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COMPLEX ANALYTIC PROPERTIES OF TUBES OVER LOCALLY HOMOGENEOUS HYPERBOLIC AFFINE MANIFOLDS

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Introduction. Let M be an affine manifold of dimension n, that is, a manifold which admits an atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is an affine transformation of $\boldsymbol{R}^n$ whenever $U_\alpha \cap U_\beta \neq \varnothing$. Then the tangent bundle T_M over M naturally admits a complex structure. Indeed, let $\{x_\alpha^1, \cdots, x_\alpha^n\}$ be the local coordinate system defined by the chart $(U_\alpha, \varphi_\alpha)$ and put $z_\alpha^i = x_\alpha^i \circ p + \sqrt{-1}dx_\alpha^i$ $(i = 1, \cdots, n)$, where p denotes the natural projection of T_M onto M. Then $\{z_\alpha^1, \cdots, z_\alpha^n\}$ is a complex local coordinate system on $p^{-1}(U_\alpha)$ and the atlas $\{(p^{-1}(U_\alpha), \{z_\alpha^1, \cdots, z_\alpha^n\})\}$ defines a complex affine structure on T_M. When M is a domain in $\boldsymbol{R}^n$, the complex manifold T_M is a usual tube domain, that is, $T_M = M + \sqrt{-1}\boldsymbol{R}^n$. In the general case, we obtain T_M by pasting tube domains together by "real" affine transformations. The complex manifold T_M will be simply called a tube over M.

When M is a domain in $\boldsymbol{R}^n$, it is well-known (e.g., Bochner-Martin [2]) that

(∗) $\quad T_M$ is a Stein manifold if and only if M is convex.

In this note, we ask whether the "if" part of (∗) remains valid for a general affine manifold M and give a partial affirmative answer to this problem.

REMARK. An affine manifold M is called convex if every pair of points of M can be joined by a geodesic with respect to the locally flat linear connection on M corresponding to the affine structure on M. It is known that an affine manifold M is convex if and only if the universal covering of M is affinely equivalent to a convex domain in $\boldsymbol{R}^n$.

Before stating our result, we fix notations and conventions which are adopted throughout this note. We denote by $\boldsymbol{R}_+$ the set of positive real numbers. For a domain Ω in $\boldsymbol{R}^n$, $G(\Omega)$ denote the group of all affine transformations of $\boldsymbol{R}^n$ leaving Ω invariant. $G(\Omega)$ acts on T_Ω as a holomorphic transformation group by the rule

$$az = f(a)z + q(a) \quad \text{for} \quad a \in G(\Omega)\ , \quad z \in T_\Omega = \Omega + \sqrt{-1}\boldsymbol{R}^n\ ,$$

where $f(a)$ and $q(a)$ denote, respectively, the linear and translation parts of the affine transformation a. This action of $a \in G(\Omega)$ on T_Ω coincides with the action of the differential of a on the tangent bundle T_Ω. A domain Ω in $\boldsymbol{R}^n$ is called homogeneous if $G(\Omega)$ acts transitively on Ω. For an affine manifold M, the natural projection of T_M onto M is denoted by p.

The purpose of this note is to prove the following:

THEOREM. *Let M be an affine manifold whose universal covering is affinely equivalent to a convex domain Ω in $\boldsymbol{R}^n$. Suppose Ω contains no complete straight lines. Then there exists a smooth strictly plurisubharmonic function ψ_M defined on an open subset of the tube T_M over M whose complement S_M in T_M is either an analytic hypersurface of T_M or an empty set. If moreover M is compact, then ψ_M is an exhaustion function.*

In the above theorem, S_M is given as the support of a divisor on T_M. When M is locally homogeneous, that is, Ω is homogeneous, it can be shown that S_M is an empty set. Hence we obtain the following:

COROLLARY. *Under the same assumption as in our theorem, suppose further that Ω is homogeneous. Then T_M contains no positive-dimensional compact analytic subsets. If moreover M is compact, then T_M is a Stein manifold.*

REMARK 1. Let Ω be a convex domain in $\boldsymbol{R}^n$ containing no complete straight lines. In connection with the assumption of the theorem and its corollary, it should be noted that, when Ω admits a discrete subgroup Γ of $G(\Omega)$ acting properly discontinuously and freely on Ω with $\Gamma\backslash\Omega$ compact, Ω is necessarily affinely equivalent to a convex cone (Vey [13]). If moreover Ω is homogeneous, then it is self-dual with respect to a suitable inner product on $\boldsymbol{R}^n$ (Koszul [6]). We note also that the tube over a self-dual homogeneous cone is a symmetric domain (Rothaus [7]).

