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AFFINE TORUS EMBEDDINGS WHICH ARE COMPLETE INTERSECTIONS

Dedicated to the memory of Professor Takehiko Miyata

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1. Introduction. Throughout this paper, let k be a field, M a free $\boldsymbol{Z}$-module of finite rank $r \geqq 1$ and N the dual $\mathrm{Hom}(M, \boldsymbol{Z})$ with the canonical pairing $\langle\ ,\ \rangle: M \times N \to \boldsymbol{Z}$. We extend this pairing $\boldsymbol{R}$-linearly to $M_{\boldsymbol{R}} \times N_{\boldsymbol{R}}$ where $M_{\boldsymbol{R}} = \boldsymbol{R} \otimes_{\boldsymbol{Z}} M$ and $N_{\boldsymbol{R}} = \boldsymbol{R} \otimes_{\boldsymbol{Z}} N$. Let σ be a strongly convex rational polyhedral cone in $N_{\boldsymbol{R}}$, i.e., $\sigma = \{\sum_{i=1}^{s} a_i n_i \mid$ any non-negative $a_i \in \boldsymbol{R}\}$ for some $n_i \in N$ $(1 \leqq i \leqq s)$ with $\sigma \cap (-\sigma) = \{0\}$. The dual cone $\sigma^\vee = \{x \in M_{\boldsymbol{R}} \mid \langle x, y\rangle \geqq 0$ for all $y \in \sigma\}$ is rational and spans $M_{\boldsymbol{R}}$ as an $\boldsymbol{R}$-vector space. The group algebra $k[M]$ of M over k, whose spectrum T_N is ragarded as a k-split torus, contains the monoid algebra $k[M \cap \sigma^\vee]$ of $M \cap \sigma^\vee$ over k as a k-subalgebra. Then $\mathrm{Spec}\, k[M \cap \sigma^\vee]$, which is denoted by X_σ, is exactly a normal affine equivariant embedding of the torus T_N. Moreover, every normal equivariant embedding of T_N is covered by such X_σ's (e.g., [4, Chap. I]). Consequently some properties on toric singularities should be characterized in terms of convex rational polyhedral cones.

Let us recall the well known hierarchy "regular" $\Rightarrow$ "local complete intersection" $\Rightarrow$ "Gorenstein" $\Rightarrow$ "Cohen-Macaulay" of conditions on X_σ. We already know the following results:

(1.1) (Mumford et al. [4]) *X_σ is nonsingular if and only if σ is nonsingular.*

(1.2) (Ishida [2]) *If $r = 3$ and X_σ is a local complete intersection, then $k[M \cap \sigma^\vee]$ is k-isomorphic to $k[x, y, z, w, u]/k[x, y, z, w, u](xz - w^b u^c, yw - u^a)$ for a triple (a, b, c) of non-negative integers.*

(1.3) (Stanley [5]) *$k[M \cap \sigma^\vee]$ is a Gorenstein ring if and only if $M \cap \mathrm{int}(\sigma^\vee) = m_\sigma + M \cap \sigma^\vee$ for an element $m_\sigma \in M$.*

(1.4) (Hochster [1]) *$k[M \cap \sigma^\vee]$ is always a Cohen-Macaulay ring.*

Moreover Stanley [6] partially and Watanabe [7] completely classified $M \cap \sigma^\vee$ such that X_σ is a local complete intersection under the assumption

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that σ is simplicial. Especially in the case where $r=2$ and σ is singular, X_σ has a unique singularity, which is a cyclic quotient singularity of $\boldsymbol{A}_k^2$ (cf. [4, Chap. I]), and hence if $k[M\cap\sigma^\vee]$ is a Gorenstein ring then it is a hypersurface (cf. [2, Example 7.8]).

The purpose of this paper is to determine completely normal torus embeddings which are local complete intersections. We now explain our result in more detail. Let us identify N (resp. M) with $\boldsymbol{Z}^r$ (resp. the dual module $(\boldsymbol{Z}^r)^\vee$ of $\boldsymbol{Z}^r$) by a fixed isomorphism (resp. its dual isomorphism). We consider a sequence $\mathfrak{g}=(g_1,\cdots,g_u)$ of length $1\leqq u<r$ with nonzero $g_i=(g_{i1},\cdots,g_{ir})\in(\boldsymbol{Z}^r)^\vee$ with respect to the basis dual to the standard basis of $\boldsymbol{Z}^r$ such that $g_{ij}=0$ $(i<j)$ and all elements of $\langle g_i, \boldsymbol{P}_\mathfrak{g}^{(i)}\rangle$ are non-negative. Here $\boldsymbol{P}_\mathfrak{g}^{(1)}=\{(1,0,\cdots,0)\}\subseteqq\boldsymbol{Z}^r$ and, for $1<i\leqq u+1$, $\boldsymbol{P}_\mathfrak{g}^{(i)}$ inductively denotes the convex hull of the union of $\boldsymbol{P}_\mathfrak{g}^{(i-1)}$ and $\{(x_1,x_2,\cdots,x_{i-1},\langle g_{i-1},x\rangle,0,\cdots,0)\in(\boldsymbol{Z}^r)_{\boldsymbol{R}}=N_{\boldsymbol{R}}\mid$ any $x=(x_1,\cdots,x_r)$ $(=(x_1,\cdots,x_{i-1},0,\cdots,0))\in\boldsymbol{P}_\mathfrak{g}^{(i-1)}\}$ in $N_{\boldsymbol{R}}$. Our main result is the following:

THEOREM 1.5. *Suppose that* $(\alpha\otimes 1_{\boldsymbol{R}})(\sigma)=\{ax\mid$ *any* $x\in\boldsymbol{P}_\mathfrak{g}^{(\dim\boldsymbol{R}\sigma)}$ *and any non-negative* $a\in\boldsymbol{R}\}$ *for an automorphism* α *of the abelian group* N *and a sequence* $\mathfrak{g}$ *of length* $\dim\boldsymbol{R}\sigma-1$. *Then* X_σ *is a local complete intersection. Conversely, suppose that* X_σ *is a local complete intersection. Then there are an automorphism* α *of* N *and a sequence* $\mathfrak{g}$ *of length* $\dim\boldsymbol{R}\sigma-1$ *such that the above equality holds.*

Concerning the assertion of this theorem, Ishida [3] showed the first half and conjectured that the latter half should hold for every σ, in terms of monoids, at the symposium on commutative algebra held at Karuizawa in 1978 (cf. Remark 2.3). He also observed that his conjecture is true when either σ is simplicial or $r\leqq 3$. The present paper was inspired by this talk.

When $\sigma^\vee$ is strongly convex, a version of our main theorem in Section 3 (cf. Theorem 3.1) gives a complete classification of algebras of invariant polynomials under linear actions of algebraic tori which are global complete intersections of given embedding dimensions. It seems to be useful in studying invariants of certain representations of reductive algebraic groups in characteristic zero.

We will collect together auxiliary notations and assertions in the next section. Stanley's criterion (1.3) for $k[M\cap\sigma^\vee]$ to be a Gorenstein ring will play a fundamental role in Section 3, when we deal with a combinatorial property on the first syzygies of $k[M\cap\sigma^\vee]$.

The following notations are standard and shall be frequently used;

$\boldsymbol{Z}$ the ring of rational integers

$\boldsymbol{Q}$	the field of rational numbers
$\boldsymbol{R}$	the field of real numbers
$\boldsymbol{R}_0$	the set consisting of all non-negative real numbers
$\boldsymbol{Z}_0$	the additive monoid consisting of all non-negative integers
$\boldsymbol{Z}_+$	the set consisting of all positive integers
$A\setminus B$	the difference set $\{x \mid x \in A, x \notin B\}$
$\mathrm{card}(X)$	the cardinality of a set X.

