ON THE FOURIER-BOREL TRANSFORMATIONS OF ANALYTIC FUNCTIONALS ON THE COMPLEX SPHERE

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Introduction. Let \( \mathcal{O}(C^{d+1}) \) and \( \text{Exp}(C^{d+1}) \) be the spaces of entire functions on \( C^{d+1} \) and entire functions of exponential type, respectively. \( \mathcal{O}'(C^{d+1}) \) and \( \text{Exp}'(C^{d+1}) \) are the spaces dual to \( \mathcal{O}(C^{d+1}) \) and \( \text{Exp}(C^{d+1}) \), respectively. For \( T \in \text{Exp}'(C^{d+1}) \) the Fourier-Borel transformation \( P_\lambda \) is defined by

\[
P_\lambda T(z) = \langle T, \exp(i\lambda \zeta \cdot z) \rangle \quad \text{for} \quad z \in C^{d+1},
\]

where \( \lambda \in C, \lambda \neq 0 \), is a fixed constant (Hashizume, Kowata, Minemura and Okamoto [2]). Martineau [4] determined the images of \( \text{Exp}'(C^{d+1}) \) and some functional spaces on \( C^{d+1} \) by the Fourier-Borel transformation \( P_\lambda \).

Let \( S = S^d \) be the unit sphere in \( R^{d+1} \) and \( \tilde{S} \) denote the complex sphere in \( C^{d+1} \). We put \( \tilde{S}(r) = \{ z \in \tilde{S}; L(z) < r \} \) and \( \tilde{S}[r] = \{ z \in \tilde{S}; L(z) \leq r \} \), where \( L(z) \) is the Lie norm on \( C^{d+1} \). \( \mathcal{O}(\tilde{S}), \mathcal{O}(\tilde{S}(r)) \) and \( \mathcal{O}(\tilde{S}[r]) \) denote the spaces of holomorphic functions on \( \tilde{S}, \tilde{S}(r) \), and \( \tilde{S}[r] \), respectively. \( \text{Exp}(\tilde{S}) \) denotes the restriction of \( \text{Exp}(C^{d+1}) \) to \( \tilde{S} \). \( \text{Exp}'(\tilde{S}), \mathcal{O}'(\tilde{S}), \mathcal{O}'(\tilde{S}(r)) \) and \( \mathcal{O}'(\tilde{S}[r]) \) are the spaces dual to \( \text{Exp}(\tilde{S}), \mathcal{O}(\tilde{S}), \mathcal{O}(\tilde{S}(r)) \) and \( \mathcal{O}(\tilde{S}[r]) \), respectively. \( \text{Exp}'(\tilde{S}) \) can be regarded as a subspace of \( \text{Exp}'(C^{d+1}) \).

Morimoto [7] determined the images of \( \mathcal{O}'(\tilde{S}) \) and \( \mathcal{O}'(\tilde{S}) \) by the Fourier-Borel transformation \( P_\lambda \) (Theorem 1.2). In this paper we will determine the images of \( \mathcal{O}'(\tilde{S}(r)) \) and \( \mathcal{O}'(\tilde{S}[r]) \) by the Fourier-Borel transformation \( P_\lambda \). The images are characterized explicitly in terms of the dual Lie norm (Theorem 3.1).

Consider a complex cone \( M = \{ z \in C^{d+1}; \sum_{j=1}^{d+1} z_j^2 = 0, z \neq 0 \} \), which can be identified with the cotangent bundle of \( S \) minus its zero section. We define for \( f' \in \text{Exp}'(\tilde{S}) \)

\[
Ff'(z) = \langle f', \exp(\xi \cdot z) \rangle \quad (z \in M).
\]

\( Ff' \) is the restriction of \( P_\lambda f' \) to \( M \). Ii [3] determined the images of \( H_{n,d} \) by \( F \), where \( H_{n,d} \) is the space of spherical harmonics of degree \( n \) in dimension \( d+1 \). Moreover if \( d \) is even, Ii [3] characterized the image of \( L^p(S) \) under this mapping \( F \). In this paper we determine the image of \( L^p(S) \) for odd \( d \) (Theorem 2.4). We also determine the images of
Exp'(S), \( \mathcal{O}'(S) \), \( \mathcal{O}'(S[r]) \), \( \mathcal{O}(S) \), \( \mathcal{O}(S[r]) \) and \( \mathcal{O}(S) \) (Theorem 2.1).

To prove our main theorems, we need, among others, Lemmas 1.3 and 1.4. Although Lemma 1.4 was proved in Li [3], we give here a new proof to it.

The outline of this paper was announced in [11]. The author would like to thank Professor M. Morimoto for his helpful suggestions.

