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SPECTRA OF MEASURES AS L_p MULTIPLIERS

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1. Preliminaries. Let G be a nondiscrete locally compact abelian group with the dual Γ, $M(G)$ the convolution measure algebra of finite regular Borel measures on G. For $\mu \in M(G)$, let $\|\mu\|$ denote the total variation norm, $\mu^1 = \mu$, $\mu^j = \mu^{j-1}*\mu$ $(j = 2, 3, \cdots)$, where $*$ denotes the convolution, $\hat{\mu}$ the Fourier-Stieltjes transform of, μ, and $\|\hat{\mu}\|_\infty = \sup\{|\hat{\mu}(\gamma)|; \gamma \in \Gamma\}$. We call μ a Hermitian measure if $\hat{\mu}(\gamma)$ is real valued on Γ. For $1 \leqq p \leqq \infty$, let $L_p(G)$ be the L_p space with respect to the Haar measure of G, $\| \quad \|_p$ the norm of $L_p(G)$. A bounded linear operator T on $L_p(G)$ is called an L_p multiplier if there exists $\hat{T} \in L_\infty(\Gamma)$ such that $T(f)\hat{} = \hat{T}\hat{f}$ for every $f \in L_p(G) \cap L_1(G)$. The set of all L_p multipliers will be written as $M_p(G)$ and the norm of $T \in M_p(G)$ is defined by

$$\|T\|_{M_p(G)} = \|T\|_{M_p} = \sup\{\|Tf\|_{L_p(G)}; \|f\|_{L_p(G)} = 1\} .$$

Then $M_p(G)$ is a commutative Banach algebra with unit δ_0 as the convolution operator, where δ_0 is the Dirac measure with unit mass at $0 \in G$. Also for $T \in M_p(G)$, let $\tilde{T}$ be the Gelfand transform

$$\|\tilde{T}\|_{\Delta M_p} = \sup\{|h(T)|; h \text{ is a complex homomorphism on } M_p(G)\} ,$$

and $\operatorname{Im} \tilde{T}$ the imaginary part of $\tilde{T}$.

Now it is known that any measure $\mu \in M(G)$ is contained in M_p(G) as a convolution operator, and $M_1(G)$ is isomorphic to $M(G)$, $M_2(G)$ to $L_\infty(\Gamma)$, $M_p(G)$ to $M_q(G)$ if $1/p + 1/q = 1$ $(1<p<\infty)$, and $M_1(G) \subseteqq M_p(G) \subseteqq M_2(G)$ $(1 \leqq p \leqq 2)$ (cf. [6]). For $T \in M_p(G)$, let $\operatorname{sp}(T, M_p)$ be the spectrum of T in $M_p(G)$, i.e., $\operatorname{sp}(T, M_p) = \{\lambda \in \boldsymbol{C}; \lambda\delta_0 - T \text{ is not invertible in } M_p(G)\}$, where $\boldsymbol{C}$ is the complex plane. Then for $\mu \in M(G)$, we have $\operatorname{closure}(\hat{\mu}(\Gamma)) = \operatorname{sp}(\mu, M_2) \subseteqq \operatorname{sp}(\mu, M_p) \subseteqq \operatorname{sp}(\mu, M(G))$ $(1 \leqq p \leqq 2)$, where $\operatorname{closure}(\hat{\mu}(\Gamma))$ is the closure of $\hat{\mu}(\Gamma)$ in the complex plane. Before stating our theorems, we make some preliminary comments. For $f \in L_1(G)$, it is well known and easy to show that $\operatorname{sp}(T_f, M_p(G)) = \hat{f}(\Gamma) \cup \{0\}$ for $1 \leqq p \leqq \infty$ if $T_f(g) = f*g$ for all $g \in L_p(G)$. However, since G is nondiscrete, the classical

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theorem of Wiener and Pitt and its generalization imply the existence of $\mu \in M(G)$ so that $\mathrm{sp}(\mu, M(G))$ properly contains closure$(\hat{\mu}(\Gamma))$. Indeed, there exists $\mu \in M(G)$ so that $\mathrm{sp}(\mu, M(G)) \neq \hat{\mu}(\Gamma) \cup \{0\}$ and $\hat{\mu}(\gamma) \to 0$ as $\gamma \to \infty$ $(\gamma \in \Gamma)$ (cf. [9]). Also [4] shows that if $1 < p < 2$, then $\mathrm{sp}(\mu, M_p(G)) = \hat{\mu}(\Gamma) \cup \{0\}$, whenever $\mu \in M(G)$ with $\hat{\mu}(\gamma) \to 0$ as $\gamma \to \infty$ $(\gamma \in \Gamma)$.