2. **Preliminaries.** Suppose that A is an epimorphic image of a regular local ring R such that the embedding dimension of A coincides with the dimension of R. Then the *homological dimension* of A is defined to be that of A as an R-module and is equal to the difference between the embedding dimension and the (Krull) dimension of A especially if A is a Cohen-Macaulay ring. A local ring A is said to be a *complete intersection* (CI, for short) if $A \simeq R/R(g_1, \cdots, g_q)$ for a regular local ring R and an R-sequence $(g_1, \cdots, g_q)$. In this case, we can choose R in such a way that q equals the homological dimension of A. A noetherian ring B or its affine scheme is defined to be a *local complete intersection* (LCI, for short) if, for every prime ideal $\mathfrak{P}$ of B, the localization $B_{\mathfrak{P}}$ of B at $\mathfrak{P}$ is a CI. Furthermore, we say that an affine k-algebra S is a *global complete intersection* (GCI, for short) over k if $S \simeq k[T_1, \cdots, T_m]/k[T_1, \cdots, T_m](F_1, \cdots, F_d)$ for a polynomial ring $k[T_1, \cdots, T_m]$ and some polynomials F_i, $1 \leqq i \leqq d$, with $d = m - \dim S$. For simplicity, we denote also by $\Phi \otimes \Psi$ the composite $A \otimes_k B \to R \otimes_k R \to R$ of the tensor product $\Phi \otimes \Psi$ of k-algebra maps $\Phi: A \to R$, $\Psi: B \to R$ with the canonical multiplication map $R \otimes_k R \to R$. A graded version of Nakayama's lemma implies the following:

LEMMA 2.1. *Let A be a noetherian $\boldsymbol{Z}_0$-graded k-algebra whose graded part of degree 0 is k. Then A is a* GCI *over k if and only if its local ring at the unique homogeneous maximal ideal is a* CI.

The proof of [6, Lemma 5.2] suggests:

LEMMA 2.2. *Let A be an affine k-domain and A' a k-subalgebra of A satisfying $A = A' \oplus \mathscr{I}$ as k-vector spaces for an ideal $\mathscr{I}$ of A. Then*:

(1) *There are a polynomial ring B over k of finite type and a k-epimorphism $\Psi: B \to A$ such that $\Psi(B') = A'$, $\Psi(\mathfrak{J}) = \mathscr{I}$ and $B = B' \oplus \mathfrak{J}$ as k-vector spaces for a polynomial subalgebra B' over k of B and an ideal $\mathfrak{J}$ of B.*

(2) *If $A_{\mathfrak{P}}$ is a* CI *for every prime ideal $\mathfrak{P}$ of A containing $\mathscr{I}$, then A' is a* LCI.

PROOF. The assertion (1) can be easily shown. Using this assertion and notation, we will show (2). Let $\mathfrak{Q}$ be a prime ideal of B' containing $B' \cap \operatorname{Ker} \Psi$. Then $B_{\mathfrak{Q}+\mathfrak{J}} = B'_{\mathfrak{Q}} \oplus \mathscr{J}_{\mathfrak{Q}+\mathfrak{J}}$ as k-vector spaces and $(B' \cap \operatorname{Ker} \Psi)_{\mathfrak{Q}}$ is an epimorphic image of $(\operatorname{Ker} \Psi)_{\mathfrak{Q}+\mathfrak{J}}$. Let $\{b_1, \cdots, b_d\}$ be a minimal system of generators of $(B' \cap \operatorname{Ker} \Psi)_{\mathfrak{Q}}$ as an ideal of $B'_{\mathfrak{Q}}$. Clearly this set is extended to a minimal system of generators of $(\operatorname{Ker} \Psi)_{\mathfrak{Q}+\mathfrak{J}}$. Since $A_{\Psi(\mathfrak{Q})+\mathscr{J}}$ is a CI, $(b_1, \cdots, b_d)$ is a $B_{\mathfrak{Q}+\mathfrak{J}}$-sequence. By the decomposition of $B_{\mathfrak{Q}+\mathfrak{J}}$ into subspaces stated above, we immediately see that $(b_1, \cdots, b_d)$ is also a $B'_{\mathfrak{Q}}$-sequence, and hence $A'_{\Psi(\mathfrak{Q})}$ is a CI. □

For a subset X of $M_{\boldsymbol{R}}$ or $N_{\boldsymbol{R}}$, let $X^{\perp}$ be the set of all elements which are orthogonal to X with respect to the $\boldsymbol{R}$-linear pairing $\langle \ , \ \rangle$, $\boldsymbol{R}_0 X$ the set of all finite sums $\sum a_i x_i$ with $a_i \in \boldsymbol{R}_0$, $\boldsymbol{R}X$ the subspace generated by X and $X^{\vee}$ the dual cone of X if X is a convex polyhedral cone. When $\sigma^{\vee}$ is strongly convex (i.e., $\sigma^{\vee} \cap (-\sigma^{\vee}) = \{0\}$), $\sigma^{\vee}$ is contained in $\sum_{i=1}^{r} \boldsymbol{R}_0 w_i$ for some $\boldsymbol{R}$-basis $\{w_1, \cdots, w_r\}$ of $M_{\boldsymbol{R}}$. Moreover, as $\sigma^{\vee}$ is rational and $M_{\boldsymbol{Q}} = \boldsymbol{Q} \otimes_{\boldsymbol{Z}} M$ is dense in $M_{\boldsymbol{R}}$, every w_i can be chosen from $M_{\boldsymbol{Q}}$. By this observation, we see that the following conditions are equivalent; (i) $\sigma^{\vee}$ is strongly convex; (ii) units (invertible elements) of $M \cap \sigma^{\vee}$ are trivial; and (iii) $M \cap \sigma^{\vee}$ is a submonoid of a finitely generated free additive monoid.

For an additive monoid $\mathscr{S}$, we shall define the notations and terminologies as follows: Denote by $k[\mathscr{S}]$ the k-vector space with the k-basis $\{e(s) \mid s \in \mathscr{S}\}$ which has the k-algebra structure defined by $e(s)e(s') = e(s+s')$, $(s, s') \in \mathscr{S} \times \mathscr{S}$. We regard $\mathscr{S} \ni s \to e(s) \in k[\mathscr{S}]$ as a homomorphism of monoids and denote by e this map. $\mathscr{S}$ is said to be *affine*, if it is a finitely generated submonoid of a torsion-free abelian group, whose subgroup generated by $\mathscr{S}$ is denoted $\langle \mathscr{S} \rangle$. $\mathscr{S}$ is said to be *normal*, if $k[\mathscr{S}]$ is normal. Every $M \cap \sigma^{\vee}$ is an affine normal submonoid of M, and conversely any affine normal monoid is expressed in the form "$M \cap \sigma^{\vee}$" (e.g., [4, Chap. I]). An element $x \in \mathscr{S}$ is said to be *fundamental* if whenever $x = y + z$ with y, z in $\mathscr{S}$ then $y = 0$ or $z = 0$. We denote by $\mathrm{FUND}(\mathscr{S})$ the set consisting of all fundamental elements in $\mathscr{S}$. When $\mathscr{S}$ is affine and without nontrivial units, $\mathrm{FUND}(\mathscr{S})$ is the unique minimal system of generators of $\mathscr{S}$ as a monoid. For an arbitrary nonzero $x \in \mathscr{S}$ and $n \in \boldsymbol{Z}_+$, let $\mathscr{S} \int_n x$ be the affine submonoid

$$\mathscr{S} + \sum_{i=1}^{n} \boldsymbol{Z}_0 e_i + \boldsymbol{Z}_0 \Big(x - \sum_{i=1}^{n} e_i \Big)$$

of $\langle \mathscr{S} \rangle \oplus \boldsymbol{Z}^n$ where $\{e_1, \cdots, e_n\}$ is the standard $\boldsymbol{Z}$-basis of $\boldsymbol{Z}^n$. Clearly

$\dim k\left[\mathscr{S}\int_n x\right] = \dim k[\mathscr{S}] + n$, and if $\mathscr{S}$ is without nontrivial units, so is $\mathscr{S}\int_n x$. For the sake of simplicity, let $\mathscr{S}\int x$ and $\mathscr{S}^{\sharp}$ respectively denote $\mathscr{S}\int_1 x$ and $\mathscr{S}\setminus(\{0\}\cup \mathrm{FUND}(\mathscr{S}))$.

REMARK 2.3. The monoid $\mathscr{S}\int x$ was initially defined by Ishida [3]. Suppose that $\mathscr{S}$ is an affine normal monoid without nontrivial units. He observed that if $k[\mathscr{S}]$ is a GCI, then so is $k\left[\mathscr{S}\int x\right]$ for any nonzero $x \in \mathscr{S}$. The first half of the assertion of Theorem 1.5 follows immediately from this. Moreover, he conjectured that if $k[\mathscr{S}]$ is a GCI, then $\mathscr{S}$ should be inductively constructed, i.e., $\mathscr{S}$ should be isomorphic to $\left(\cdots\left(\left(\boldsymbol{Z}_0\int x_1\right)\int x_2\right)\int\cdots\right)\int x_n$ as a monoid for some $x_1 \in \boldsymbol{Z}_0\setminus\{0\}$, $x_{i+1} \in \left(\cdots\left(\boldsymbol{Z}_0\int x_1\right)\int\cdots\right)\int x_i\setminus\{0\}$ $(1 < i < n)$ and $n \in \boldsymbol{Z}_0$ (cf. [3]).

