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ON A PROBLEM OF DOOB ABOUT ANGULAR AND FINE CLUSTER VALUES

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Abstract. In 1965, Joseph L. Doob proved that if f is a superharmonic function on a half-space and if $A(p)$ and $F(p)$ denote respectively the angular and fine cluster sets of f at a boundary point p, then one of the following cases holds at almost every boundary point p.

(i) $-\infty \in F(p) \cap A(p)$ and $F(p) \subsetneq A(p)$.

(ii) f does not have an angular limit at p but has a finite fine limit and an equal normal limit there.

(iii) f has a finite angular limit and an equal fine limit at p.

He then asked whether in (i) the set $F(p)$ can be a proper subinterval of $A(p)$ on a P set of positive measure.

In this note, we study this problem in the two dimensional case. We construct a Nevanlinna's function for which (i) holds for a countably dense set of boundary points. Our result is sharp in the sense that the P set cannot be improved to be of positive measure. It is not clear whether the construction is possible for any P set of measure zero.

1. Introduction. Let H be the right half-plane and let $f(z)$ be a function defined in H. We say that the function f has an angular cluster value v at a boundary point p, if there is a Stolz angle $\Delta(p)$ (i.e., an angle lying in H with one vertex at p) and a sequence $\{p_n\}$ of points in $\Delta(p)$ such that

$$\lim_{n\to\infty} p_n = p \quad \text{and} \quad \lim_{n\to\infty} f(p_n) = v \ .$$

We shall now follow Brelot [2, p. 327] to introduce the notion of thin set in the sense of Cartan and Brelot. A set E will be said to be ordinarily thin at a point p, if either p is not a limit point of E or there exists a superharmonic function $S(z)$ such that

$$S(p) < \varliminf_{z\to p} S(z) \ , \qquad \text{where} \quad z \in E - p \ .$$

The first case is trivial and therefore only the second case will be considered in the sequel.

In contrast to the ordinary thinness, we shall now introduce the

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minimal thinness in the sense of Ferrand and Naïm, see [16], [17]. Clearly, the Martin boundary of H is simply the set $\partial H \cup \{\infty\} = \partial \bar{H}$. If $p \in \partial \bar{H}$, we denote by $K_p(z)$ a minimal harmonic function on H with pole at p. Following Brelot [3, p. 36], we define the reduced function of K_p relative to a set $E \subset H$ by

$$R^E_{K_p}(z) = \inf\{f \in S^+ \colon f(z) \geqq K_p(z) \quad \text{on} \quad E\} ,$$

where S^+ is the set of all non-negative superharmonic functions on H. We say that E is minimally thin at p if it satisfies

$$K_p(z) \not\equiv R^E_{K_p}(z) \qquad \text{for} \quad z \in H .$$

Notice that both of the above two functions are conformally invariant hence so is the minimal thinness. Thus, in the sequel, we shall work in the unit disk instead of a half plane. Also notice that if a set is ordinarily thin at a boundary point of H, then it is minimally thin there due to a theorem of Jackson [13, Theorem 4]. From this, we can see what we really need in the sequel is ordinary thinness.

With the notion of thinness, we can now follow Doob [8, p. 113] to define the fine cluster value. For this, we let $D(|z| < 1)$ and $C(|z| = 1)$ be the unit disk and circle, respectively. We say that a function $f(z)$ has a fine cluster value v at a point $p \in C$, if there is a set $E \subset D$ which is not minimally thin at p and

$$\lim_{z \to p} f(z) = v , \qquad \text{where} \quad z \in E .$$

In this case, the point p is called a fine limit point of the set E.

It remains to introduce the notion of Nevanlinna's class N. A function $f \in N$, if it can be represented as

$$f(z) = g(z)/h(z) ,$$

where g and h are bounded holomorphic on D. We denote by N^+ the subclass of N containing those functions holomorphic on D.

With the help of the above definitions, we are now able to state our main result as follows.

THEOREM 1. *Let $\{p_n\}$ be a countably dense subset of C. Then there is a function $f \in N^+$ such that at each point p_n the fine cluster set $F(p_n) = \infty$ while the angular cluster set $A(p_n)$ is the whole extended plane.*

In view of Doob's problem [8, p. 123], we may ask whether there exists a superharmonic function satisfying condition (i). Theorem 1 answers this question in the affirmative due to the fact that the function $-|f(z)|$ is superharmonic and satisfies (i).