On the other hand, Igari [5] proved that for $1 < p < \infty (p \neq 2)$ there exists $\mu \in M(G)$ such that $\mathrm{sp}(\mu, M_p) \neq$ closure$(\hat{\mu}(\Gamma))$. In fact, he showed that each operating function from $M(G)$ to $M_p(G)$ is extended to an entire function (cf. [3]). Also Zafran [13] constructed $T \in M_p(G) \setminus M(G)$ $(1 < p < \infty, p \neq 2)$ such that $\hat{T}(\gamma) \to 0$ as $\gamma \to \infty$ and $\mathrm{sp}(T, M_p) \neq \hat{T}(\Gamma) \cup \{0\}$. Later in [14] he showed that each operating function on $C_0M_p(G) = \{T \in M_p(G); \hat{T}(\gamma) \to 0$, as $\gamma \to \infty$, $\hat{T}$ is continuous on $\Gamma\}$ is extended to an entire function (cf. [15]).

Now for a Hermitian measure $\mu \in M(G)$ with $\|\mu\| = 1$, Sarnak [11] proved that the spectrum of μ in $M_p(G)$ is contained in some area of the unit disk, but generally $\mathrm{sp}(\mu, M_p) \subsetneqq \mathrm{sp}(\mu, M(G))$ $(1 < p < \infty)$. Indeed, when $G = \boldsymbol{T}$ (unit circle group), he proved that there exists a Hermitian measure such that $\mathrm{sp}(\mu, M(G)) = \{z \in \boldsymbol{C}; |z| \leqq \|\mu\|\} \supsetneqq \mathrm{sp}(\mu, M_p(G)) =$ closure $(\hat{\mu}(\Gamma))$ for all $1 < p < \infty$ (cf. [2]).

In this paper, we will give some results concerning spectra of measures in $M_p(G)$ by the method of [5]. Then we will obtain a Hermitian measure μ on G such that $\mathrm{sp}(\mu, M_p(G)) \supsetneqq$ closure$(\hat{\mu}(\Gamma))$ for all $1 \leqq p < 2$. In §2, by an application of [5, Lemma 1] we will show that there exists a Hermitian measure μ on each nondiscrete locally compact abelian group such that $\|\mathrm{Im}\,\tilde{\mu}\|_{\Delta M_p} > 0$ for all $1 \leqq p < 2$. Also in §3, we will investigate the spectrum of the measure in §2. When $G = \boldsymbol{T}$, we will prove in §4 that only entire functions can operate on the algebra which contains $L_1(\mathrm{G})$ and the measure in §2.

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2. Measures as L_p multipliers on locally compact abelian groups. In this section we will show the existence of certain measures which have suitable spectra as L_p multipliers when G is a nondiscrete locally compact abelian group. Let $\Delta(r)$ be the direct sum of countably many copies of the cyclic group $\boldsymbol{Z}(r)$ of order r $(r \geqq 2)$ and $D(r)$ be the dual to $\Delta(r)$. We refer the reader to [5] and [8] for the proof of the following.

LEMMA (cf. [5, Lemma 1]). *Let Γ be the dual of G, and Γ be $\boldsymbol{Z}$ or $\Delta(r)$. Then for any* $1 \leqq p < 2$ and a *positive integer j there exist a constant K_p* (>1) *depending only on G and p, and a nonnegative trigono-*

metric polynomial $\phi = \phi_{p,j}$ *on* G *such that*

(i) $\hat{\phi} \geqq 0$ on Γ, $\|\phi\|_1 = 1$, and

(ii) $\|\exp(ij(\phi\lambda_G))\|_{M_p(G)} > K_p^j$,

where λ_G *is the normalized Haar measure on* G, *and* $\exp(i\mu) \in M(G)$ *is defined by* $\exp(i\mu)^\wedge = \exp(i\hat{\mu})$ *for* $\mu \in M(G)$.