LEMMA 2.4. *Let x be a nonzero element of an affine monoid $\mathscr{S}$ without nontrivial units. For any $n \in \boldsymbol{Z}_+$, we have*:

(1) *The following three conditions are equivalent*; (i) $x \notin \mathrm{FUND}(\mathscr{S})$; (ii) $\mathrm{FUND}\left(\mathscr{S}\int_n x\right) \supseteqq \mathrm{FUND}(\mathscr{S})$; *and* (iii) $k\left[\mathscr{S}\int_n x\right]$ *is minimally generated by* $\mathrm{card}(\mathrm{FUND}(\mathscr{S})) + n + 1$ *elements as a k-algebra.*

(2) *$\mathscr{S}$ is normal if and only if so is $\mathscr{S}\int_n x$.*

(3) *$k[\mathscr{S}]$ is a* GCI *if and only if so is $k\left[\mathscr{S}\int_n x\right]$.*

(4) *$\mathscr{S}\int_n x$ is isomorphic to $\left(\cdots\left(\left(\mathscr{S}\int x_1\right)\int x_2\right)\int\cdots\right)\int x_n$ as a monoid, where $x_1 = x$, $x_2 \in \mathrm{FUND}\left(\mathscr{S}\int x_1\right)\setminus \mathrm{FUND}(\mathscr{S})$ and*

$$x_{i+1} \in \mathrm{FUND}\left(\left(\cdots\left(\mathscr{S}\int x_1\right)\int\cdots\right)\int x_i\right)\setminus \mathrm{FUND}\left(\left(\cdots\left(\mathscr{S}\int x_1\right)\int\cdots\right)\int x_{i-1}\right), \quad 1 < i < n .$$

PROOF. (1) follows easily from the definition of $\mathscr{S}\int_n x$.

(2): Suppose that $\mathscr{S}$ is normal. Let us express an element $y \in \left\langle\mathscr{S}\int_n x\right\rangle = \langle\mathscr{S}\rangle \oplus \boldsymbol{Z}^n$ as $y = u + \sum_{i=1}^n \eta_i e_i$ with $u \in \langle\mathscr{S}\rangle$ and $\eta_i \in \boldsymbol{Z}$, $1 \leqq i \leqq n$, and assume $my \in \mathscr{S}\int_n x$ for an $m \in \boldsymbol{Z}_+$. There exist $v \in \mathscr{S}$ and $\xi_i \in \boldsymbol{Z}_0$ $(1 \leqq i \leqq n+1)$ such that

$$my = v + \sum_{i=1}^n \xi_i e_i + \xi_{n+1}\left(x - \sum_{i=1}^n e_i\right) .$$

Since $\mathscr{S}$ is normal, by the above identities we may assume $u = 0$, which implies that $\xi_{n+1} = 0$ and $\eta_i = \xi_i - \xi_{n+1} \in \boldsymbol{Z}_0$ $(1 \leqq i \leqq n)$. Thus $\mathscr{S}\int_n x$ is saturated in $\left\langle \mathscr{S}\int_n x \right\rangle$ and is normal (e.g., [4, Chap. I, Lemma 1]). The converse can be similarly shown.

(3): (We can generalize this assertion, but it is not necessary.) Let $\Psi\colon A \to k[\mathscr{S}]$ be an $\langle \mathscr{S} \rangle$-graded epimorphism from an $\langle \mathscr{S} \rangle$-graded polynomial k-algebra A of dimension equal to card(FUND($\mathscr{S}$)) and B an $(n+1)$-dimensional polynomial k-algebra $k[X_1, \cdots, X_{n+1}]$. We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(1 \otimes \alpha) \arrow[r] & k[\mathscr{S}] \otimes_k B \arrow[r, "1\otimes\alpha"'] & k\left[\mathscr{S}\int_n x\right] \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\Psi \otimes \alpha) \arrow[r] \arrow[u] & A \otimes_k B \arrow[r, "\Psi\otimes\alpha"'] \arrow[u, "\Psi\otimes 1"] & k\left[\mathscr{S}\int_n x\right] \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

with exact rows, where $\alpha\colon B \to k\left[\mathscr{S}\int_n x\right]$ is a k-algebra map defined by $\alpha(X_i) = e(e_i)$ $(1 \leqq i \leqq n)$ and $\alpha(X_{n+1}) = e(x - \sum_{i=1}^n e_i)$. Clearly $\mathrm{Ker}(1 \otimes \alpha)$ is generated by $e(x) \otimes 1 - 1 \otimes \prod_{i=1}^{n+1} X_i$. Let $\{g_1, \cdots, g_d\}$ be a minimal system of $\langle \mathscr{S} \rangle$-homogeneous generators of $\mathrm{Ker}\,\Psi$ and $y \in \Psi^{-1}(e(x))$ a monomial of a regular system of $\langle \mathscr{S} \rangle$-homogeneous parameters of A. Suppose

$$g_d \otimes 1 = \sum_{i=1}^{d-1} a_i(g_i \otimes 1) + a_d\left(y \otimes 1 - 1 \otimes \prod_{i=1}^{n+1} X_i\right)$$

for some homogeneous elements a_i $(1 \leqq i \leqq d)$ in $A \otimes_k B$ and let us apply $1 \otimes \mu$ to both sides of this identity, where μ is a k-endomorphism of B sending all X_i's to zero. Then $\mathrm{Ker}\,\Psi$ contains $(1 \otimes \mu)(a_d)$ or one of prime divisors of y in A. But the latter case does not occur, because $\dim A =$ card(FUND($\mathscr{S}$)). Thus $(1 \otimes \mu)(a_d)$ belongs to $A(g_1, \cdots, g_{d-1})$, which contradicts the choice of $\{g_1, \cdots, g_d\}$. From this observation, we deduce that $\{g_1 \otimes 1, \cdots, g_d \otimes 1, y \otimes 1 - 1 \otimes \prod_{i=1}^{n+1} X_i\}$ is a minimal system of generators of $\mathrm{Ker}\,(\Psi \otimes \alpha)$. Consequently we obtain the equivalence in (3), as desired.

(4): We inductively see that $\mathrm{FUND}\left(\left(\cdots\left(\mathscr{S}\int x_1\right)\int\cdots\right)\int x_n\right) / \mathrm{FUND}(\mathscr{S})$ consists of $n+1$ elements and the sum of all elements of this set equals x. The assertion follows immediately from this observation. $\square$

For any $n \in M$ and an M-graded module $L = \bigoplus_{i \in M} L_i$ over a M-graded k-algebra A, $L(n)$ denotes the M-graded A-module whose underlying A-module is L and the M-grading is given by $L(n)_i = L_{n+i}$, $i \in M$. When A is a Cohen-Macaulay ring and possesses a dualizing complex $\mathscr{K}^{\cdot}(A)$ in the category of M-graded A-modules, the unique non-vanishing M-graded module $H^d(\mathscr{K}^{\cdot}(A))$ is said to be an *M-graded canonical module* of A and is denoted by $\Omega_M(A)$. Moreover if A is a Gorenstein ring and has a unique M-homogeneous maximal ideal $\mathfrak{m}$ with $A/\mathfrak{m} \simeq k$, then $\Omega_M(A)$ is isomorphic to $A(a)$ for some $a \in M$.

The *interior* of $\sigma^\vee$, which is denoted by $\mathrm{int}(\sigma^\vee)$, equals $\{x \in \sigma^\vee \mid \langle f, x\rangle > 0$ for all nonzero $f \in \sigma\}$. We have $M \cap \mathrm{int}(\sigma^\vee) = M \cap \sigma^\vee \cap \boldsymbol{Z}_+^n$, if M is a subgroup of $\boldsymbol{Z}^n$ satisfying $M \cap \sigma^\vee = M \cap \boldsymbol{Z}_0^n$ and $M \cap \sigma^\vee \cap \boldsymbol{Z}_+^n \neq \varnothing$.

