

where c_n is the Fourier coefficient of $W(x, t)$, i.e.,

$$\begin{aligned} c_n &= N^{-1}\int_0^N W(x, t)\, W(xN^{-1}, n)dx = N^{-1}\int_0^N W(x, t \oplus nN^{-1})dx \\ &= N^{-1}\sum_{k=0}^{N-1} W(k, t \oplus nN^{-1})\int_0^1 W(x, t \oplus nN^{-1})dx \\ &= N^{-1}D(t \oplus nN^{-1}; N)J(t \oplus nN^{-1}, 1) , \end{aligned}$$

where we use the scaling property,

$$W(xN^{-1}, n) = W(x, nN^{-1}) \quad \text{(a.e.)} .$$

The partial sum

$$s_n(x) = \sum_{k=0}^{n-1} c_k W(xN^{-1}, k)$$

converges to $W(x, t)$ on $[0, N)$, since $W(x, t)$ is of bounded variation and continuous from the right on R_+ (cf. Chrestenson [6]). Hence

$$W(x, t) = \lim_{n\to\infty} s_n(x) = N^{-1}\sum_{k=[t]N}^{([t]+1)N-1} D(t \oplus kN^{-1}; N)\, W(xN^{-1}, k) \qquad (x \in [0, N)) . \tag{5.1}$$

Therefore

$$\begin{aligned} X(t) &= \int_0^N W(x, t)\zeta(dx) \\ &= N^{-1}\sum_{k=[t]N}^{([t]+1)N-1} D(t \oplus kN^{-1}; N)\int_0^N W(xN^{-1}, k)\zeta(dx) \\ &= N^{-1}\sum_{k=[t]N}^{([t]+1)N-1} D(t \oplus kN^{-1}; N)X(kN^{-1}) \quad (t \in R_+) . \end{aligned} \tag{5.2}$$

Thus we have the following.

THEOREM 5.1. *Suppose that a measurable DSP* $\{X(t), t \in R_+\}$ *satisfies the assumptions* (A), (B), *and* (C). *If its spectral measure is sequency band-limited, i.e.,* $m(A \cap [N, \infty)) = 0$ $(A \in \mathscr{B}(R_+))$ *for some positive integer* N *then* (5.2) *holds.*

COROLLARY 5.1. *If, in particular,* $N = 2^n$ *in Theorem* 5.1 *above,* *then*

$$X(t) = X([t2^n]2^{-n}) \quad (t \in R_+) . \tag{5.3}$$

PROOF. This follows from the fact $D(t \oplus k2^{-n}; 2^n) = 2^n$ when $t \oplus k2^{-n} < 2^{-n}$, i.e., $k = [t2^n]$; $= 0$ otherwise.

The Corollary 5.1 shows that if a *DSP* is sequency band-limited, then it may be considered to be essentially a discrete-parameter *DSP*.

6. Linear dyadic processes and DSP. Let η be a random measure on $\mathscr{B}(R_+)$ with $E\eta(A) = 0$ and

$$E\eta(A)\overline{\eta(B)} = \sigma^2\int_{A\cap B} dx \quad (A, B \in \mathscr{B}(R_+)) .$$

For $\Phi(t) \in L^2(R_+)$, define a stochastic process by the stochastic integral in quadratic mean

$$Y(t) = \int_0^\infty \Phi(t \oplus s)\eta(ds) \quad (t \in R_+) . \tag{6.1}$$

We shall call such a process *a linear dyadic process* (*LDP*) [5], [16] and [20]. We shall characterize *LDP*'s. Let $\{Y(t), t \in R_+\}$ be an *LDP* as above. Let J_A be the Walsh-Fourier transform of the characteristic function $\chi_A(x)$ of the bounded set $A \in \mathscr{B}(R_+)$, i.e.,

$$J_A(t) = \int_0^\infty \chi_A(x) W(x, t)dx = \int_A W(x, t)dx .$$

For every bounded set $A \in \mathscr{B}(R_+)$, define a random variable ξ by the integral

$$\xi(A) = \int_0^\infty J_A(t)\eta(dt) .$$

It is easily seen that ξ is an orthogonal random measure, since we have by Parsevals relation (Selfridge [25]),

$$E\xi(A)\overline{\xi(B)} = \sigma^2\int_0^\infty J_A(t)\overline{J_B(t)}dt = \sigma^2\int_0^\infty \chi_A(x)\overline{\chi_B(x)}dx = \sigma^2\int_{A\cap B} dx .$$

We shall show that for every function $f \in L^2(R_+)$,

$$\int_0^\infty F(t)\eta(dt) = \int_0^\infty f(x)\xi(dx) , \tag{6.2}$$

where

$$F(t) = \underset{R\to\infty}{\text{l.i.m.}} \int_0^R f(x) W(x, t)dx .$$

First consider the simple function

$$f(x) = \sum_{k=1}^{n} a_k J_{A_k}(x) ,$$

where the A_k's are bounded and disjoint sets belonging to $\mathscr{B}(R_+)$. By definition,

$$\int_0^\infty F(t)\eta(dt) = \sum_{k=1}^{n} a_k \int_0^\infty J_{A_k}(t)\eta(dt) = \sum_{k=1}^{n} a_k \xi(A_k) = \int_0^\infty f(x)\xi(dx) .$$