PROOF. By [5, Lemma 1] and [8], there exists a Hermitian probability measure $\mu = \mu_{p,j}$ on G such that $\|\exp(ij\mu/2)\|_{M_p(G)} > K_p^j$, where $K_p > 1$ is a constant depending only on p and G. Put $\nu = (\delta_0 + \mu)/2$. Then ν is a probability measure, $\hat{\nu} \geqq 0$ on Γ, and

$$\|\exp(ij\nu)\|_{M_p(G)} = \|\exp(ij\mu/2)\|_{M_p(G)} > K_p^j .$$

Therefore we obtain the trigonometric polynomial $\phi = \phi_{p,j}$ on G with the desired properties by convolving ν with an appropriate trigonometric polynomial (cf. [1]). q.e.d.

REMARK 1 (cf. [8]). (i) For $\Gamma = \Delta(r)$ for some r, we may choose

$$K_p = \left[\sum_{\gamma \in \Delta}\left|\int \exp\left(iR(x, \gamma_0)\right)(x, \gamma)dx\right|^p\right]^{1/(2p)},$$

where dx is the normalized Haar measure of $D(r)$ and γ_0 is an element of order r.

(ii) For $\Gamma = \boldsymbol{Z}$, we may choose

$$K_p = \left[\sum_{m \in \boldsymbol{Z}}\left|\int \exp(i(\cos x - mx))dx/(2\pi)\right|^p\right]^{1/(2p)} = \left[\sum_{m \in \boldsymbol{Z}}|J_m(1)|^p\right]^{1/(2p)}$$

where $dx/(2\pi)$ is the normalized Haar measure of $\boldsymbol{T}$ and $J_m(x)$ is the Bessel function.

REMARK 2. By Riesz-Thorin's convexty theorem we choose $K_p \leqq \exp(2/p - 1)$ in Lemma. Thus we have $K_p \to 1$ as $p \to 2 (p < 2)$.

THEOREM 1. *Let* G *be an infinite compact abelian group. Then there exists a probability measure* $\mu \in M(G)$, *with nonnegative Fourier-Stieltjes transform, such that for real number* $1 \leqq p < 2$

$$\|\operatorname{Im} \tilde{\mu}\|_{\Delta M_p(G)} > 0 .$$

In particular, we get $\operatorname{sp}(\mu, M_p(G)) \supsetneqq \operatorname{closure}(\hat{\mu}(\Gamma))$ *for all* p $(1 \leqq p < 2)$.

PROOF. Let $\boldsymbol{Q}$ be the set of all rational numbers. For $1 \leqq p < 2$ and $T \in M_p(G)$, we write $\|T\|_{M(p,\Gamma)}$ for $\|T\|_{M_p(G)}$. For each natural number n, let p_n be a rational number satisfying $1 \leqq p_n < 2$ such that $\{p_n; n \geqq 1\} =$

$\boldsymbol{Q}\cap\{p; 1\leqq p<2\}$ and that each $p\in \boldsymbol{Q}$ with $1\leqq p<2$ appears infinitely often among the p_n's.

Case 1. $\Gamma=\Delta(r)$ for some r. For natural numbers $m<n$, we write

$$G(m, n)=\prod_{k=m+1}^{n} \boldsymbol{Z}(r)\ , \quad \text{and} \quad \Gamma(m, n)=\prod_{k=m+1}^{n} \boldsymbol{Z}(r)^{\wedge}\ .$$

We shall identify $G(m, n)$ and $\Gamma(m, n)$ with the naturally corresponding subgroups of G and Γ, respectively.

Now we choose natural numbers n_j $(j\geqq 0)$ as follows. Put $n_0=1$, and suppose that $n_0<n_1<\cdots<n_{j-1}$ have been choosen for some $j\geqq 1$. By the above Lemma with $p=p_j$, there exist $n_j>n_{j-1}$ and a probability measure $\mu_j\in M(G(n_{j-1}, n_j))$ such that

$$(1)\qquad \hat{\mu}_j\geqq 0 \quad \text{on} \quad \Gamma(n_{j-1}, n_j)\ , \quad \text{and}$$

$$(2)\qquad \|\exp(ij\mu_j)\|_{M_p(G(n_{j-1},n_j))}>K_p^j \qquad \text{for} \quad p=p_j\ .$$