Notice that Theorem 1 is sharp in the sense that it cannot hold on a subset of C of positive Lebesgue measure. In fact, any function $f \in N^+$ must have angular limits almost everywhere on C, see [5, Theorem 2.18]. More generally, we have the following easy consequence of Doob's theorem [7, Theorem 7.3].

THEOREM 2. *Let f be a superharmonic function in D and let S be the subset of C such that the fine cluster set $F(p) = v$ for each $p \in S$. If the Lebesgue measure $|S| > 0$, then $f = v$ identically.*

It seems to us that the following result should be true: There is no holomorphic function f on D such that

$$\infty \in F(p) \cap A(p) \quad \text{and} \quad F(p) \subsetneqq A(p)$$

holds on a subset of C of positive measure. In other words, the answer to the aforementioned problem of Doob should be negative.

On the other hand, there does exist a meromorphic function on D such that $F(p) \subsetneqq A(p)$ holds at every point p on C, and in fact $F(p)$ is a singleton while $A(p)$ is the extended plane, see [12].

2. **Wiener criterion.** According to a theorem of Brelot [2, p. 327], we know that the notion of thinness is equivalent to that of irregularity. It follows from the Wiener criterion [18] that a set E is thin at a point $p \in \bar{E}$ if and only if

$$\sum_{n=1}^{\infty} n W(E_n) < \infty , \tag{1}$$

where $W(E_n)$ is the Wiener capacity of the set

$$E_n = E \cap \{z: e^{-(n+1)} \leqq |z - p| < e^{-n}\} .$$

We shall now introduce the metric property of $W(E)$. To see this, we first observe that the relation between the Wiener capacity $W(E)$ and the logarithmic capacity $L(E)$ of a set E is the following, see [2, p. 321],

$$W(E) = 1/(\log(1/L(E))) . \tag{2}$$

Moreover, if E is a disk of diameter $|E|$, then we have

$$L(E) = |E|/2 . \tag{3}$$

The function W has the following subadditive property

$$W\Big(\bigcup_n E_n\Big) \leqq \sum_n W(E_n) . \tag{4}$$

3. A set thin on C. In order to prove Theorem 1, we shall first construct a sequence $\{z_n\}$ of points in D and a sequence $\{D_n\}$ of disks in D with centers at z_n such that the union $E = \cup D_n$ is thin at every point on C. For this, we let $\{p_n = e^{i\theta_n}\}$ be a countably dense subset of C and let $a_j = 1 - e^{-j}$, $j = 1, 2, \cdots$ We define

$$(5)\qquad \begin{cases} z_1 = a_1 e^{i\theta_1}\,, \quad z_2 = a_2 e^{i\theta_2}\,, \\ z_3 = a_3 e^{i\theta_1}\,, \quad z_4 = a_4 e^{i\theta_2}\,, \quad z_5 = a_5 e^{i\theta_3}\,, \\ \cdots\cdots \\ z_n = a_n e^{i\theta_1}\,, \quad z_{n+1} = a_{n+1} e^{i\theta_2}, \cdots, z_t = a_t e^{i\theta_{k+1}}\,, \end{cases}$$

where $n = k(k+1)/2$, $t + 1 = (k+1)(k+2)/2$ and k denotes the k-th row in the above triangle array.

Notice that $|z_n| < |z_{n+1}|$ for $n = 1, 2, \cdots$, and for convenience $\{z_n\}$ will be called a monotone sequence.

LEMMA 1. *Let $\{z_n\}$ be a monotone sequence defined by* (5) *and let D_n be the disk with center at z_n and radius e^{-n^b}, where $b > 3$ is fixed and $n = 1, 2, \cdots$. Then the union $E = \cup D_n$ is ordinarily thin at every point on C and therefore it is minimally thin there.*

PROOF. Let p be a point on C and let E_n be the set defined by (1). Then by (4) and (5) we have

$$(6)\qquad W(E_k) \leqq W\Big(\bigcup_{n=k}^{\infty} D_n\Big) \leqq \sum_{n=k}^{\infty} W(D_n) \qquad \text{for} \quad k = 1, 2, \cdots .$$

Combining (2), (3), (6) and the hypothesis, we obtain

$$W(E_k) \leqq \sum_{n=k}^{\infty} n^{-b} \leqq k^{-(b-1)} \qquad \text{for} \quad k = 1, 2, \cdots ,$$

and therefore (1) becomes

$$\sum_{k=1}^{\infty} k\, W(E_k) \leqq \sum_{k=1}^{\infty} k^{-(b-2)} < \infty .$$

This proves the lemma.