For every pair of simple functions f and g we may show that

$$E\int_0^\infty F(t)\eta(dt)\overline{\int_0^\infty G(t)\eta(dt)} = E\int_0^\infty f(x)\xi(dx)\overline{\int_0^\infty g(x)\xi(dx)} ,$$

where $G(t)$ is the Walsh-Fourier transform of $g(x)$ in the $L^2(R_+)$ sense. This relation can be extended to $L^2(R_+)$, since the set of all simple functions is dense in $L^2(R_+)$.

Now we recall that $\Phi \in L^2(R_+)$ and its Walsh-Fourier transform ϕ is also belongs to $L^2(R_+)$. The application of (6.2) to Φ and ϕ therefore gives

$$Y(t) = \int_0^\infty \Phi(t \oplus s)\eta(ds) = \int_0^\infty W(x, t)\phi(x)\xi(dx) . \tag{6.3}$$

Put

$$\zeta(A) = \int_A \phi(x)\xi(dx) \quad (A \in \mathscr{B}(R_+)) ;$$

then $E\zeta(A) = 0$ and

$$E\zeta(A)\overline{\zeta(B)} = \sigma^2\int_{A\cap B} |\phi(x)|^2 dx \quad (A, B \in \mathscr{B}(R_+)) .$$

Hence, rewriting (6.3) as

$$Y(t) = \int_0^\infty W(x, t)\zeta(dx) , \tag{6.4}$$

we have the following result.

THEOREM 6.1. *Let $\{Y(t), t \in R_+\}$ be an LDP in the form of* (6.1). *Then it is a DSP in the form of* (6.4) *with the spectral density* $\sigma^2|\phi(x)|^2$. *The representation* (6.4) *is unique.*

We shall also assert the converse statement.

THEOREM 6.2. *Let $\{X(t), t \in R_+\}$ be a DSP with spectral density. Then it is also an LDP.*

PROOF. Suppose that

$$X(t) = \int_0^\infty W(x, t)\zeta(dx)$$

and

$$E\zeta(A)\overline{\zeta(B)} = \int_{A\cap B} f(x)dx \quad (A, B \in \mathscr{B}(R_+)) ,$$

where f is the spectral density of $\{X(t), t \in R_+\}$. Let h be a measurable function such that $|h(x)|^2 = f(x)$, so that $h \in L^2(R_+)$. Let ζ_1 be a random measure that is orthogonal to ζ, with

$$E\zeta_1(A)\overline{\zeta_1(B)} = \int_{A\cap B} dx \quad (A, B \in \mathscr{B}(R_+)) .$$

Put

$$\xi(A) = \int_0^\infty \chi_A(x)/h(x)\zeta(dx) + \int_0^\infty \chi_A(x)h_1(x)\zeta_1(dx) ,$$

where $h_1(x) = 0$ when $h(x) \neq 0$; $= 1$ when $h(x) = 0$, and $1/h(x)$ is taken as zero if $h(x) = 0$; then

$$\begin{aligned} E\xi(A)\overline{\xi(B)} &= \int_0^\infty (\chi_A(x)\chi_B(x)/|h(x)|^2)f(x)dx + \int_0^\infty \chi_A(x)\chi_B(x)|h_1(x)|^2 dx \\ &= \int_{A\cap B} dx . \end{aligned}$$

Define a random measure

$$\eta(A) = \int_0^\infty J_A(x)\xi(dx) \quad (A \in \mathscr{B}(R_+)) ,$$

then

$$\eta(A \oplus t) = \int_0^\infty J_A(x) W(x, t)\xi(dx) .$$

Hence, noting that

$$X(t) = \int_0^\infty W(x, t)\zeta(dx) = \int_0^\infty W(x, t)h(x)\xi(dx)$$

we have in the same way as (6.2)

$$X(t) = \int_0^\infty H(s)\eta(ds \oplus t) = \int_0^\infty H(s \oplus t)\eta(ds) ,$$

where H is the Walsh-Fourier transform of h in the $L^2(R_+)$ sense.

The discrete-parameter counterparts of Theorems 6.1 and 6.2 are shown by Nagai [19].

7. Summary. It is known that the (ordinary) mean-continuous stationary processes assume their spectral representations. Measurability of the processes ensures their mean-continuity. The W-mean-continuity of DSP's, however, requires not only measurability of the processes but also the Assumption (A). Moreover, two supplementary assumptions (B) and (C) are required in order to represent the DSP's by Walsh functions, owing to the discontinuity of these functions. The exponentionl functions, in terms of which the stationary processes are represented, are, however, continuous everywhere. For DSP's the assumptions (A), (B) and (C) are relevant to their spectral representations if detailed analysis of the processes is to be implemented.

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