Identify G with the product group $\prod_{j=1}^{\infty} G(n_{j-1}, n_j)$, and put $\mu=\mu_1\times\mu_2\times\cdots$, the product measure of all $\mu_j(j\geqq 1)$. Clearly, μ is a probability measure on G with $\hat{\mu}\geqq 0$. Writing $\Gamma_j=\Gamma(n_{j-1}, n_j)\subset\Gamma$ for each $j\geqq 1$, we also have

$$\begin{aligned}\|\exp(ij\mu)\|_{M_p(G)}&=\|\exp(ij\hat{\mu})\|_{M(p,\Gamma)}\\ &\geqq \|\exp(ij\hat{\mu}_j)\|_{M(p,\Gamma_j)} \qquad \text{for} \quad p=p_j\ ,\end{aligned}$$

where the first inequality is obvious, since $\hat{\mu}=\hat{\mu}_j$ on Γ_j (cf. [10, Corollary 4.6]) and the second inequality follows from (2). Since each element of $\boldsymbol{Q}\cap\{p; 1\leqq p<2\}$ appears infinitely often in $\{p_j\}$, it is routine to show that

$$(3)\qquad \lim_{n\to\infty}(\|\exp(in\mu)\|_{M_p(G)})^{1/n}\geqq K_p>1$$

for all $p\in\boldsymbol{Q}\cap\{p; 1\leqq p<2\}$. But $\boldsymbol{Q}\cap\{p; 1\leqq p<2\}$ is dense in $\{p; 1\leqq p<2\}$, so we get

$$\lim_{n\to\infty}\|\exp(in\mu)\|_{M_p(G)}^{1/n}>1 \qquad \text{for all} \quad 1\leqq p<2$$

by (3) and Riesz-Thorin's theorem.

Case 2. $\Gamma=\boldsymbol{Z}$. For each positive integer j, the Lemma yields a nonnegative trigonometric polynomial ϕ_j on $\boldsymbol{T}$ such that

$$(4)\qquad \hat{\phi}_j\geqq 0 \quad \text{on} \quad \boldsymbol{Z}\ , \qquad \|\phi_j\|_1=1\ , \quad \text{and}$$

$$(5)\qquad \|\exp(ij\phi_j)\|_{M_p(T)}>K_p^j \quad \text{for} \quad p=p_j\ .$$

Choose a trigonometric polynomial f_j on $\boldsymbol{T}$ such that

$$(6)\qquad \|f_j\|_p=1 \quad \text{and} \quad \left\|\sum_k \exp(ij\hat{\phi}_j(k))\hat{f}_j(k)\exp(ikt)\right\|_p>K_p^j$$

for $p = p_j$. Also choose a natural number m_j so that

$$(7) \qquad (\operatorname{supp}\hat{\phi}_j) \cup (\operatorname{supp}\hat{f}_j) \subset \{-m_j, -m_j + 1, \cdots, m_j - 1, m_j\} .$$

Now let $r_1 = 1, r_2, r_3, \cdots$ be an increasing sequence of natural numbers such that $r_n \to \infty$ very rapidly. Let λ_T be the normalized Haar measure of $\boldsymbol{T}$. Then by the proof of [7, Lemma 5], the measures defined by

$$d\mu_n(t) = [\phi_1(r_1 t)\phi_2(r_2 t) \cdots \phi_n(r_n t)]d\lambda_T(t)$$

all are probability measures, and converges weak* to a probility measure $\mu \in M(\boldsymbol{T})$ such that

$$(8) \qquad \hat{\mu}(k_1 r_1 + k_2 r_2 + \cdots + k_n r_n) = \prod_{j=1}^{n} \hat{\phi}_j(k_j)$$

whenever the k_j are integers such that

$$(9) \qquad |k_j| \leqq m_j \ (j = 1, 2, \cdots, n) , \quad \text{and}$$

$$(10) \qquad \hat{\mu}(m) = 0 \quad \text{for all other integers} \quad m .$$

Now let a natural number j be given. Define a trigometric polynomial g_j by setting $g_j(t) = f_j(r_j t)$ for $t \in \boldsymbol{T}$. Then

$$(11) \qquad \|g_j\|_p = 1 \quad \text{and} \quad \operatorname{supp}\hat{g}_j \subset \{r_j k; k = -m_j, \cdots, m_j\}$$

by (6) and (7). Moreover, $\hat{g}_j(kr_j) = \hat{f}_j(k)$ for all $k \in \boldsymbol{Z}$. It follows that by (6) and (8)

$$\begin{aligned} &\left\| \sum_k \exp(ij\hat{\mu}(k))\hat{g}_j(k) \exp(ikt) \right\|_p \\ &= \left\| \sum_k \exp(ij\hat{\mu}(kr_j))\hat{g}_j(kr_j) \exp(ikr_j t) \right\|_p \\ &= \left\| \sum_k \exp(ij\hat{\phi}_j(k))\hat{f}_j(k) \exp(ikt) \right\|_p > K_p^j \quad \text{for} \quad p = p_j . \end{aligned}$$