The above lemma intuitively says that the configuration of an ordinarily thin set is "small" as viewed from its limit point on C. The same geometric meaning is not true for minimally thin set and in fact, there can be a "large" set when it tends to a boundary point tangentially. More precisely, we shall state and prove the following corollary of Brelot and Doob's theorem [4, Theorem 2].

LEMMA 2. *Let H be the right half plane and let $T(\alpha)$ be the sub-*

domain of H bounded by the imaginary axis and the curve defined by the equation

$$x = y^{1+\alpha}, \qquad \text{for some} \quad \alpha > 0 .$$

Then the union of $T(\alpha)$ and its conjugate $\bar{T}(\alpha)$ is minimally thin at the origin.

PROOF. The result follows immediately from the aforementioned theorem of Brelot and Doob, due to the fact that

$$\int_0^\varepsilon \frac{x}{y^2} dy < \infty , \qquad \text{where} \quad \varepsilon > 0 .$$

Notice that what we really need is a thin set in a disk instead of a half plane. For this, we shall now estimate the growth of a minimal harmonic function whose pole is located at a boundary point, say, $p = 1$.

LEMMA 3. *Let $T(\alpha)$ be the domain defined in Lemma 2, where $0 < \alpha < 1$, and let $T^*(\alpha)$ be the image of $T(\alpha)$ under the mapping $w = (1 - z)/(1 + z)$. Then the growth of the following minimal harmonic function*

$$h(w) = (1 - |w|^2)/|1 - w|^2 , \quad \text{for} \quad w \in D ,$$

tends to infinity with the following order

$$h(w) = O((1 - |w|^2)^{-(1-\alpha)/(1+\alpha)}) ,$$

where $w \to 1$ and $w \in D - T^*(\alpha)$.

PROOF. In view of Lemma 2, we may represent the curve on the boundary of $T(\alpha)$ by

$$z = x + iy , \qquad \text{where} \quad x = y^{1+\alpha} , \qquad \alpha > 0 .$$

Then by a simple computation, we have

$$\begin{aligned} 1 - |w|^2 &= 4d^{-1}[(1 + y^{1+\alpha})(y^{1+\alpha} + y^{2+2\alpha} + y^2) - y^2] , \\ \text{and} \quad |1 - w|^2 &= 4d^{-1}[(y^{1+\alpha} + y^{2+2\alpha} + y^2)^2 + y^2] , \end{aligned}$$

where $d = [(1 + y^{1+\alpha})^2 + y^2]^2$.

Since $0 < \alpha < 1$ and $d \to 1$ as $y \to 0$, we obtain

$$\lim_{y \to 0} \frac{(1 - |w|^2)^{2/(1+\alpha)}}{|1 - w|^2} = 4^{(1-\alpha)/(1+\alpha)} .$$

This concludes the result.

4. Blaschke product. As before, let $\{p_n = e^{i\theta_n}\}$ be a countably dense subset of C and let $\{z_n\}$ be the monotone sequence defined by (5). Then the series

$$\sum_{n=1}^{\infty}(1-|z_n|)=\sum_{n=1}^{\infty}e^{-n}<\infty\ .$$

This defines the following Blaschke product, see [5, p. 28]

$$B(z, z_n)=\prod_{n=1}^{\infty}\frac{\bar{z}_n}{z_n}\frac{z-z_n}{1-\bar{z}_n z}\ . \tag{7}$$

In order to estimate the above Blaschke product, we shall need the following well-known inequalities of Harnack.

LEMMA 4. *If $0\leqq a\leqq b\leqq c\leqq d<1$ and $z, w\in D$, then*

$$\frac{c-b}{1-bc}\leqq\frac{d-b}{1-bd}\leqq\frac{d-a}{1-ad}\ , \tag{8}$$

$$\left|\frac{|z|-|w|}{1-|zw|}\right|\leqq\left|\frac{z-w}{1-\bar{z}w}\right|\leqq\frac{|z|+|w|}{1+|wz|}\ . \tag{9}$$

With the help of Lemma 4 and a technique of Bagemihl and Seidel [1], we are now able to prove the following:

LEMMA 5. *Under the hypothesis of Lemma 1, if $B(z, z_n)$ is the Blaschke product defined by (7), then for each $w\notin E$ and $|z_m|\leqq|w|\leqq|z_{m+1}|$ we have*

$$|B(w, z_n)|\geqq e^{-2(m+1)^b}\ ,\qquad\text{where}\quad b>3\ .$$

PROOF. According to (5), we can see that

$$|z_n|=a_n=1-e^{-n}\ ,\qquad n=1, 2, \cdots \tag{10}$$

For convenience, we shall separate the Blaschke product $B(z, z_n)$ into the following five subproducts:

$$B(z, z_n)=B_1(z)F_m(z)F_{m+1}(z)F_{m+2}(z)B_2(z)\ , \tag{11}$$

where $F_n(z)=(z-z_n)/(1-\bar{z}_n z)$, $n=m, m+1, m+2$,

$$B_1(z)=\prod_{n=1}^{m-1}F_n(z)\quad\text{and}\quad B_2(z)=\prod_{n=m+3}^{\infty}F_n(z)\ .$$

Since $|w|>|z_{m-1}|\geqq|z_n|$, for $n=1, 2, \cdots, m-1$, it follows from (8) that for $n\leqq m-1$,

$$\frac{|z_m|-|z_{m-1}|}{1-|z_m z_{m-1}|}\leqq\frac{|w|-|z_{m-1}|}{1-|wz_{m-1}|}\leqq\frac{|w|-|z_n|}{1-|wz_n|}\ .$$

This together with (9) and (10) yields

$$|B_1(w)|\geqq\prod_{n=1}^{m-1}\frac{|w|-|z_n|}{1-|wz_n|}\geqq\left(\frac{|z_m|-|z_{m-1}|}{1-|z_m z_{m-1}|}\right)^{m-1}\geqq 4^{-(m-1)}\ . \tag{12}$$

Next, we define disks by

$$D(z_m) = \{z\colon |z - z_m| < e^{-m^b}\} .$$

Then for $w \notin D(z_m)$ we have

$$(13) \qquad |F_m(w)| = \frac{|w - z_m|}{1 - |wz_m|} \geqq |w - z_m| \geqq e^{-m^b} .$$

Similarly, for $w \notin D(z_{m+1})$ we have

$$(14) \qquad |F_{m+1}(w)| \geqq e^{-(m+1)^b} .$$

Turning to the fourth product, we get

$$(15) \qquad |F_{m+2}(w)| \geqq \frac{|z_{m+2}| - |w|}{1 - |wz_{m+2}|} > \frac{1}{4} .$$

Finally, we observe that for $n > m + 2$,

$$|F_n(w)| \geqq 1 - e^{-(n-m-2)} .$$

It follows that

$$(16) \qquad |B_2(w)| \geqq \exp\left(-2\sum_{n=1}^{\infty} e^{-n}\right) > e^{-2} .$$

Combining (11), (12), (13), (14), (15), and (16), we obtain for $w \notin E$ and $|z_m| \leqq |w| \leqq |z_{m+1}|$

$$|B(w, z_n)| > 4^{-m} e^{-[2+m^b+(m+1)^b]} > e^{-2(m+1)^b} .$$

This completes the proof.

5. Proof of Theorem 1. The method here is somewhat like that of [10 (Th. 10)] or [11]. Let $\{p_n = e^{i\theta_n}\}$ be a countably dense subset of C, $\{z_n\}$ the monotone sequence associated with $\{p_n\}$, $\{D_n\}$ the sequence of disks with center at z_n and radius e^{-n^b}, $b > 3$, and $B(z, z_n)$ the Blaschke product whose zeros are precisely the sequence $\{z_n\}$. We then define the function

$$(17) \qquad f(z) = B(z, z_n)S(z, p_n) ,$$

where

$$S(z, p_n) = \exp\left[\sum_{n=1}^{\infty} (p_n + z) \Big/ ((p_n - z)n^2)\right] .$$

Since the inverse S^{-1} is an inner function, the function f is a bounded characteristic of Nevanlinna, see [5, p. 40]. For convenience, we denote

$$S_n(z) = \exp\left[(p_n + z)/((p_n - z)n^2)\right] , \qquad n = 1, 2, \cdots .$$

Then clearly we have

(18) $$|S_n(z)| > 1 \quad \text{for each} \quad z \in D \quad \text{and} \quad n = 1, 2, \cdots .$$

We shall now prove that the function f defined by (17) has the fine limit ∞ at every point p_k, $k = 1, 2, \cdots$. For this, we let

$$g(z) = B(z, z_n)S(z, p_k) , \quad \text{where} \quad k \quad \text{is fixed} .$$

Then by (17) and (18) we find that

$$|f(z)| \geqq |g(z)| , \qquad \text{for each} \quad z \in D .$$

It follows that if g has the fine limit ∞ at p_k, then so does f.