THEOREM 2.5 ([4, Chap. I, Theorems 9 and 14], [5]). *$\Omega_M(k[M \cap \sigma^\vee])$ can be identified with the ideal $\bigoplus_{x \in M \cap \mathrm{int}(\sigma^\vee)} ke(x)$ of $k[M \cap \sigma^\vee]$.*

Let $\omega(M \cap \sigma^\vee)$ be an element of $M \cap \mathrm{int}(\sigma^\vee)$ which satisfies $z = 0$ whenever $\omega(M \cap \sigma^\vee) = y + z$ with $y \in M \cap \mathrm{int}(\sigma^\vee)$ and $z \in M \cap \sigma^\vee$. By Stanley's theorem (1.3), $k[M \cap \sigma^\vee]$ is a Gorenstein ring if and only if $\omega(M \cap \sigma^\vee) + M \cap \sigma^\vee = M \cap \mathrm{int}(\sigma^\vee)$.

Recall that a *directed graph* $\mathscr{D}$ consists of a finite non-empty set $\mathrm{VER}(\mathscr{D})$ and a set $\mathrm{DED}(\mathscr{D})$ of ordered pairs of distinct elements of $\mathrm{VER}(\mathscr{D})$. The elements of $\mathrm{VER}(\mathscr{D})$ and $\mathrm{DED}(\mathscr{D})$ are respectively called *vertices* and *directed edges* of $\mathscr{D}$. For $e = (x, y) \in \mathrm{DED}(\mathscr{D})$ with x, $y \in \mathrm{VER}(\mathscr{D})$, let us set $\mathfrak{i}(e) = x$ and $\mathfrak{f}(e) = y$. An alternating sequence $(x_0, e_1, x_1, e_2, \cdots, e_n, x_n)$ $(n \geqq 2)$ of vertices and directed edges (i.e., a *directed path*) is said to be a *directed circuit of length n* in $\mathscr{D}$, if $x_{j-1} = \mathfrak{i}(e_j)$, $x_j = \mathfrak{f}(e_j)$ $(1 \leqq j \leqq n)$, $x_n = x_0$ and $x_i \neq x_j$ for any $0 \leqq i < j \leqq n$ with $(i, j) \neq (0, n)$. We then express this sequence by the sequence $(x_0, x_1, \cdots, x_{n-1})$ of distinct vertices. $\mathscr{D}$ is said to be *acyclic*, unless it contains directed circuits. The following elementary characterization of acyclicity of directed graphes is probably well known.

LEMMA 2.6. *Let $\mathscr{D} = (\mathrm{VER}(\mathscr{D}), \mathrm{DED}(\mathscr{D}))$ be a directed graph. Then $\mathscr{D}$ is acyclic if and only if there is a linear ordering $\leqq$ on $\mathrm{VER}(\mathscr{D})$ satisfying $\mathfrak{i}(e) \prec \mathfrak{f}(e)$ for all $e \in \mathrm{DED}(\mathscr{D})$.*

PROOF. Suppose that $\mathscr{D}$ is acyclic. Then there is a vertex x in $\mathscr{D}$ which is unequal to $\mathfrak{f}(e)$ for every $e \in \mathrm{DED}(\mathscr{D})$. Let $\mathscr{D}'$ be a directed subgraph of $\mathscr{D}$ defined by $\mathrm{VER}(\mathscr{D}') = \mathrm{VER}(\mathscr{D}) \setminus \{x\}$, $\mathrm{DED}(\mathscr{D}') = \{e \in \mathrm{DED}(\mathscr{D}) \mid \mathfrak{i}(e) \neq x\}$. Because $\mathscr{D}'$ is acyclic, we can inductively define a linear ordering on $\mathrm{VER}(\mathscr{D})$, as desired. The converse of this assertion is trivial. $\square$

3. The main theorem.

The latter half of the assertion of Theorem 1.5 is a consequences of the following:

THEOREM 3.1. *For a non-negative integer h, $k[M \cap \sigma^\vee]$ is a LCI* *whose local ring at the prime ideal, maximal in the set of proper M-homogeneous ideals, is of homological dimension h if and only if $M\cap\sigma^\vee$ is isomorphic to* $\left(\cdots\left(\left(\boldsymbol{Z}_0^{n_0}\int_{n_1} x_1\right)\int_{n_2} x_2\right)\int_{n_3}\cdots\right)\int_{n_h} x_h \oplus \boldsymbol{Z}^{r'}$ *as a monoid where* $n_i\in\boldsymbol{Z}_+$ $(0\leqq i\leqq h)$, $r'=r-\dim\boldsymbol{R}\sigma$, $x_1\in(\boldsymbol{Z}_0^{n_0})^\#$ *and* $x_{j+1}\in\left(\left(\cdots\left(\boldsymbol{Z}_0^{n_0}\int_{n_1} x_1\right)\int_{n_2}\cdots\right)\int_{n_j} x_j\right)^\#$ $(1\leqq j<h)$.

PROOF OF THEOREM 1.5. Suppose that $(\alpha\otimes 1_{\boldsymbol{R}})(\sigma)=\boldsymbol{R}_0\boldsymbol{P}_{\mathfrak{g}}^{(r'')}$ for an automorphism α of N and a sequence $\mathfrak{g}$ satisfying the conditions in Theorem 1.5. Without loss of generality, we may assume that α is the identity. Let $\{e_1^*,\cdots,e_r^*\}$ be the $\boldsymbol{Z}$-basis of $(\boldsymbol{Z}^r)^\vee=M$ dual to the standard basis of $\boldsymbol{Z}^r=N$. Set $\Xi_1=\boldsymbol{Z}_0e_1^*+\sum_{i=2}^r\boldsymbol{Z}e_i^*$ and

$$\Xi_i=\sum_{j=1}^i\boldsymbol{Z}_0e_j^*+\sum_{j=2}^i\boldsymbol{Z}_0(g_{j-1}-e_j^*)+\sum_{j=i+1}^r\boldsymbol{Z}e_j^*\qquad(2\leqq i\leqq r'')\,,$$

where $r''=\dim\boldsymbol{R}\sigma$. Then we inductively have $(\boldsymbol{R}_0\Xi_i)^\vee=\boldsymbol{R}_0\boldsymbol{P}_{\mathfrak{g}}^{(i)}$ for $1\leqq i\leqq r''$. Because $\Xi_{r''}$ is normal (cf. (2) of Lemma 2.4) and generates M, $\Xi_{r''}=M\cap((\boldsymbol{R}_0\Xi_{r''})^\vee)^\vee$ (e.g., [4, Chap. I]), and consequently $\Xi_{r''}=M\cap\sigma^\vee$. By this equality and (3) of Lemma 2.4, we see that $k[M\cap\sigma^\vee]$ is a LCI.

Conversely, suppose that $k[M\cap\sigma^\vee]$ is a LCI. Then, by (4) of Lemma 2.4 and Theorem 3.1, M has a $\boldsymbol{Z}$-basis $\{\varepsilon_1^*,\cdots,\varepsilon_r^*\}$ and contains nonzero g_i $(1\leqq i<r'')$ such that $g_i\in\Gamma_i$ and $M\cap\sigma^\vee=\Gamma_{r''}+\sum_{i=r''+1}^r\boldsymbol{Z}\varepsilon_i^*$. Here $r''=\dim\boldsymbol{R}\sigma$, $\Gamma_1=\boldsymbol{Z}_0\varepsilon_1^*$ and

$$\Gamma_i=\Gamma_{i-1}+\boldsymbol{Z}_0\varepsilon_i^*+\boldsymbol{Z}_0(g_{i-1}-\varepsilon_i^*)\qquad(2\leqq i\leqq r'')\,.$$

Put $\delta_i=(\boldsymbol{R}_0(\Gamma_i+\sum_{j=i+1}^r\boldsymbol{Z}\varepsilon_j^*))^\vee$ $(1\leqq i\leqq r'')$ and let $\{\varepsilon_1,\cdots,\varepsilon_r\}$ be the $\boldsymbol{Z}$-basis of N dual to $\{\varepsilon_1^*,\cdots,\varepsilon_r^*\}$. Clearly $\delta_{r''}=(\sigma^\vee)^\vee=\sigma$ and $g_i\in(\sum_{j=i+1}^r\boldsymbol{R}\varepsilon_j)^\perp\cap M\cap\delta_i^\vee=\Gamma_i$ (e.g., [4, Chap. I]). We may assume that $\{\varepsilon_1,\cdots,\varepsilon_r\}$ is the standard basis of $\boldsymbol{Z}^r=N$. Then $\mathfrak{g}=(g_1,\cdots,g_{r''-1})$ satisfies the conditions in Theorem 1.5 and the convex polytopes $\boldsymbol{P}_{\mathfrak{g}}^{(i)}$'s are well defined. We can inductively show $\boldsymbol{R}_0\boldsymbol{P}_{\mathfrak{g}}^{(i)}=\delta_i$ for $1\leqq i\leqq r''$, which implies $\boldsymbol{R}_0\boldsymbol{P}_{\mathfrak{g}}^{(r'')}=\sigma$. □