This, combined with (11), yields

$$\|\exp(ij\mu)\|_{M_p(T)} > K_p^j \quad \text{for} \quad p = p_j \quad \text{and} \quad j = 1, 2, \cdots .$$

As in Case 1, we conclude that μ has the required properties.

Case 3. Let Γ be an unbounded ordered group. For each positive integer j, the Lemma yields trigonometric polynomials f_j and g_j on $\boldsymbol{T}$ such that

$$(12) \qquad f_j \geqq 0, \|f_j\|_1 = 1, \hat{f}_j \geqq 0 ,$$

$$(13) \qquad \|g_j\|_p \leqq 1 , \quad \|g_j * \exp(ijf_j\lambda_T)\|_p > K_p^j \text{ for } p = p_j .$$

Also choose a natural number N_j so that

$$(14) \qquad (\operatorname{supp}\hat{f}_j) \cup (\operatorname{supp}\hat{g}_j) \subset \{-N_j, \cdots, -1, 0, 1, \cdots, N_j\} .$$

Then by the proof of [7, Lemma 5], there exist $\{\gamma_s\}_{s=1}^{\infty} \subset \Gamma(\mathrm{ord}(\gamma_s) \geqq 3, s \geqq 1)$ and a probability measure $\mu \in M(G)$ which has the following properties: (i) when

$$\phi_j = \sum_{|k| \leqq N_j} \hat{f}_j(k)(x, k\gamma_j) \quad \text{for} \quad j = 1, 2, \cdots, \quad \text{and}$$

$$d\mu_n = \phi_1 \cdots \phi_n d\lambda_G \quad \text{for} \quad n = 1, 2, \cdots,$$

the probability measures $\{d\mu_n\}$ converge weak* to μ.

(ii) We have

$$\hat{\mu}(k_1\gamma_1 + \cdots + k_n\gamma_n) = \prod_{j=1}^{n} \hat{f}_j(k_j)\,, \tag{15}$$

whenever the k_j are integers such that

$$|k_j| \leqq N_j \quad (j = 1, \cdots, n)\,. \tag{16}$$

(iii)

$$\hat{\mu}(\gamma) = 0 \quad \text{on} \quad \Gamma \Big\backslash \bigcup_{n-1}^{\infty} \{k_1\gamma_1 + \cdots + k_n\gamma_n : |k_j| \leqq N_j, 1 \leqq j \leqq n\}\,. \tag{17}$$

Now let a natural number j be given. Define a trigonometric polynomial $\psi_j(x) = \sum_{|k| \leqq N_j} \hat{g}_j(k)(x, k\gamma_j)$ on G. Then

$$\mathrm{supp}\,\hat{\psi}_j \subset \{k\gamma_j : |k| \leqq N_j\} \quad \text{and} \quad \hat{\psi}_j(k\gamma_j) = \hat{g}_j(k) \tag{18}$$

for all $k \in \boldsymbol{Z}$. It follows that by (17) and (18)

$$\exp(ij\hat{\mu}(k\gamma_j))\hat{\psi}_j(k\gamma_j) = \exp(ij\hat{f}_j(k))\hat{g}_j(k) \tag{19}$$

for all $k \in \boldsymbol{Z}$, and $j = 1, 2, \cdots$.

For the polynomial Q on $\boldsymbol{T}$ of order m, put $Q^*(x) = \sum_{|k| \leqq m} \hat{Q}(k)(x, k\gamma)$ for any $\gamma \in \Gamma$ of order $\geqq 3$. Then it is well known that

$$3\|Q\|_{L_p(T)} \geqq \|Q^*\|_{L_p(G)} \geqq (1/2)\|Q\|_{L_p(T)} \tag{20}$$

for $1 \leqq p < 2$ (cf. [3], [5]).