Moreover, by a rotation, we may, without loss of generality, assume that p_k is located at 1. This gives

(19) $$|g(z)| = |B(z, z_n)| \exp [(1 - |z|^2)/(|1 - z|^2 k^2)] .$$

Now, let $T^*(\alpha)$ be the domain defined by Lemma 3. Then by Lemma 2 and the conformal invariance of minimal thinness, we know that the set $T^*(\alpha)$ is minimally thin at the point 1. Moreover, from Lemma 1 we also know that the set $E = \cup D_n$ is also minimally thin at 1, hence so does the union $E \cup T^*(\alpha)$. Thus it is sufficient to prove that

(20) $$\lim_{z \to 1} |g(z)| = \infty , \qquad \text{where} \quad z \in D - E \cup T^*(\alpha) .$$

According to Lemma 3, there is a positive constant c such that

(21) $$\frac{1 - |z|^2}{|1 - z|^2} \geqq c(1 - |z|)^{-(1-\alpha)/(1+\alpha)} , \qquad 0 < \alpha < 1 .$$

Since the point z satisfies

$$|z_m| \leqq |z| \leqq |z_{m+1}| \qquad \text{for some} \quad m ,$$

where

$$|z_n| = 1 - e^{-n} , \qquad n = 1, 2, \cdots$$

it follows from (21) that

(22) $$\exp [(1 - |z|^2)/(|1 - z|^2 k^2)] \geqq \exp [ck^{-2} e^{m(1-\alpha)/(1+\alpha)}]$$

where $z \in D - E \cup T^*(\alpha)$ and $|z_m| \leqq |z| \leqq |z_{m+1}|$.

On the other hand, by virtue of Lemma 5, we have

(23) $$|B(z, z_n)| \geqq \exp [-2(m + 1)^b] , \qquad b > 3 ,$$

where z satisfies the same restriction of (22). Since $m \to \infty$ as $z \to 1$, the conclusion (20) follows from (19), (22), and (23). This establishes that the function f has fine limit ∞ at every point p_n.

Finally, we shall prove that the angular cluster set $A(p_n)$ of f at the point p_n is the extended plane. Suppose on the contrary that there is a value $v \notin A(p_n)$. Then there is a neighborhood $N(v)$ of v and a Stolz angle $\Delta(p_n)$ with one vertex at p_n such that range of f over $\Delta(p_n)$ is disjoint from $N(v)$. Let $z = z(w)$ be a conformal mapping from D_w onto $\Delta(p_n)$. Then the composite function $f(z(w)) = F(w)$ is normal in D_w in the sense of Lehto and Virtanen [15, p. 53]. Since f has fine limit ∞ at every p_n, we can see that there is a Jordan arc J ending at p_n along which the function $f(z)$ tends to infinity. This implies that the function $F(w)$ tends to infinity along the image arc $W(J)$, ending at the point $W(p_n)$, where $w = w(z)$ is the inverse function of $z = z(w)$. It follows from [1, Theorem 2] that the function $F(w)$ as well as the function $f(z)$ has angular limit ∞. This, however, contradicts the fact that the radius $\overline{Op_n}$ contains infinitely many zeros of f due to (5). This completes the proof.

In view of (21), we obtain immediately the following:

COROLLARY 1. *Let f_n be a function on D defined by*

$$f_n(z) = (1 - z)^n \exp[(1 + z)/(1 - z)] .$$

Then f_n has fine limit ∞ at 1.

Notice that the inverse f_n^{-1} has fine limit 0 at 1. This yields a result of Doob [7, p. 531] when $n = 1$. Clearly, the function f_n tends to 0 along each circular path to 1. Of course, the union of all those paths is minimally thin at 1.