The rest of this paper is devoted to the proof of Theorem 3.1. When $M\cap\sigma^\vee\simeq\mathscr{S}\oplus\boldsymbol{Z}^a$ for an $a\in\boldsymbol{Z}_+$ and an affine submonoid $\mathscr{S}$, $k[M\cap\sigma^\vee]$ is a LCI if and only if so is $k[\mathscr{S}]$ (e.g., Lemma 2.2). Thus the "if" part follows immediately from (1) and (3) of Lemma 2.4 and it suffices to show the "only if" part under the assumption that $\sigma^\vee$ is strongly convex (see the proof of the "only if" part of [4, Chap. I, Theorem 4]). Hereafter, assume that $\sigma^\vee$ is strongly convex and $k[M\cap\sigma^\vee]$ is a singular LCI (and so a GCI). We need the following further notations and terminologies.

Put $m=\mathrm{card}(\mathrm{FUND}(M\cap\sigma^\vee))$. Let R be an m-dimensional polynomial k-algebra $k[T_1,\cdots,T_m]$ and Φ a k-algebra epimorphism from R to $k[M\cap\sigma^\vee]$

satisfying $\{\Phi(T_1), \cdots, \Phi(T_m)\} = \{e(x) | x \in \mathrm{FUND}(M \cap \sigma^\vee)\}$. By [1, Proposition 1], there is a free abelian group Z^n of rank n which contains M as a subgroup such that $M \cap Z_0^n = M \cap \sigma^\vee$ and $M \cap \sigma^\vee \cap Z_+^n \neq \emptyset$. We fix this Z^n and regard $k[M \cap \sigma^\vee]$ as a Z^n-graded algebra in a natural way. Define a unique Z^n-gradation on R so that Φ is a Z^n-graded map of degree $0 \in Z^n$. Put $I = \{1, \cdots, n\}$ and $J = \{1, \cdots, m\}$. When x is an element of the i-th homogeneous part of a Z^n-graded object with $i = (i_1, \cdots, i_n) \in Z^n$, we put $\deg(x) = i$, $\|\deg(x)\| = \sum_{j=1}^n |i_j|$ and $\mathrm{supp}(x) = \{j \in I | i_j \neq 0\}$. For a monomial $y = aT_1^{j_1} \cdots T_m^{j_m}$ with $j = (j_1, \cdots, j_m) \in Z_0^m$ and $a \in k^* = k \setminus \{0\}$, $\log_{\mathscr{T}}(x)$ and $\mathrm{supp}_{\mathscr{T}}(x)$ stand respectively for j and $\{i \in J | j_i \neq 0\}$. Conversely T^j denotes the monomial $T_1^{j_1} \cdots T_m^{j_m}$ in R, and $\mathscr{T}$ denotes the multiplicative monoid consisting of all T^j's in R.

Recall that a monomial L in $\mathscr{T}$ is said to be *square-free*, if L is a product of distinct T_i's. An element F of R is said to be *standard* if $F = L_1 - L_2$ with distinct $L_i \in \mathscr{T}$ $(i = 1, 2)$ and L_1 square-free. In this case we denote L_1 (resp. L_2) by α_F (resp. β_F).

For a finite set $\mathscr{P}$ of standard Z^n-homogeneous elements in R, let $\mathscr{G}_{\mathscr{P}}$ be the directed graph defined by $\mathrm{VER}(\mathscr{G}_{\mathscr{P}}) = \mathscr{P}$ and $\mathrm{DED}(\mathscr{G}_{\mathscr{P}}) = \{(F_1, F_2) \in \mathscr{P} \times \mathscr{P} | F_1 \neq F_2 \text{ and } \mathrm{supp}_{\mathscr{T}}(\alpha_{F_1}) \cap \mathrm{supp}_{\mathscr{T}}(\beta_{F_2}) \neq \emptyset\}$. Furthermore, a sequence $((L_1, L_1'), (L_2, L_2'), \cdots, (L_u, L_u'))$ in $\mathscr{T} \times \mathscr{T}$ is defined to be a *$\mathscr{P}$-path from $x \in \mathscr{T}$ to $y \in \mathscr{T}$* if $x \in RL_1'$, $x\prod_{j=1}^{i-1} L_j \in R \prod_{j=1}^{i} L_j'$ $(2 \leqq i \leqq u)$, $x \prod_{j=1}^{u} L_j \in Ry \prod_{j=1}^{u} L_j'$ and, for each $1 \leqq i \leqq u$, $L_i - L_i'$ or $L_i' - L_i$ belongs to $\mathscr{P}$.

Since $k[M \cap \sigma^\vee]$ is a GCI (e.g., Lemma 2.1), $\mathrm{Ker}\,\Phi$ is minimally generated by $d = m - r$ Z^n-homogeneous elements. For any $I' \subseteq I$, we define the following notations: Put $\mathscr{S}_{I'} = \{x \in M \cap \sigma^\vee | \mathrm{supp}(e(x)) \subseteqq I'\}$, $J_{I'} = \{j \in J | \mathrm{supp}(T_j) \subseteqq I'\}$ and, for a set $\mathscr{P}$ of Z^n-homogeneous elements of a Z^n-graded object, $\mathscr{P}_{\cap I'} = \{F \in \mathscr{P} | \mathrm{supp}(F) \subseteqq I'\}$. Let $\mathrm{SYZ}_1(I')$ be the set consisting of all minimal systems of Z^n-homogeneous and standard generators of $k[T_j | j \in J_{I'}] \cap \mathrm{Ker}\,\Phi$ as an ideal. (When this ideal coincides with the zero ideal, we can regard $\mathrm{SYZ}_1(I')$ as $\{\emptyset\}$.) Obviously $\mathscr{S}_{I'}$ is an affine normal submonoid of $M \cap \sigma^\vee$ and $\{e(s) | s \in \mathrm{FUND}(\mathscr{S}_{I'})\} = \{\Phi(T_j) | j \in J_{I'}\}$. For $K' \in \Phi^{-1}(e(\omega(\mathscr{S}_{I'}))) \cap \mathscr{T}$, a system $\mathscr{P} \in \mathrm{SYZ}_1(I')$ is said to be (I', K')-tiled, if $J_{I'}$ is a disjoint union of all $\mathrm{supp}_{\mathscr{T}}(\alpha_F)$'s, $F \in \mathscr{P}$, and $\mathrm{supp}_{\mathscr{T}}(K')$. We will show the existence of a tiled system of relations of $k[M \cap \sigma^\vee]$ in R, which will play an essential role in our proof of Theorem 3.1.

LEMMA 3.2. *Let I' be a subset of I and $\mathscr{P}$ a minimal system of Z^n-homogeneous generators of* $\mathrm{Ker}\,\Phi$. *Then*:

(1) *$\mathscr{P}_{\cap I'}$ minimally generates $k[T_j | j \in J_{I'}] \cap \mathrm{Ker}\,\Phi$ as an ideal.*

(2) *$k[\mathscr{S}_{I'}]$ is a* GCI.

(3) $\mathrm{SYZ}_1(I')$ *is non-empty.*

(4) $\deg(e(\omega(\mathscr{S}_{I'}))) = \sum_{j \in J_{I'}} \deg(T_j) - \sum_{F \in \mathscr{P} \cap I'} \deg(F)$.

PROOF. Both (1) and (2) follow immediately from the proof of Lemma 2.2. When $k[\mathscr{S}_{I'}]$ is a polynomial ring over k, (3) is trivial and (4) follows from the well known isomorphism $\Omega_{Z^n}(k[\mathscr{S}_{I'}]) \simeq k[\mathscr{S}_{I'}](-\sum_{j \in J_{I'}} \deg(T_j))$ of Z^n-graded $k[\mathscr{S}_{I'}]$-modules. Thanks to these assertions, we need to show (3) and (4) only in the case where $I' = I$ (recall that $k[M \cap \sigma^\vee]$ is assumed to be a *singular* LCI). Let F_i $(1 \leqq i \leqq d)$ be all elements of $\mathscr{P}$.