Then it follows by (13), (19) and (20) that

$$\begin{aligned} 3\|\exp(ij\mu)\|_{M_p(G)} &\geqq \|\exp(ij\mu) * \psi_j\|_{L_p(G)} \\ &\geqq (1/2)\|\exp(ijf_j\lambda_T) * g_j\|_{L_p(T)} > (1/2)K_p^j \end{aligned}$$

for $p = p_j$. This yields

$$\|\exp(ij\mu)\|_{M_p(G)} > (1/6)K_p^j \quad \text{for} \quad p = p_j \quad \text{and} \quad j \geqq 1\,.$$

As in Case 1, we conclude that μ has the required properties.

Case 4. Suppose G is an infinite compact abelian group. Then Γ contains $\boldsymbol{Z}$, $\Delta(r)$ for some r or an unbounded ordered group. Since $\|\hat{T}\|_{M(p,\Gamma)} \geqq \|\hat{T}\|_{M(p,\Lambda)}$ for all $T \in M_p(G)$ with all closed subgroups Λ of Γ by [10, Corollary

4.6], this completes the proof. q.e.d.

THEOREM 2. *Let G be a nondiscrete locally compact abelian group. Then there exists a probability measure $\mu \in M(G)$ with nonnegative Fourier-Stieltjes transform, such that for real number $1 \leqq p < 2$*

$$\|\operatorname{Im} \tilde{\mu}\|_{\Delta M_p(G)} > 0 .$$

In particular, we get $\operatorname{sp}(\mu, M_p(G)) \supsetneqq \operatorname{closure}(\hat{\mu}(\Gamma))$ *for all p $(1 \leqq p < 2)$.*

PROOF. By Theorem 1, we may assume G to be noncompact. Since G is nondiscrete, by the structure theorem (cf. [9]) G contains an open subgroup of the form $G_0 = \boldsymbol{R}^n \times H$, where $n \geqq 0$ and H is compact.

Case 1. Suppose H is an infinite group. Then there exists $\mu_0 \in M(H)$ having the properties of Theorem 1. By Theorem 1 and [10, Lemma 3.1] there exists a probability measure $\mu \in M(G)$ with nonnegative Fourier-Stieltjes transform such that

$$\|\exp(ij\mu)\|_{M_p(G)} = \|\exp(ij\mu_0)\|_{M_p(H)}$$

for all $1 \leqq p < 2$ and $j = 1, 2, \cdots$. Therefore we get

$$\lim_{m\to\infty} \|\exp(im\mu)\|_{M_p(G)}^{1/m} > 1 \quad \text{for all} \quad 1 \leqq p < 2.$$

Case 2. Suppose H is a finite group. Since G is nondiscrete, n is a posivive integer. Then by Theorem 1 there exists $\mu_0 \in M(\boldsymbol{T})$ having the properties of Theorem 1. By [9] and [10, Corollary 4.6], there exists a probability measure $\mu_1 \in M(\boldsymbol{R}^n)$ with nonnegative Fourier-Stieltjes transform such that

$$\|\exp(ij\mu_1)\|_{M_p(\boldsymbol{R}^n)} \geqq \|\exp(ij\mu_0)\|_{M_p(\boldsymbol{T})}$$

for all $j = 1, 2, \cdots$. So by [10, Lemma 3.1], there exists a probability measure $\mu \in M(G)$ with nonnegative Fourier-Stieltjes transform such that

$$\|\exp(ij\mu)\|_{M_p(G)} \geqq \|\exp(ij\mu_0)\|_{M_p(\boldsymbol{T})}$$

for all $j = 1, 2, \cdots$. Therefore we get the desired results. q.e.d

3. Spectra of measures. Let $\mu \in M(G)$ be a Hermitian measure with $\|\mu\| = 1$. Then by Riesz-Thorin's theorem, we have

$$\|\operatorname{Im} \tilde{\mu}\|_{\Delta M_p} \leqq 2/p - 1 \quad \text{for} \quad 1 \leqq p < 2. \quad \text{(cf. Remark 2)}$$

Hence we have $\|\operatorname{Im} \tilde{\mu}\|_{\Delta M_p} \to 0$ as $p \to 2$ $(p < 2)$.

Sarnak [11] obtained the next result which is better that the above result.