Following Collingwood and Piranian [6], we shall call a point $p \in C$ a Julia point of f, if the range $f(\Delta(p))$ covers the whole extended plane except at most two points, where $\Delta(p)$ is an arbitrary Stolz angle in D with one vertex at p. Then by the same argument as in Theorem 1 and the method in [10, Theorem 10], we obtain the following:

COROLLARY 2. *Let $\{p_n\}$ be a countably dense subset of C. Then there is a function $f \in N^+$ such that $F(p_n) = \infty$ and each p_n is a Julia point of f.*

6. Angular and relatively angular limits. As before, let H be the right half-plane, p a point on the boundary ∂H of H, $\Delta(p)$ a Stolz angle in H with one vertex at p, and f a function defined in H. If $f(z)$ tends to a value v as $z \to p$ and $z \in \Delta(p)$, then we say that f has angular limit v relative to $\Delta(p)$. If $f(z)$ tends to the same value v as $z \to p$, $z \in \Delta$ for each Stolz angle Δ, then v will be called the angular limit of f at p. It was proved by Brelot and Doob [4, p. 410] that for almost every point $p \in \partial H$, where f has a relatively angular limit v, the function f has the angular limit v. We shall now apply this theorem to prove the following distributive property of sequences of points in D.

THEOREM 3. *Let $Z=\{z_n=r_n e^{i\theta_n}\}$ be a sequence of points in D for which the set $\{e^{i\theta_n}\}$ is dense on C. Let P be the subset of C such that for each $p\in P$, there is a wide Stolz angle $\Delta(p)\subset D$ containing infinitely many points of Z, and Q the subset of C such that for each $q\in Q$, there is a narrow Stolz angle $\delta(q)\subset D$ containing no points of Z. Then the Lebesgue measure $|P\cap Q|=0$.*

PROOF. According to the aforementioned theorem of Brelot and Doob, it is sufficient to construct a function f such that at each point $p\in P\cap Q$ the function f has a relatively angular limit but no angular limit at p. For this, we first divide D into the following disjoint rings

$$D=\bigcup_{n=0}^{\infty} R_n,\ R_n=\{z\colon 1-2^{-n}\leqq |z|<1-2^{-n-1}\}\ .$$

Then we write

$$Z_n=Z\cap R_n=\{z_{nk}\}\ ,\qquad k=1,2,\cdots,k(n)\ .$$

Inductively, we can choose a sequence $\{a_n\}$ of sufficiently small numbers such that $a_{n+1}<a_n/2$ and each disk D_{nk} of radius a_n with center at a point $z_{nk}\in Z_n$ is contained in D and disjoint from each other. We then define the following meromorphic function

$$f(z)=\sum_{n=0}^{\infty}\sum_{k=1}^{k(n)}\frac{a_n}{(z-z_{nk})2^{n+k}}\ . \tag{24}$$

We now let

$$T=\bigcup_{n,k} D_{nk}\ ,\quad \text{where}\quad D_{nk}\subset D\quad \text{for each}\quad n\quad \text{and}\quad k\ .$$

Clearly, for $z\in T^c$, the complement of T, we have

$$a_n/|z-z_{nk}|\leqq 1\ .$$

This implies that the series defined by (24) converges for all $z\in T^c$, and therefore the function f is continuous relative to the set T^c.

For each point $p\in P\cap Q$, there is a narrow Stolz angle $\delta(p)\subset D$ containing no points of Z. Moreover, from the condition $a_{n+1}<a_n/2$, it is easy to see that there is a subangle $\delta^*(p)\subset\delta(p)$ such that $\delta^*(p)\subset T^c$. This in turn implies that the function f has an angular limit relative to the angle $\delta^*(p)$. Since $p\in P$, the function f has no angular limit at p. This concludes that the measure of $P\cap Q$ is zero due to the aforementioned theorem of Brelot and Doob.

Notice that by choosing a_n to be sufficiently small the set T will be thin at every point on C and therefore the function f defined by (24) will have fine limit at every point on C, see [12].

Also notice that the above Theorem 3 is sharp as will be seen from the following two examples, where the sets P and Q are defined in Theorem 3.

EXAMPLE 1. If $\{z_n\}$ is a sequence of points in D satisfying $\sum(1-|z_n|)<\infty$, then the measures $|P|=0$ and $|Q|=2\pi$, so that $|P\cap Q|=0$.