REMARK 2. Let M be a Hessian manifold in the sense of Shima [9]. Then the tube T_M over M naturally becomes a Kähler manifold (Shima [10], Cheng-Yau [3]). Matsushima posed a question, which is closely related to our problem: When is T_M a Stein manifold? We formulate this question as follows:

Let M be a complete Hessian manifold. Then is T_M a Stein manifold?

An affine manifold M is called hyperbolic if the universal covering of M is affinely equivalent to a convex domain in $\boldsymbol{R}^n$ containing no complete straight lines (cf. Koszul [5]). It can be shown that every hyperbolic affine manifold admits a canonical Hessian metric. Therefore our corollary shows that, for a compact Hessian manifold M, the answer to the above problem is affirmative, when M is hyperbolic and locally homogeneous. We note that every compact, or more generally quasi-compact, Hessian manifold is convex (Shima [11]).

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1. Preliminaries. By a convex cone V in $\boldsymbol{R}^n$, we mean a non-empty open set in $\boldsymbol{R}^n$ satisfying the following conditions:

(a) If $x \in V$ and $\lambda \in \boldsymbol{R}_+$, then $\lambda x \in V$.

(b) If $x, x' \in V$, then $x + x' \in V$.

(c) V contains no complete straight lines.

The group $G(V)$ then consists of all linear transformations of $\boldsymbol{R}^n$ leaving V invariant. Let $\langle \sharp, \sharp \rangle$ be an inner product on $\boldsymbol{R}^n$ and let V^* be the dual cone of V with respect to this inner product, that is,

$$V^* = \{u \in \boldsymbol{R}^n \mid \langle x, u \rangle > 0 \quad \text{for all} \quad x \in \bar{V} - \{0\}\} ,$$

where $\bar{V}$ denotes the closure of V in $\boldsymbol{R}^n$. We define a function Φ_V on T_V by

$$\Phi_V(z) = \int_{V^*} \exp(-\langle z, u \rangle) du \quad (z \in T_V) ,$$

where du denotes the Lebesgue measure on $\boldsymbol{R}^n$ and $\langle z, u \rangle = \langle x, u \rangle + \sqrt{-1}\langle y, u \rangle$ for $z = x + \sqrt{-1}y \in T_V = V + \sqrt{-1}\boldsymbol{R}^n$; the restriction of the function Φ_V to V, viewed as the zero-section of T_V, is denoted by ϕ_V. Note that Φ_V is determined up to positive constant multiple depending on the choice of the inner product $\langle \sharp, \sharp \rangle$ on $\boldsymbol{R}^n$. The function Φ_V coincides with a constant multiple of the so-called Cauchy kernel associated with the tube domain T_V (cf. Stein-Weiss [12], Mumford et al. [1]) and ϕ_V is called the characteristic function of the convex cone V (cf. Vinberg [14]). Since the integral $\int_{V^*} \exp(-\langle z, u \rangle) du$ converges absolutely and uniformly on any compact set in T_V, Φ_V is holomorphic on T_V and hence ϕ_V is real-analytic on V. The functions Φ_V and ϕ_V have the following properties:

(C1) $\Phi_V(az) = |\det a|^{-1} \Phi_V(z)$ for all $z \in T_V$, $a \in G(V)$.

(C2) $\Phi_V(x + \sqrt{-1}y)$ tends to 0 locally uniformly on $x \in V$ as $\|y\| = (\langle y, y \rangle)^{1/2}$ $(y \in \boldsymbol{R}^n)$ tends to ∞.