PROPOSITION 1 ([11]). *Let G be a locally compact abelian group,*

$\mu \in M(G)$ a Hermitian measure with $\|\mu\| = 1$ and $1 \leqq p \leqq 2$. Then $\mathrm{sp}(\mu, M_p(G))$ *is contained in the region bounded by $\gamma_p = \{z; \theta(z) = \pi/2 + \pi(p' - 1)/p\}$ where $\theta(z)$ is the angle subtended at z by the line segment* $[-1, 1]$.

Morever, when $G = \boldsymbol{T}$ and μ is the measure given by convolution by Cantor-Lebesgue-type, he proved that $\mathrm{sp}(\mu, M_p(\boldsymbol{T})) = \mathrm{closure}(\hat{\mu}(\boldsymbol{Z}))$, and $\mathrm{sp}(\mu, M(\boldsymbol{T})) \supsetneqq \mathrm{sp}(\mu, M_p(\boldsymbol{T}))$ $(1 < p \leqq 2)$. He also showed the same result for certain Riesz products (cf. [2]).

PROPOSITION 2 ([12, Lemma 2.2]). *Let G be a locally compact abelian group, $\mu \in M(G)$, and $1 \leqq p < \infty$. If a complex number λ is an isolated point of* $\mathrm{sp}(\mu, M_p(G))$, *then λ is in the closure of $\hat{\mu}(\Gamma)$.*

By §2 and Proposition 2, we get the following, which may be of some interest in view of Proposition 1.

THEOREM 3. *Let G be a nondiscrete locally compact abelian group. Then there exists a probability measure μ, with nonnegative Fourier-Stieltjes transform having the following properties: There exists a sequence $\{p_j\}_{j=1}^{\infty}$ of real numbers such that $p_j \to 2$ $(p_j < 2)$ as $j \to \infty$,* that

$$\|\mathrm{Im}\,\tilde{\mu}\|_{\Delta M_{p_j}} > \|\mathrm{Im}\,\tilde{\mu}\|_{\Delta M_{p_{j+1}}} \quad \textit{for all} \quad j$$

and that $\mathrm{sp}(\mu, M_{p_j}) \setminus \mathrm{sp}(\mu, M_{p_{j+1}})$ *is uncountable $(j \geqq 1)$. In particular,* we *have*

$$\mathrm{sp}(\mu, M_{p_j}) \supsetneqq \mathrm{sp}(\mu, M_{p_{j+1}}) \supsetneqq \mathrm{closure}(\hat{\mu}(\Gamma)) \quad (j \geqq 1)\ .$$

PROOF. By Theorem 2, there exists a probability measure $\mu \in M(G)$ having the properties of Theorem 2. We put $N_p = \|\mathrm{Im}\,\tilde{\mu}\|_{\Delta M_p}$ for all p $(1 \leqq p < 2)$. By the remark at the beginning of this section, we have $N_p \to 0$ as $p \to 2$ $(p < 2)$. Then by Theorem 2, there exists a sequence $\{p_1 < p_2 < \cdots < 2\}$ of real numbers such that

$$N_{p_1} > N_{p_2} > \cdots > N_{p_j} > \cdots \geqq 0\ .$$

Moreover by $N_{p_j} > N_{p_{j+1}}$ $(j \geqq 1)$ and Proposition 2, $\mathrm{sp}(\mu, M_{p_j}(G)) \setminus \mathrm{sp}(\mu, M_{p_{j+1}}(G))$ contains a nonempy perfect set. Thus it is uncountable. q.e.d.

4. Individual symbolic calculus. In this section, we consider the operating function of μ obtained in Theorem 1 for $G = \boldsymbol{T}$.

PROPOSITION 3. *Let Φ be a 2π-periodic continuous function on $\boldsymbol{R}$, and μ as in Theorem 1. Also let p be any fixed positive number with $1 < p < 2$. Assume that $\Phi(\lambda\hat{\mu} + a + \hat{f}) \in M_p(\boldsymbol{T})^\wedge$ for $\lambda, a \in \boldsymbol{R}$, and $\hat{f}$ real-valued $(f \in L_1(\boldsymbol{T}))$, where $M_p(\boldsymbol{T})^\wedge = \{\hat{T}; T \in M_p(\boldsymbol{T})\}$. Then Φ is extended to an entire function.*