(3): We may assume that each F_i is expressed as $F_i = \alpha_i - \beta_i$ with α_i, $\beta_i \in \mathscr{T}$. Suppose $\mathrm{SYZ}_1(I) = \varnothing$. Then there is an index i_0 with $1 \leqq i_0 \leqq d$ such that neither α_{i_0} nor β_{i_0} are square-free. Hence $\alpha_{i_0} = x\alpha'$ and $\beta_{i_0} = y\beta'$ for some α', β', x and y in $\mathscr{T}$ satisfying $\mathrm{supp}(F_{i_0}) = \mathrm{supp}(\alpha') = \mathrm{supp}(\beta')$. Let z be an element of $\Phi^{-1}(e(\omega(\mathscr{S}_{\mathrm{supp}(F_{i_0})}))) \cap \mathscr{T}$. Because $k[\mathscr{S}_{\mathrm{supp}(F_{i_0})}]$ is a Gorenstein ring, by (1.3) we can choose monomials x', y' from $\mathscr{T}$ in such a way that both $\alpha' - zx'$ and $\beta' - zy'$ belong to $\mathrm{Ker}\,\Phi$. Clearly

$$F_{i_0} = (\alpha' - zx')x - (\beta' - zy')y + z(xx' - yy') .$$

Thus $xx' - yy' \in \mathrm{Ker}\,\Phi$, and F_{i_0} is in the ideal product of $\mathrm{Ker}\,\Phi$ and the Z^n-homogeneous maximal ideal of R. This contradicts the minimality of the system $\mathscr{P}$.

(4): (This assertion was essentially obtained in [5].) Since $(F_1, \cdots, F_d)$ is a Z^n-homogeneous R-sequence,

$$\begin{aligned} k[M \cap \sigma^\vee](-\deg(e(\omega(M \cap \sigma^\vee)))) &\simeq \Omega_{Z^n}(k[M \cap \sigma^\vee]) \\ &\simeq (\Omega_{Z^n}(R/R(F_1, \cdots, F_{d-1}))/F_d\Omega_{Z^n}(R/R(F_1, \cdots, F_{d-1})))(\deg(F_d)) \\ &\simeq (\Omega_{Z^n}(R)/(F_1, \cdots, F_d)\Omega_{Z^n}(R))\Big(\sum_{i=1}^{d} \deg(F_i)\Big) \\ &\simeq (R/R(F_1, \cdots, F_d))\Big(\sum_{i=1}^{d} \deg(F_i) - \sum_{i=1}^{m} \deg(T_i)\Big) \end{aligned}$$

as Z^n-graded R-modules. Hence the identity in (4) follows directly from these isomorphisms. □

LEMMA 3.3. *Let K be a monomial in $\Phi^{-1}(e(\omega(M \cap \sigma^\vee))) \cap \mathscr{T}$. If a monomial $x \in \mathscr{T}$ is not divisible by K in R and satisfies* $\mathrm{supp}(x) = I$, *then there is a $\mathscr{P}$-path from x to K for any $\mathscr{P} \in \mathrm{SYZ}_1(I)$.*

PROOF. Let F_i $(1 \leqq i \leqq d)$ be all elements of a fixed system $\mathscr{P} \in \mathrm{SYZ}_1(I)$. According to (1.3), there exists a monomial x' satisfying $x - Kx' \in \mathrm{Ker}\,\Phi$. Then $x - Kx'$ is expressed as

$$x - Kx' = \sum_{(i,j) \in \mathscr{A}} u_{ij} F_i ,$$

where $\mathscr{Q}$ is a finite subset of $\{1, \cdots, d\} \times \boldsymbol{Z}_+$ and $u_{ij} \in R$, $(i, j) \in \mathscr{Q}$, are nonzero monomials of $\{T_1, \cdots, T_m\}$. Let Θ_{ij}, $(i, j) \in \mathscr{Q}$, denote $\{\log_{\mathscr{T}}(u_{ij}\alpha_{F_i}), \log_{\mathscr{T}}(u_{ij}\beta_{F_i})\}$ and $\mathscr{G}$ be a graph (i.e., a finite one-dimensional simplicial complex) of which the set of vertices is $\mathscr{Q}$ and the set of edges is $\{\{(i, j), (i', j')\} \mid$ distinct (i, j), (i', j') in $\mathscr{Q}$ with $\Theta_{ij} \cap \Theta_{i'j'} \neq \varnothing\}$. Put $\gamma_0 = \log_{\mathscr{T}}(x)$. Let (i_0, j_0) be a vertex of $\mathscr{G}$ satisfying $\gamma_0 \in \Theta_{i_0 j_0}$ and $\mathscr{G}'$ a maximal connected subgraph of $\mathscr{G}$ containing (i_0, j_0) as a vertex.

Suppose $\log_{\mathscr{T}}(Kx') \notin \Theta_{ij}$ for every vertex (i, j) of $\mathscr{G}'$. Then we have

$$x = \sum_{(i,j) \in \mathrm{VER}(\mathscr{G}')} u_{ij} F_i \in \operatorname{Ker} \Phi ,$$

where $\mathrm{VER}(\mathscr{G}')$ denotes the set of all vertices of $\mathscr{G}'$. Hence a T_i must belong to $\operatorname{Ker} \Phi$, a contradiction.

From $\mathscr{G}'$ we choose a path, which is represented as in Figure 1 in

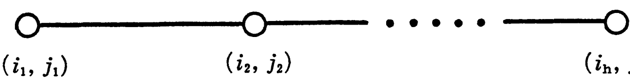

FIGURE 1

an obvious way, of the shortest length in such a way that $\gamma_0 \in \Theta_{i_1 j_1}$ and $\log_{\mathscr{T}}(Kx') \in \Theta_{i_h j_h}$. Put $\gamma_h = \log_{\mathscr{T}}(Kx')$. For each $1 \leqq q < h$, we see that $\Theta_{i_q j_q}$ and $\Theta_{i_{q+1} j_{q+1}}$ intersect exactly at one element and denote by γ_q this element. Then $\Theta_{i_q j_q} = \{\gamma_{q-1}, \gamma_q\}$ $(1 \leqq q \leqq h)$. Put

$$L_q = T^{\gamma_q - \log_{\mathscr{T}}(u_{i_q j_q})} , \quad L'_q = T^{\gamma_{q-1} - \log_{\mathscr{T}}(u_{i_q j_q})}$$

for $1 \leqq q \leqq h$. Clearly $L_q - L'_q$ or $L'_q - L_q$ belongs to $\mathscr{P}$. Since we inductively have $\log_{\mathscr{T}}(x \prod_{u=1}^{q} L_u / (\prod_{u=1}^{q} L'_u)) = \gamma_q$, the sequence $((L_1, L'_1), \cdots, (L_h, L'_h))$ is a $\mathscr{P}$-path from x to K. $\square$

PROPOSITION 3.4. *For any* $I' \subseteqq I$ *and* $K \in \Phi^{-1}(e(\omega(\mathscr{S}_{I'}))) \cap \mathscr{T}$, *there exists a system* $\mathscr{P} \in \mathrm{SYZ}_1(I')$ *which is* (I', K)*-tiled.*

PROOF. Let us prove this by induction on $\mathrm{card}(I')$. When $k[\mathscr{S}_{I'}]$ is a polynomial ring over k, by Lemma 3.2, we see that $\Phi^{-1}(e(\omega(\mathscr{S}_{I'}))) = \prod_{i \in J_{I'}} T_i$, $\mathrm{SYZ}_1(I') = \{\varnothing\}$ and this empty system $\varnothing \in \mathrm{SYZ}_1(I')$ is (I', K)-tiled. Thus we may assume that $I = I'$ (recall that $k[M \cap \sigma^\vee]$ is assumed to be a *singular* LCI). For an arbitrary $\mathscr{P} \in \mathrm{SYZ}_1(I)$, let $\Delta_{\mathscr{P}}$ (resp. $\nabla_{\mathscr{P}}$) denote the fraction $\prod_{i \in J} T_i / \Gamma_{\mathscr{P}}$ (resp. $K \prod_{F \in \mathscr{P}} \alpha_F / \Gamma_{\mathscr{P}}$) in R where $\Gamma_{\mathscr{P}}$ is a product of distinct T_i's such that $i \in \bigcup_{F \in \mathscr{P}} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K)$. Let $\mathscr{Q} \in \mathrm{SYZ}_1(I)$ be a system satisfying $\|\deg(\Delta_{\mathscr{Q}})\| = \min\{\|\deg(\Delta_{\mathscr{P}})\| \mid \mathscr{P} \in \mathrm{SYZ}_1(I)\}$. When $\|\deg(\Delta_{\mathscr{Q}})\| = 0$, we have $J = \bigcup_{F \in \mathscr{Q}} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K)$ and, by (4) of Lemma 3.2, easily infer that $\mathscr{Q}$ is (I, K)-tiled. So let us assume that $\|\deg(\Delta_{\mathscr{Q}})\| > 0$. Put $I'' = \mathrm{supp}(\Delta_{\mathscr{Q}})$.