(C3) $\phi_V(ax) = |\det a|^{-1} \phi_V(x)$ for all $x \in V$, $a \in G(V)$.

(C4) $\phi_V > 0$ and $\log \phi_V$ is a convex function on V, that is, the Hessian $(\partial^2 \log \phi_V(x)/\partial x^i \partial x^j)$ of $\log \phi_V$ $(x = (x^1, \cdots, x^n))$ is positive-definite at every point of V.

(C5) $\phi_V(x)$ tends to ∞ as $x \in V$ approaches $\partial V = \bar{V} - V$.

(C1) is a consequence of the change of variable in the integral $\Phi_V(az)$. (C3) follows immediately from (C1). (C2) follows from the Riemann-Lebesgue theorem. The first assertion of (C4) is obvious by definition. For the second assertion of (C4) and (C5), see Vinberg [14].

The following lemma is essentially due to Rothaus [7].

LEMMA. *Let V be a convex cone in $\boldsymbol{R}^n$. If V is homogeneous, then the function Φ_V never vanishes on T_V.*

PROOF. For $s \in \boldsymbol{C}$ with $\operatorname{Re} s \geqq 1$, we define a function h_s on V by

$$h_s(x) = \phi_V(x)^{-s} \int_{V^*} \exp(-\langle x, u\rangle)\phi_{V^*}(u)^{1-s} du \quad (x \in V) .$$

It follows from (C3) that the function h_s is $G(V)$-invariant. Hence, as V is assumed homogeneous, h_s is a constant function on V, which we denote by $\Delta(s)$. Here $\Delta(s)$ is a holomorphic function of $s \in \boldsymbol{C}$ for $\operatorname{Re} s \geqq 1$ and called the Gamma function of V when V is self-dual. Once $\Delta(s)$ is defined, the rest of the proof follows from Rothaus [7, Theorem 2.3, p. 195].

2. **Proof of Theorem and Corollary.** Let Ω be a convex domain in $\boldsymbol{R}^n$ containing no complete straight lines. We define a convex cone $V(\Omega)$ in $\boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ by

$$V(\Omega) = \{(\lambda x, \lambda) \in \boldsymbol{R}^n \times \boldsymbol{R} \mid x \in \Omega, \lambda \in \boldsymbol{R}_+\} . \tag{1}$$

Then there exists a natural affine embedding ι of Ω into $V(\Omega)$ defined by

$$\iota\colon \Omega \ni x \mapsto (x, 1) \in V(\Omega) . \tag{2}$$

Let ρ be the group homomorphism of $A(n, \boldsymbol{R})$ into $GL(n+1, \boldsymbol{R})$ given by

$$A(n, \boldsymbol{R}) \ni a \mapsto \begin{pmatrix} f(a) & q(a) \\ 0 & 1 \end{pmatrix} \in GL(n+1, \boldsymbol{R}) , \tag{3}$$

where $A(n, \boldsymbol{R})$ denotes the group of all affine transformations of $\boldsymbol{R}^n$. Then we have $\rho(G(\Omega)) \subset G(V(\Omega))$; the pair (ρ, ι) of the homomorphism $\rho\colon G(\Omega) \to G(V(\Omega))$ and the map $\iota\colon \Omega \to V(\Omega)$ is equivariant, that is,

$$\iota \circ a = \rho(a) \circ \iota \quad \text{for every} \quad a \in G(\Omega) . \tag{4}$$

In view of (1) and (2), this shows that the subgroup $\rho(G(\Omega)) \cdot \{\lambda 1_{n+1} \mid \lambda \in \boldsymbol{R}_+\}$ of $G(V(\Omega))$ acts transitively on $V(\Omega)$ if $G(\Omega)$ acts transitively on Ω, where 1_{n+1} denotes the identity matrix of degree $n+1$. Therefore, when Ω is homogeneous, $V(\Omega)$ is also homogeneous. We denote by $d\iota$ the differential of the map $\iota: \Omega \to V(\Omega)$. Then, since ι is an affine map, $d\iota$ gives a holomorphic embedding of T_Ω into $T_{V(\Omega)}$. Moreover, by differentiating both sides of (4), we obtain

$$(5) \qquad d\iota \circ a = \rho(a) \circ d\iota \quad \text{for every} \quad a \in G(\Omega) .$$