1. Preliminaries. Let \( d \) be a positive integer and \( d \geq 2 \). \( S = S^d = \{ x \in \mathbb{R}^{d+1}; \| x \| = 1 \} \) denotes the unit sphere in \( \mathbb{R}^{d+1} \), where \( \| x \|^2 = x_1^2 + x_2^2 + \cdots + x_{d+1}^2 \). \( ds \) denotes the unique \( O(d+1) \) invariant measure on \( S \) with \( \int_S ds = 1 \), where \( O(k) \) is the orthogonal group of degree \( k \). \( \| \|_2 \) is the \( L^2 \)-norm on \( S \). \( H_{n,d} \) is the space of spherical harmonics of degree \( n \) in dimension \( d + 1 \). For spherical harmonics, see Müller [8].

The Lie norm \( L(z) \) and the dual Lie norm \( L^*(z) \) on \( C^{d+1} \) are defined as follows:

\[
L(z) = L(x + iy) := \frac{1}{2} \left( ||x||^2 + ||y||^2 + 2(||x||^2 ||y||^2 - (x \cdot y)^2)^{1/2} \right),
\]

\[
L^*(z) = L^*(x + iy) := \sup \{ ||\xi \cdot z||; L(\xi) \leq 1 \}
= (1/\sqrt{2}) \left( ||x||^2 + ||y||^2 + ((||x||^2 - ||y||^2)^2 + 4(x \cdot y)^2)^{1/2} \right)^{1/2},
\]

where \( z, \xi \in C^{d+1} \), and \( z \cdot \xi = z_1 \xi_1 + z_2 \xi_2 + \cdots + z_{d+1} \xi_{d+1}, \) \( x, y \in \mathbb{R}^{d+1} \), (see Drużkowski [1]).

We put

\[
\mathcal{B}(r) := \{ z \in C^{d+1}; L(z) < r \} \quad \text{for} \quad 0 < r \leq \infty
\]

and

\[
\mathcal{B}[r] := \{ z \in C^{d+1}; L(z) \leq r \} \quad \text{for} \quad 0 \leq r < \infty.
\]

Let us denote by \( \mathcal{O}(\mathcal{B}(r)) \) the space of holomorphic functions on \( \mathcal{B}(r) \). Then \( \mathcal{O}(\mathcal{B}(r)) \) is an FS space. \( \mathcal{O}(\mathcal{B}(\infty)) = \mathcal{O}(C^{d+1}) \) is the space of entire functions on \( C^{d+1} \). Let us define

\[
\mathcal{O}(\mathcal{B}[r]) := \text{ind lim}_{r > r} \mathcal{O}(\mathcal{B}(r)).
\]

Then \( \mathcal{O}(\mathcal{B}[r]) \) is a DFS space.

Let \( N \) be a norm on \( C^{d+1} \). For \( r > 0 \) we put

\[
X_{r,N} := \{ f \in \mathcal{O}(C^{d+1}); \sup_{z \in C^{d+1}} |f(z)| \exp(-rN(z)) < \infty \}.
\]

Then \( X_{r,N} \) is a Banach space with respect to the norm

\[
\| f \|_{r,N} = \sup_{z \in C^{d+1}} |f(z)| \exp(-rN(z)).
\]
Define

\[ \text{Exp}(C^{d+1}: (r: N)) := \text{proj lim}_{r' > r} X_{r', N}, \quad \text{for } 0 \leq r < \infty, \]
\[ \text{Exp}(C^{d+1}: [r: N]) := \text{ind lim}_{r' < r} X_{r', N}, \quad \text{for } 0 < r \leq \infty. \]

\[ \text{Exp}(C^{d+1}: (r: N)) \text{ is an FS space and } \text{Exp}(C^{d+1}: [r: N]) \text{ is a DFS space.} \]

\[ \text{Exp}(C^{d+1}) = \text{Exp}(C^{d+1}: [\infty: N]) \text{ is independent of the choice of the norm } N \text{ and is called the space of entire functions of exponential type.} \]

\[ \text{Exp}(C^{d+1}), \text{Exp}'(C^{d+1}), \text{Exp}'(C^{d+1}: [r: N]), \text{Exp}'(B[r]) \text{ denote the spaces dual to} \]
\[ \text{Exp}(C^{d+1}), \text{Exp}'(C^{d+1}), \text{Exp}'(B[r]) \text{ and } \text{Exp}'(B[r]), \text{respectively.} \]

\[ S = \{ z \in C^{d+1}; z_1^2 + z_2^2 + \cdots + z_{d+1}^2 = 1 \} \text{ is the complex sphere. For } \]
\[ 1 < r \leq \infty \text{ we put} \]
\[ S(r) := B(r) \cap \tilde{S} = \{ z = x + iy \in \tilde{S}; \|y\| < (r - 1/r)/2 \} \]
and for \[ 1 \leq r < \infty \]
\[ S[r] = B[r] \cap \tilde{S} = \{ z = x + iy \in \tilde{S}; \|y\| \leq (r - 1/r)/2 \}. \]