Suppose $I'' = I$. According to Lemma 3.3, there is a $\mathscr{Q}$-path $((L_1, L_1'), (L_2, L_2'), \cdots, (L_h, L_h'))$ from $\Delta_{\mathscr{Q}}$ to K. Since $\Delta_{\mathscr{Q}}$ is divisible by L_1' in R, L_1' is square-free and $L_1' = \beta_F$ for some $F \in \mathscr{Q}$. Put $\mathscr{Q}^{(1)} = (\mathscr{Q} \setminus \{L_1 - L_1'\}) \cup \{L_1' - L_1\}$. Obviously $\mathscr{Q}^{(1)} \in \mathrm{SYZ}_1(I)$, and

$$\mathrm{supp}_{\mathscr{T}}(\Delta_{\mathscr{Q}^{(1)}}) \subseteqq (\mathrm{supp}_{\mathscr{T}}(\Delta_{\mathscr{Q}} \cup \mathrm{supp}_{\mathscr{T}}(L_1)) \setminus \mathrm{supp}_{\mathscr{T}}(L_1') \subseteqq \mathrm{supp}_{\mathscr{T}}(\Delta_{\mathscr{Q}} L_1/L_1') \ .$$

Hence $\Delta_{\mathscr{Q}} L_1/L_1'$ is divisible by $\Delta_{\mathscr{Q}^{(1)}}$ in R, which shows $\|\deg(\Delta_{\mathscr{Q}^{(1)}})\| \leqq \|\deg(\Delta_{\mathscr{Q}} L_1/L_1')\| = \|\deg(\Delta_{\mathscr{Q}})\|$. By the choice of $\mathscr{Q}$, we must have $\Delta_{\mathscr{Q}^{(1)}} = \Delta_{\mathscr{Q}} L_1/L_1'$. Obviously $((L_2, L_2'), \cdots, (L_h, L_h'))$ is a $\mathscr{Q}^{(1)}$-path from $\Delta_{\mathscr{Q}^{(1)}}$ to K. For $i < h$, let us inductively put $\mathscr{Q}^{(i+1)} = (\mathscr{Q}^{(i)} \setminus \{L_{i+1} - L_{i+1}'\}) \cup \{L_{i+1}' - L_{i+1}\}$. Then we can similarly and inductively show that $L_{i+1} - L_{i+1}' \in \mathscr{Q}^{(i)}$, $\mathscr{Q}^{(i+1)} \in \mathrm{SYZ}_1(I)$ and $\Delta_{\mathscr{Q}^{(i+1)}} = \Delta_{\mathscr{Q}^{(i)}} L_{i+1}/L_{i+1}'$. On the other hand, K is a divisor of $\Delta_{\mathscr{Q}} \prod_{i=1}^{h} L_i/(\prod_{i=1}^{h} L_i')$ in R. But this contradicts the definition of $\Delta_{\mathscr{Q}^{(h)}}$, because $\Delta_{\mathscr{Q}^{(h)}} = \Delta_{\mathscr{Q}^{(h-1)}} L_h/L_h' = \cdots = \Delta_{\mathscr{Q}} \prod_{i=1}^{h} L_i/(\prod_{i=1}^{h} L_i')$. Thus I'' is a non-empty proper subset of I.

For any $\mathscr{P} \in \mathrm{SYZ}_1(I)$ and $j \in J$, $j \in \mathrm{supp}_{\mathscr{T}}(\nabla_{\mathscr{P}})$ if and only if the square of T_j is a divisor of $K \prod_{F \in \mathscr{P}} \alpha_F$ in R. Moreover, by the identity in (4) of Lemma 3.2, we have $\deg(\Delta_{\mathscr{P}}) = \deg(\nabla_{\mathscr{P}})$ and $\mathrm{supp}(\Delta_{\mathscr{P}}) = \mathrm{supp}(\nabla_{\mathscr{P}})$. Clearly $\Delta_{\mathscr{Q}}$ is a divisor of $\prod_{j \in J_{I''}} T_j$ in R and

$$(*) \qquad J \setminus J_{I''} \subseteqq \bigcup_{F \in \mathscr{Q} \setminus \mathscr{Q}_{\cap I''}} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K) \ .$$

Assume $\mathscr{Q}_{\cap I''} = \emptyset$. By (1) of Lemma 3.2, Φ induces a Z^n-graded k-isomorphism $k[T_j \mid j \in J_{I''}] \simeq k[\mathscr{S}_{I''}]$. Thus we have $\Phi(\prod_{j \in J_{I''}} T_j) = e(\omega(S_{I''}))$, which implies that $\prod_{j \in J_{I''}} T_j$ is a divisor of $\Delta_{\mathscr{Q}}$ in R (cf. (1.3)), i.e., $\prod_{j \in J_{I''}} T_j = \Delta_{\mathscr{Q}}$. Consequently, $J \setminus J_{I''} = \bigcup_{F \in \mathscr{Q}} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K)$. Since $\nabla_{\mathscr{Q}}$ is a divisor of $K \prod_{F \in \mathscr{Q}} \alpha_F$ in R, $\mathrm{supp}(\nabla_{\mathscr{Q}})$ does not coincide with I'', a contradiction. Hence $\mathscr{Q}_{\cap I''} \neq \emptyset$, i.e., $k[\mathscr{S}_{I''}]$ is not a polynomial ring over k (cf. (1) of Lemma 3.2).

Let K' be any monomial in $\Phi^{-1}(e(\omega(\mathscr{S}_{I''}))) \cap \mathscr{T}$. By our induction hypothesis, there exists a non-empty system $\mathscr{P}_1 \in \mathrm{SYZ}_1(I'')$ which is (I'', K')-tiled. Put $\mathscr{Q}' = \mathscr{P}_1 \cup (\mathscr{Q} \setminus \mathscr{Q}_{\cap I''})$. Clearly $\mathscr{Q}' \in \mathrm{SYZ}_1(I)$ (cf. (1) of Lemma 3.2) and $\mathrm{supp}_{\mathscr{T}}(\Delta_{\mathscr{Q}'})$ is contained in

$$(J_{I''} \setminus (\bigcup_{F \in \mathscr{P}_1} \mathrm{supp}_{\mathscr{T}}(\alpha_F))) \cup ((J \setminus J_{I''}) \setminus (\bigcup_{F \in \mathscr{Q} \setminus \mathscr{Q}_{\cap I''}} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K))) \ .$$

By (*) and the definition of $\mathscr{P}_1$, we see that the last set coincides with $\mathrm{supp}_{\mathscr{T}}(K')$. As $\Phi(\Delta_{\mathscr{Q}}) \in \Omega_{Z^n}(k[\mathscr{S}_{I''}])$ and $\Delta_{\mathscr{Q}'}$ is square-free, $\|\deg(\Delta_{\mathscr{Q}'})\| \leqq \|\deg(K')\| \leqq \|\deg(\Delta_{\mathscr{Q}})\|$. From the choice of $\mathscr{Q}$, we deduce that $\Delta_{\mathscr{Q}} \in \Phi^{-1}(e(\omega(\mathscr{S}_{I''}))) \cap \mathscr{T}$ and $\Delta_{\mathscr{Q}'} = K'$. The last equality implies

$$(**) \qquad \mathrm{supp}_{\mathscr{T}}(K') \cap (\bigcup_{F \in \mathscr{Q} \setminus \mathscr{Q}_{\cap I''}} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K)) = \emptyset \ ,$$

which is independent of the choice of $K' \in \Phi^{-1}(e(\omega(\mathscr{S}_{I''}))) \cap \mathscr{T}$.