We now define a function Φ_Ω on T_Ω by

$$\Phi_\Omega = \Phi_{V(\Omega)} \circ d\iota ;$$

the restriction of the function Φ_Ω to Ω, viewed as the zero-section of T_Ω, is denoted by ϕ_Ω. When Ω is a convex cone, the function Φ_Ω defined above coincides with the one defined in §1 up to positive constant multiple. Indeed, we have $V(\Omega) = \Omega \times \boldsymbol{R}_+$ as a convex cone. On the other hand, it can be shown that, for the product $V = V_1 \times V_2$ of convex cones V_1 and V_2, we have $\Phi_V(z) = c\Phi_{V_1}(z_1)\Phi_{V_2}(z_2)$ for some $c \in \boldsymbol{R}_+$, where Φ_V, Φ_{V_1}, Φ_{V_2} denote the functions defined in §1 and $z = (z_1, z_2) \in T_V = T_{V_1} \times T_{V_2}$. Hence our assertion follows from the fact that the map $d\iota: T_\Omega \to T_{V(\Omega)}$ is given by $d\iota(z) = (z, 1)$ $(z \in T_\Omega)$. It is clear from the definition that Φ_Ω is holomorphic on T_Ω, while ϕ_Ω is real-analytic on Ω. (C1)$\sim$(C5) in §1 hold for Φ_Ω and ϕ_Ω. This follows from the corresponding properties of $\Phi_{V(\Omega)}$ and $\phi_{V(\Omega)}$. Here, in view of (5) and (3), $\det a$ is replaced by $\det f(a)$ in (C1) and (C3). The lemma in §1 also remains valid for Φ_Ω. Indeed, if Ω is homogeneous, then, as previously remarked, $V(\Omega)$ is also homogeneous. Hence, applying the lemma in §1 to $\Phi_{V(\Omega)}$, we see that $\Phi_{V(\Omega)}$ never vanishes on $T_{V(\Omega)}$, which clearly implies that Φ_Ω never vanishes on T_Ω.

Let Ω be as above. We put $S_\Omega = \{z \in T_\Omega \mid \Phi_\Omega(z) = 0\}$. Then, since the function Φ_Ω is not identically zero by, e.g., (C4), $T_\Omega - S_\Omega$ is a non-empty open subset of T_Ω. By (C1), we also see that the sets $T_\Omega - S_\Omega$ and S_Ω are $G(\Omega)$-invariant. We define a function ψ_Ω on $T_\Omega - S_\Omega$ by

$$\psi_\Omega(z) = \log \phi_\Omega(p(z)) - \log |\Phi_\Omega(z)| \quad \text{for} \quad z \in T_\Omega - S_\Omega .$$

Note that, for $z \in T_\Omega$, $p(z)$ is the real part of z with respect to the complex structure $T_\Omega = \Omega + \sqrt{-1}\boldsymbol{R}^n$. Since

$$\partial\bar{\partial}\psi_\Omega(z) = \partial\bar{\partial} \log \phi_\Omega(p(z)) = \frac{1}{4} \sum_{i,j=1}^{n} \frac{\partial^2 \log \phi_\Omega(x)}{\partial x^i \partial x^j} dz^i \wedge d\bar{z}^j ,$$

where $z = (z^1, \cdots, z^n)$, $p(z) = x = (x^1, \cdots, x^n)$ and $\operatorname{Re} z^i = x^i$, and since the matrix $(\partial^2 \log \phi_\Omega(x)/\partial x^i \partial x^j)$ is positive-definite at every point of Ω by (C4), ψ_Ω is a smooth strictly plurisubharmonic function on $T_\Omega - S_\Omega$. Moreover, (C1) and (C3) imply that the function ψ_Ω is $G(\Omega)$-invariant, because $p(az) = ap(z)$ for all $z \in T_\Omega$, $a \in G(\Omega)$.