To see this, we need only apply the Blaschke product $B(z, z_n)$ defined by (7). Clearly this product has angular limits at every point of a subset $Q^*\subset C$ whose measure $|Q^*|=2\pi$. If $p\in Q^*\cap P$, then the product B must have angular limit 0 at p. It follows from a uniqueness theorem [5, Theorem 2.5] that the measure $|Q^*\cap P|=0$. Since the set

$$Q^*-Q^*\cap P\subset Q\,, \qquad \text{we must have} \quad |Q|=2\pi\,.$$

EXAMPLE 2. If $\{z_n\}$ is a rearrangement of the following sequence

$$z_{nk}=(1-2^{-n})e^{i2\pi k/2^n}\,, \qquad k=0,1,\cdots,2^n-1\,,$$

then the measures $|P|=2\pi$ and $|Q|=0$ so that $|P\cap Q|=0$. Moreover, we have

$$\cup\{e^{i2\pi k/2^n}\}\subset Q\,.$$

To see this, let $p\in C$ and let $\Delta(p)$ be a Stolz angle symmetric at p whose subtended angle is of 2θ. If $\tan\theta>2\pi$, then it is easy to see that $\Delta(p)$ contains a point z_{nk}, for each $n=N, N+1,\cdots$, where N is sufficiently large and k depends on n and p. This yields $|P|=2\pi$ and $|Q|=0$.

To prove the second assertion, we may just consider the point at 1. Let s be the segment between 1 and z_{n1} and let θ_n be the angle between s and the segment [0, 1]. Then by a simple computation we find that $\tan\theta_n\to 2\pi$ as $n\to\infty$. This concludes the result.

From the above two examples, we can see that for any two positive numbers α and β with $\alpha+\beta=2\pi$, there is a sequence $\{z_n\}$ of points in D such that $|P|=\alpha$ and $|Q|=\beta$.

7. Angular and fine cluster values. In this section, we shall study a topological property between angular and fine cluster values at a boundary point.

THEOREM 4. *Let $A(p)$ and $F(p)$ be the angular and fine cluster values of a function f at $p\in C$. Then $F(p)$ is closed in $A(p)$.*

PROOF. First, we shall prove a necessary and sufficient condition for a value $v \in F(p)$. For this, we consider the lemniscatic domain of f related to v defined by

$$E(f, v, \varepsilon) = \{z: z \in D \quad \text{and} \quad |f(z) - v| < \varepsilon\} .$$

We shall show that a value $v \in F(p)$ if and only if for each $\varepsilon > 0$ the set $E(f, v, \varepsilon)$ is not minimally thin at p.

Suppose now the set $E(f, v, \varepsilon)$ is minimally thin at p for some $\varepsilon_0 > 0$. Then clearly $E(f, v, \varepsilon)$ is also minimally thin at p for any $\varepsilon < \varepsilon_0$. This implies that any subset $E \subset D$ satisfying

$$\lim_{z \to p} f(z) = v , \qquad \text{where} \quad z \in E ,$$

must be minimally thin at p, so that $v \notin F(p)$.

Conversely, if the set $E(f, v, \varepsilon)$ is not minimally thin at p for each $\varepsilon > 0$, then by the Wiener criterion, see Brelot and Doob [4, p. 399], we have

$$\sum_{n=1}^{\infty} \gamma_n(\varepsilon) e^{-2n} = \infty , \qquad \gamma_n(\varepsilon) = R_{Ga}^{E_n(\varepsilon)}(b) , \tag{25}$$

where a and b are fixed, Ga is the Green function with pole at a, R is the reduced function and

$$E_n(\varepsilon) = E(f, v, \varepsilon) \cap \{z: e^{-n-1} \leqq |z - p| < e^{-n}\} .$$

Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to zero. Then by (25), inductively we can choose a sequence of disjoint compact subsets $E_{nk}(\varepsilon_n)$, $k = 1, 2, \cdots, k(n)$ tending to p and satisfying

$$\sum_{k=1}^{k(n)} \gamma_{nk}(\varepsilon_n) e^{-2n} \geqq 1 , \qquad \text{where} \quad k(n) \quad \text{depends on} \quad n .$$

Let $E = \bigcup_{n,k} E_{nk}$. Then clearly E satisfies (25) and $f(z)$ tends to v as $z \to p$, where $z \in E$. This yields that $v \in F(p)$.

We are now able to prove our theorem. Wet let $v \in A(p) - F(p)$. Then by what we have proved the lemniscatic domain $E(f, v, \varepsilon)$ is minimally thin at p for some ε. Observe that for any point w with $|w - v| < \varepsilon/2$ and any z with $|f(z) - w| < \varepsilon/2$, we have

$$|f(z) - v| < \varepsilon .$$

This implies that

$$E(f, w, \varepsilon/2) \subset E(f, v, \varepsilon)$$

and therefore the set $E(f, w, \varepsilon/2)$ is also minimally thin at p. We thus conclude that the set $F(p)$ is closed in $A(p)$.