Let $\mathscr{P}_2 \in \mathrm{SYZ}_1(I'')$ be a $(I'', \Delta_{\mathscr{Q}})$-tiled system and set $\mathscr{Q}'' = \mathscr{P}_2 \cup (\mathscr{Q} \setminus \mathscr{Q}_{\cap I''}) \in \mathrm{SYZ}_1(I)$. By the above observations, $\Delta_{\mathscr{Q}''} = \Delta_{\mathscr{Q}}$ and $\nabla_{\mathscr{Q}''} \in \Phi^{-1}(e(\omega(\mathscr{S}_{I''}))) \cap \mathscr{T}$ (recall that $\deg(\Delta_{\mathscr{Q}''}) = \deg(\nabla_{\mathscr{Q}''})$). Then, applying (**) to $K' = \nabla_{\mathscr{Q}''}$, we have

$$\mathrm{supp}_{\mathscr{T}}(\nabla_{\mathscr{Q}''}) \cap \Big(\bigcup_{F \in \mathscr{Q}'' \setminus \mathscr{P}_2} \mathrm{supp}_{\mathscr{T}}(\alpha_F) \cup \mathrm{supp}_{\mathscr{T}}(K)\Big) = \varnothing .$$

Consequently, $\nabla_{\mathscr{Q}''}$ can be expressed as a product of all T_i's whose squares are divisors of $\prod_{F \in \mathscr{P}_2} \alpha_F$. Since $\mathrm{supp}_{\mathscr{T}}(\alpha_F)$, $F \in \mathscr{P}_2$, are disjoint, we must have $\deg(\Delta_{\mathscr{Q}}) = \deg(\Delta_{\mathscr{Q}''}) = \deg(\nabla_{\mathscr{Q}''}) = 0$, a contradiction. □

We now fix a monomial $K \in \Phi^{-1}(e(\omega(M \cap \sigma^{\vee}))) \cap \mathscr{T}$ and a (I, K)-tiled system $\mathscr{P} \in \mathrm{SYZ}_1(I)$.

PROPOSITION 3.5. $\mathscr{G}_{\mathscr{P}}$ is acyclic.

PROOF. Assume that $\mathscr{G}_{\mathscr{P}}$ is not acyclic. Let $(F_1, \cdots, F_u)$ $(u > 1)$ be a directed circuit of the shortest length u in $\mathscr{G}_{\mathscr{P}}$. Then we see that $i \equiv j - 1 \pmod u$ for $1 \leqq i, j \leqq u$ if $(F_i, F_j) \in \mathrm{DED}(\mathscr{G}_{\mathscr{P}})$. Let x_i, $1 \leqq i < u$, (resp. x_u) be a product of all T_j's with $j \in \mathrm{supp}_{\mathscr{T}}(\alpha_{F_i}) \cap \mathrm{supp}_{\mathscr{T}}(\beta_{F_{i+1}})$ (resp. $j \in \mathrm{supp}_{\mathscr{T}}(\alpha_{F_u}) \cap \mathrm{supp}_{\mathscr{T}}(\beta_{F_1})$) and put $\alpha'_i = \alpha_{F_i}/x_i$ $(1 \leqq i \leqq u)$, $\beta'_i = \beta_{F_i}/x_{i-1}$ $(1 < i \leqq u)$ and $\beta'_1 = \beta_{F_1}/x_u$. Clearly

$$\begin{aligned} F_u \prod_{i=1}^{u-1} \alpha'_i &\equiv x_u \prod_{i=1}^{u} \alpha'_i + x_{u-2}\beta'_{u-1}\beta'_u \prod_{i=1}^{u-2} \alpha'_i \pmod{RF_{u-1}} \\ &\equiv x_u \prod_{i=1}^{u} \alpha'_i - x_{u-3}\beta'_{u-2}\beta'_{u-1}\beta'_u \prod_{i=1}^{u-3} \alpha'_i \pmod{R(F_{u-2}, F_{u-1})} \\ &\cdots\cdots\cdots\cdots \\ &\equiv x_u\Big(\prod_{i=1}^{u} \alpha'_i + (-1)^u \prod_{i=1}^{u} \beta'_i\Big) \pmod{R(F_1, \cdots, F_{u-1})} , \end{aligned}$$

and hence the prime ideal $R(F_1, \cdots, F_u)$ contains $\prod_{i=1}^u \alpha'_i + (-1)^u \prod_{i=1}^u \beta'_i$. As $\deg(\prod_{i=1}^u \alpha'_i) = \deg(\prod_{i=1}^u \beta'_i)$ and $\sum_{i=1}^u F_i \neq 0$, we see that $\prod_{i=1}^u \alpha'_i \neq 1$. Moreover, $\prod_{i=1}^u \alpha'_i$ and $\prod_{i=1}^u \beta'_i$ are relatively prime in R. Thus $\prod_{i=1}^u \alpha'_i$ is divisible by $\alpha_{F_{i_0}}$ or $\beta_{F_{i_0}}$ in R for some $1 \leqq i_0 \leqq u$ (recall that $\prod_{i=1}^u \alpha'_i + (-1)^u \prod_{i=1}^u \beta'_i \in R(F_1, \cdots, F_u)$). Since $\mathrm{supp}_{\mathscr{T}}(\alpha_{F_i})$'s are disjoint and $\mathrm{supp}_{\mathscr{T}}(\alpha'_i) \neq \mathrm{supp}_{\mathscr{T}}(\alpha_{F_i})$, the first case does not occur. Consequently, we can choose an index i_1 with $1 \leqq i_1 \leqq u$ in such a way that $\mathrm{supp}_{\mathscr{T}}(\alpha'_{i_1}) \cap \mathrm{supp}_{\mathscr{T}}(\beta_{F_{i_0}}) \neq \varnothing$. As $(F_{i_1}, F_{i_0}) \in \mathrm{DED}(\mathscr{G}_{\mathscr{P}})$, we must have $i_1 \equiv i_0 - 1 \pmod u$, which contradicts the definition of α'_{i_1}. □

PROOF OF THEOREM 3.1. Let us complete the proof of the theorem by induction on r. Thanks to Lemma 2.6 and Proposition 3.5, we can define a linear ordering $\leqq$ on $\mathscr{P}$ satisfying $\mathfrak{i}(e) \prec \mathfrak{f}(e)$ for every $e \in \mathrm{DED}(\mathscr{G}_{\mathscr{P}})$.

Let F_d be the largest element of $\mathscr{P}$ with respect to this ordering $\leqq$ and put $J_{d-1} = J\setminus \mathrm{supp}_{\mathscr{T}}(\alpha_{F_d})$ and $\mathscr{S}_{d-1} = \{s \in M\cap\sigma^\vee | e(s) \in \Phi(k[T_j | j \in J_{d-1}] \cap \mathscr{T})\}$, respectively. From the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\Phi_{|A}\otimes 1) & \longrightarrow & A\otimes_k B & \xrightarrow[\Phi_{|A}\otimes 1]{} & k[\mathscr{S}_{d-1}]\otimes_k B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{can.}} & & \downarrow{\scriptstyle 1\otimes\Phi_{|B}} & & \\ 0 & \longrightarrow & RP & \longrightarrow & R & \xrightarrow[\Phi]{} & k[M\cap\sigma^\vee] & \longrightarrow & 0 \end{array}$$

with exact rows, we immediately deduce

$$\begin{aligned} k[M\cap\sigma^\vee] &\simeq k[\mathscr{S}_{d-1}]\otimes_k B/(k[\mathscr{S}_{d-1}]\otimes_k B(1\otimes\alpha_{F_d} - \Phi(\beta_{F_d})\otimes 1)) \\ &\simeq k\Big[\mathscr{S}_{d-1}\int_u e^{-1}(\Phi(\beta_{F_d}))\Big], \end{aligned}$$

where $A=k[T_j | j \in J_{d-1}]$, $B=k[T_j | j \in \mathrm{supp}_{\mathscr{T}}(\alpha_{F_d})]$ and $u=\mathrm{card}(\mathrm{supp}_{\mathscr{T}}(\alpha_{F_d}))-1$. Thus $M\cap\sigma^\vee$ is isomorphic to $\mathscr{S}_{d-1}\int_u e^{-1}(\Phi(\beta_{F_d}))$ as a monoid. Hence the assertion follows from (1.1), Lemma 2.4 and our induction hypothesis. $\square$

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Note added in proof. By a slight modification in Lemma 3.3, we can somewhat simplify the proof of Proposition 3.4.