We now prove our theorem. Let Γ be the covering transformation group of the covering $\Omega \to M$. Then we have $\Gamma \subset G(\Omega)$ by assumption. It follows that Γ acts properly discontinuously and freely on T_Ω and $\Gamma \backslash T_\Omega = T_M$ as a complex manifold. Therefore, in view of (C1), the function Φ_Ω can be regarded as a non-trivial holomorphic section of a flat line bundle over T_M. We denote by S_M the support of the divisor determined by Φ_Ω; S_M is either a closed analytic hypersurface of T_M or an empty set. Then $T_\Omega - S_\Omega$ is a Γ-invariant open subset of T_Ω and we have $T_M - S_M = \Gamma \backslash (T_\Omega - S_\Omega)$. Since ψ_Ω is a $G(\Omega)$- and hence Γ-invariant function on $T_\Omega - S_\Omega$, ψ_Ω induces a function ψ_M on $T_M - S_M$, which is smooth and strictly plurisubharmonic, because ψ_Ω is smooth and strictly plurisubharmonic. This proves the first assertion of the theorem. To prove the second, let $c \in \boldsymbol{R}$ and put $E = \{z \in T_M - S_M \mid \psi_M(z) < c\}$. Then, from (C2) and the definition of ψ_Ω and S_Ω, we see that, for any $x \in M$, there exists a neighborhood U_x of x such that $p^{-1}(U_x) \cap E$ is relatively compact in $T_M - S_M$. Since M is compact by assumption, there exist a finite number of points $x_1, \cdots, x_k$ of M such that $M = \cup_{i=1}^k U_{x_i}$. Thus E is relatively compact in $T_M - S_M$, because $E = \cup_{i=1}^k (p^{-1}(U_{x_i}) \cap E)$ and each set $p^{-1}(U_{x_i}) \cap E$ is relatively compact in $T_M - S_M$. Hence ψ_M is an exhaustion function, which completes the proof of the theorem.

Next we prove the corollary. Since Ω is homogeneous by assumption, we see by the lemma that the function Φ_Ω never vanishes on T_Ω, which implies that, in the theorem, S_M is an empty set and hence ψ_M is a smooth strictly plurisubharmonic function defined on the whole of T_M. Therefore T_M contains no positive-dimensional compact analytic subsets. If M is compact, then, since ψ_M is an exhaustion function, T_M is a Stein manifold by a theorem of Grauert [4].

EXAMPLE. Let Ω be the cone of positive real numbers and let M be an affine manifold $\Gamma \backslash \Omega$ with $\Gamma = \{\lambda^k \mid k \in \boldsymbol{Z}\}$ $(\lambda \neq 1 \in \boldsymbol{R}_+)$. Then T_Ω is the right-half plane in the complex plane and T_M is a half torus. The function ψ_Ω defined in the proof of the theorem is given by

$$\psi_\Omega(z) = \log(1/\mathrm{Re}\, z) - \log|1/z| \quad (z \in T_\Omega) .$$

This function induces a strictly subharmonic function ψ_M on the half torus T_M.

REMARK. From the proof of the theorem and the second half of the corollary, we conclude the existence of invariant holomorphic functions on a symmetric tube domain (cf. Remark 1 in the introduction): Let Ω be a self-dual homogeneous cone. Let Γ be a discrete subgroup of $G(\Omega)$

acting properly discontinuously and freely on Ω. Suppose $M: = \Gamma\backslash\Omega$ is compact. Then there exists a non-constant Γ-invariant holomorphic function on T_Ω.

In the above situation, combined with a result of Serre [8], our corollary also shows

$$H^1(\Gamma, O(T_\Omega)) = 0 ,$$

where $O(T_\Omega)$ denotes the ring of holomorphic functions on T_Ω and is regarded as a Γ-module by the rule $a\cdot f = f - f\circ a^{-1}$ $(a \in \Gamma, f \in O(T_\Omega))$.

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