Notice that for an arbitrary function f the associated fine cluster set $F(p)$ needs not be open in $A(p)$. In fact, if f is the function defined by Theorem 1, then the function $1/f \in N$ for which the sets $F(p_n) = \{0\}$ and $A(p_n)$ is the whole extended plane, where $\{p_n\}$ is a countably dense subset of C. However, for some functions we do have the openness of $F(p)$ in $A(p)$.

THEOREM 5. *If f is a normal function in D, then the fine cluster set $F(p)$ is both closed and open in $A(p)$ for each $p \in C$.*

PROOF. The first assertion has been proved by Theorem 4. To prove the second, we consider a point v in $F(p)$. If $v \notin A(p)$, we are done. On the other hand, if $v \in A(p)$, then by a theorem of Doob [7, Theorem 4.1] we can see that every value in $A(p)$ is also a value in $F(p)$. Thus the set $A(p)$ is a neighborhood of v in $A(p)$ so that $F(p)$ is open in $A(p)$.

Notice that by use of another theorem of Doob [7, Theorem 4.3], we obtain immediately the following result which is neither a corollary of Theorem 5 nor implying Theorem 5.

THEOREM 6. *If f is a normal function in D then $F(p) = A(p)$ for almost every point $p \in C$.*

8. **Problems.** First, in Theorem 1, we have only proved our result for a countably dense subset of boundary points. In contrast to the results of Lohwater and Piranian [14, Theorem 1], we may ask whether Theorem 1 is still true if the countable set is replaced by a set of measure zero of type F_σ and of first category.

Second, in Theorem 3 we may ask a necessary and sufficient condition of the sets P and Q for which Theorem 3 holds, where the measure $|P \cap Q| = 0$.

Third, in Theorem 5, if f is normal in D, is it true that the set $F(p)$ is both closed and open in the cluster set $C(p)$ of f at p?

Finally, if f is holomorphic in D, is it true that the set $F(p)$ is open in $A(p)$ for almost every point on C? Of course, this is false if f is meromorphic in D, see [12]. If this would be the case, then our conjecture made in the "Introduction" would be true. In the following, we shall explain the reason why we expect this to be true.

Notice that the lemniscatic domain can be either connected or disconnected in an arbitrary small neighborhood of p. The first case corresponds to an asymptotic value $v \in F(p)$ proved the asymptotic path is not minimally thin at p. On the other hand, if f is holomorphic in D such that f

has no asymptotic values at p, then the set $F(p)$ should be open in $A(p)$. More precisely, we shall sketch the proof of the following known result by a theorem of Erdös and Hwang [9].

THEOREM 7. *If $E(p_n, 0, \varepsilon^n)$ is the lemniscatic domain of a polynomial*

$$P_n(z) = \sum_{k=0}^{n} a_k z^k , \qquad a_n \neq 0 ,$$

then the logarithmic capacity Cap $E(p_n, 0, \varepsilon^n) = |a_n|\varepsilon$.

PROOF. Let $q_n(z) = p_n(z)/a_n$ and let $E(q_n, 0, \varepsilon^n)$ be the lemniscatic domain of q_n. If the theorem were false, then there would be a polynomial $r_n(z)$ of degree n with one as the coefficient of z^n such that the corresponding lemniscatic domain $E(r_n, 0, \varepsilon^n) \subset E(q_n, 0, \varepsilon^n)$. This however contradicts our aforementioned theorem.

Now, if a function f is holomorphic in D, then it can be expanded as

$$f(z) = \lim_{n\to\infty} p_n(z) , \qquad \text{where} \quad p_n(z) = \sum_{n=0}^{n} a_k z^k .$$

According to Theorem 7, we have

$$\text{Cap}\, E(p_n, v, \varepsilon) = \text{Cap}\, E(p_n, w, \varepsilon) = |a_n|\varepsilon^{1/n} .$$

This shows that the logarithmic capacity is invariant from the value v to w. From this, it should be true that if f is holomorphic in D for which $v \in F(p)$ and v is not an asymptotic value of f at p, then the nonthinness of a set $E(f, v, \varepsilon)$ should be transited to a set $E(f, w, \varepsilon)$ for any w in a sufficiently small neighborhood of v. In other words, if $v \in F(p)$ and v is not an asymptotic value, then v should be an interior point of $F(p)$. If this would be the case, then by the "Ambiguous Theorem" of Bagemihl, see [5, Theorem 4.12], we can see that the conjecture we made before would not be ambiguous.

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