PROOF. Step 1. We show that for any real number λ, there exists $C_\lambda > 0$ depending only on λ such that $\|\Phi\circ(\lambda\mu + a\delta_0)\|_{M_p(T)} \leqq C_\lambda$ for all $a \in [-\pi, \pi]$. If the above result is false, we may suppose that there exists a sequence $\{a_n\}$ such that $|a_p| < 1/2^n$, and $\|\Phi\circ(\lambda\mu + a_n\delta_0 + b\delta_0)\|_{M_p} \to \infty$ as $n \to \infty$. Then there exists a sequence $\{k_n\}$ of positive integers and an $S_n \in M_p(\boldsymbol{T})$ such that $S_n = \chi_{[-k_n, k_n]}$, the characteristic function of $[-k_n, k_n]$, and

$$\|S_n\circ\Phi\circ(\lambda\mu + a_n\delta_0 + b\delta_0)\|_{M_p} \to \infty \quad \text{as} \quad n \to \infty\ .$$

There exist a sequence $\{N_n\}$ of positive integers and $\{Q_n\}$ trigonometric polynomials on $\boldsymbol{T}$ such that

(1) the $[-k_n, k_n] + N_n$ are pairwise disjoint,

(2) $\hat{Q}_n = 1$ on $[-k_n, k_n] + N_n$,

(3) $\operatorname{supp}\hat{Q}_n \cap \operatorname{supp}\hat{Q}_m = \varnothing\, (n \neq m)$ and

(4) $\|Q_n\|_1 \leqq 2$ for all n .

Also we define $g = \sum a_n Q_n \in L_1(\boldsymbol{T})$ and $B_n \in M_p(\boldsymbol{T})$ with $\hat{B}_n = \chi_{[-k_n, k_n]+N_n}$, Then by assumption, we have $\Phi\circ(\lambda\mu + g + b\delta_0) \in M_p(\boldsymbol{T})$.

Hence by the Fourier-Stieltjes transform, we get

$$B_n\circ\Phi\circ(\lambda\mu + g + b\delta_0) = B_n\circ\Phi\circ(\lambda\mu + a_nQ_n + b\delta_0) = B_n\circ\Phi\circ(\lambda\mu + a_n\delta_0 + b\delta_0)\ .$$

Hence $\|B_n\circ\Phi\circ(\lambda\mu + g + b\delta_0)\|_{M_p} = \|S_n\circ\Phi\circ(\lambda\mu + a_n\delta_0 + b\delta_0\|_{M_p}$. On the other hand, by M. Riesz's theorem we have $\|B_n\|_{M_p} \leqq A_p$ $(n \geqq 1)$, where A_p is a constant depending only on p. Thus we have

$$\|S_n\circ\Phi\circ(\lambda\mu + a_n\delta_0 + b\delta_0)\|_{M_p} \leqq A_p\|\Phi\circ(\lambda\mu + g + b\delta_0)\|_{M_p}$$

for all n, and we obtain a contradiction.

Step 2. Let the Fourier series of Φ be $\sum a_n \exp(int)$. Then for any positive integer j, we have $a_n = O(\exp(-j|n|))$ for all $n \in \boldsymbol{Z}$.

Indeed, let j be a positive integer, and n a nonnegative integer. Then we get

$$\begin{aligned} a_n \exp(inj\hat{\mu}) &= (1/2\pi)\int\Phi(x)\exp(-in(x - j\hat{\mu}))dx \\ &= (1/2\pi)\int\Phi(x + j\hat{\mu})\exp(-jnx)dx\ . \end{aligned}$$

By Step 1, we get

$$\|a_n \exp(\mathrm{inj}\hat{\mu})\|_{M_p} \leqq (1/2\pi)\int\|\Phi(x + j\hat{\mu})\|_{M_p}dx \leqq C_j\ ,$$

where C_j is a constant depending only on j. Hence $|a_n| \leqq C_j\|\exp(inj\hat{\mu})\|_{M_p}^{-1}$.

On the other hand, by Theorem 1 we get $\|\exp(inj\hat{\mu})\|_{M_p}^{-1} \leqq 6\exp(-nj\log K_p)$ (cf. Remark 1) and $|a_n| \leqq 6C_j \exp(-nj\log K_p)$. When n is negative integer, we analogously get $|a_n| \leqq 6C_{-j}\exp(-|n|j\log K_p)$. Therefore by Steps 1 and 2, we get the desired result. q.e.d.

REMARK (Added on December 25, 1984). After submitting this paper we have been informed, by Professor S. Igari that Lamberton [16] independently proved our Theorem 1 when G is the unit circle group.

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