It is clear that \[ S = S[1] \text{ and } \tilde{S} = S(\infty). \]

Let us denote by \( \mathcal{O}(S(r)) \) the space of holomorphic functions on \( S(r) \) equipped with the topology of uniform convergence on every compact subset of \( S(r). \) We put
\[ \mathcal{O}(S[r]) := \text{ind lim}_{r' > r} \mathcal{O}(S(r')). \]

\( \mathcal{O}(S[r]) \) is an FS space and \( \mathcal{O}(S[r]) \) is a DFS space. \( \mathcal{O}(S[1]) \) is the space of real analytic functions on \( S. \) \( \text{Exp}(S) \) denotes the restriction to \( \tilde{S} \) of \( \text{Exp}(C^{d+1}). \) \( \mathcal{O}'(S(r)), \mathcal{O}'(S[r]) \) and \( \text{Exp}'(S) \) denote the spaces dual to \( \mathcal{O}(S(r)), \mathcal{O}(S[r]) \) and \( \text{Exp}(S), \) respectively. We have the following
sequence of functional spaces on \( \tilde{S} \) (cf. Morimoto [6], [7]):

\[ (1.1) \quad \text{Exp}'(S) \supset \mathcal{O}'(S) \supset \mathcal{O}'(S[r]) \supset \mathcal{O}'(S(r)) \supset \mathcal{O}'(S[1]). \]

If \( f \) is a function or a functional on \( S, \) we denote by \( f_n \) the \( n \)-th spherical harmonic component of \( f; \)
\[ (1.2) \quad f_n(s) = N(n, d) \langle f, P_{n,d}(\cdot s) \rangle \quad \text{for } s \in S, \]
where
\[ (1.3) \quad N(n, d) = \dim H_{n,d} = \frac{(2n + d - 1)(n + d - 2)!}{n!(d - 1)!} \]
and \( P_{n,d} \) is the Legendre polynomial of degree \( n \) and of dimension \( d + 1. \)

We put \( L_\alpha(x) = |x|^\alpha P_{n,d}(\alpha \cdot x/|x|) \) for fixed \( \alpha \in S. \) Then \( L_\alpha \) is the unique homogeneous harmonic polynomial of degree \( n \) with the following
properties:

\[(1.4) \quad L_n(Ax) = L_n(x) \quad \text{for all} \quad A \in O(d + 1) \quad \text{such that} \quad A\alpha = \alpha.\]

\[(1.5) \quad L_n(\alpha) = 1.\]

We see that \(f_n\) belongs to \(H_n,d\) for \(n = 0, 1, \ldots\). We can characterize the functional spaces in (1.1) by the behavior of the spherical harmonic development as follows.

**Lemma 1.1** (Morimoto [7, Theorems 5.1 and 6.1]). If \(f_n\) is the \(n\)-th spherical harmonic component of \(f\), then

\[(1.6) \quad f \in \text{Exp}'(\tilde{S}) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} = 0,\]

\[(1.7) \quad f \in \mathcal{O}'(\tilde{S}) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} < \infty,\]

\[(1.8) \quad f \in \mathcal{O}'(\tilde{S}[r]) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} \leq r \quad (1 \leq r < \infty),\]

\[(1.9) \quad f \in \mathcal{O}'(\tilde{S}(r)) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} < r \quad (1 < r \leq \infty),\]

\[(1.10) \quad f \in L^2(\tilde{S}) \iff \{|f_n|\}_{n=0,1,2,\ldots} \in l^2,\]

\[(1.11) \quad f \in \mathcal{O}(\tilde{S}[r]) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} \leq 1/r \quad (1 < r \leq \infty),\]

\[(1.12) \quad f \in \mathcal{O}(\tilde{S}(r)) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} < 1/r \quad (1 \leq r < \infty),\]

\[(1.13) \quad f \in \mathcal{O}(\tilde{S}) \iff \limsup_{n \to \infty} \|f_n\|_{2/n}^{1/n} = 0.\]

The Fourier-Borel transformation \(P_1\) for a functional \(T \in \text{Exp}'(C^{d+1})\) is defined by

\[P_1 T(z) := \langle T, \exp(i\lambda \xi \cdot z) \rangle \quad \text{for} \quad z \in C^{d+1},\]

where \(\lambda \in C, \lambda \neq 0,\) is a fixed constant. We define the transformation \(P_i\) for a functional \(f' \in \text{Exp}'(\tilde{S})\) by

\[P_i f'(x) := \langle f', \exp i\lambda(\xi \cdot z) \rangle.\]

The following is known:

**Theorem 1.2** (Morimoto [7, Theorem 7.1]). The transformation \(P_i\) establishes the linear topological isomorphisms

\[(1.14) \quad P_i : \text{Exp}'(\tilde{S}) \sim \mathcal{O}_1(C^{d+1}),\]

\[(1.15) \quad P_i : \mathcal{O}(\tilde{S}) \sim \text{Exp}_1(C^{d+1}),\]

where we put
FOURIER-BOREL TRANSFORMATIONS

\[ \mathcal{O}(C^{d+1}) := \{ F \in \mathcal{O}(C^{d+1}); (\Delta_x + \lambda^2)F(z) = 0 \}, \]

\[ \text{Exp}_d(C^{d+1}) := \text{Exp}(C^{d+1}) \cap \mathcal{O}(C^{d+1}), \]

and \( \Delta_x = (\partial/\partial z_x)^2 + (\partial/\partial z_y)^2 + \cdots + (\partial/\partial z_{d+1})^2. \)

We define a complex cone \( M \) by

\[ M = \{ z \in C^{d+1}; z_1^2 + z_2^2 + \cdots + z_{d+1}^2 = 0, z \neq 0 \}. \]

\( M \) is identified with the cotangent bundle of \( S \) minus its zero section (cf. II [3], Rawnsley [9], [10]). \( P_n(C^{d+1}) \) denotes the space of homogeneous polynomials of degree \( n \) on \( C^{d+1} \). \( \text{Holo}(M) \) and \( P_n(M) \) denote the restriction to \( M \) of \( \mathcal{O}(C^{d+1}) \) and \( P_n(C^{d+1}) \), respectively. We define the subset \( N \) of \( M \) by

\[ N = \{ z = x + iy \in M; ||x|| = ||y|| = 1 \}, \]

where \( x, y \in R^{d+1} \). The unit cotangent bundle to \( S \) is identified with the subset \( N \) and we have \( N \cong O(d+1)/O(d-1) \). \( dN \) denotes the unique \( O(d+1) \) invariant measure on \( N \) with \( \int_N 1dN = 1 \). We define the inner product

\[ \langle \varphi, \psi \rangle_N := \int_N \varphi(z)\overline{\psi(z)}dN \]

and the norm

\[ ||\varphi||_N = \langle \varphi, \varphi \rangle_N^{1/2}. \]

**Lemma 1.3.** If \( \alpha \) and \( \beta \) belong to \( S \), the following formula is valid.

\[ \int_N (z \cdot \alpha)^n(z \cdot \beta)^\overline{m}dN = \frac{n! \Gamma((d+1)/2)}{\Gamma(n + (d+1)/2) \delta_{nm}P_n(\alpha \cdot \beta)}. \]

**Proof.** Denote by \( F(\alpha, \beta) \) the left hand side of (1.16). Then for any orthogonal matrix \( A \)

\[ F(A\alpha, A\beta) = \int_N (z \cdot A\alpha)^n(z \cdot A\beta)^\overline{m}dN \]

\[ = \int_{z = x + iy \in N} (x \cdot A\alpha + iy \cdot A\alpha)^n(x \cdot A\beta + iy \cdot A\beta)^\overline{m}dN \]

\[ = \int_N (A^{-1}z \cdot \alpha)^n(A^{-1}z \cdot \beta)^\overline{m}dN. \]

Since \( dN \) is \( O(d+1) \)-invariant we get

\[ (1.17) \quad F(A\alpha, A\beta) = F(\alpha, \beta) \]

for any \( A \in O(d+1) \). As a function of \( \alpha \), \( F(\alpha, \beta) \) belongs to \( H_{n,d} \), since
Similarly, as a function of $\beta$, $F(\alpha, \beta)$ belongs to $H_{n,d}$.

Suppose $n \neq m$. There exists an $A \in O(d + 1)$ such that $A\alpha = \beta$ and $A\beta = \alpha$. Then (1.17) gives

(1.18)  \quad F(\alpha, \beta) = F(\beta, \alpha).

If we fix $\alpha$, (1.18) implies that $F(\alpha, \beta) \in H_{n,d} \cap H_{m,d}$. Since $H_{n,d} \cap H_{m,d} = \{0\}$, we have

(1.19)  \quad F(\alpha, \beta) = 0 \quad \text{if} \quad n \neq m.

Next we assume $n = m$. For all $A \in O(d + 1)$ such that $A\alpha = \alpha$ we have from (1.17) $F(\alpha, A\beta) = F(A\alpha, A\beta) = F(\alpha, \beta)$. Therefore $F(\alpha, \beta)$, as a function of $\beta$, is a homogeneous harmonic polynomial of degree $n$ and satisfies (1.4). So we obtain

(1.20)  \quad F(\alpha, \beta) = CP_{n,d}(\alpha \cdot \beta),

where

$$C = \int_N |z \cdot \alpha|^n dN = \frac{n! \Gamma((d + 1)/2)}{\Gamma(n + (d + 1)/2)}$$

(c.f. Rawnsley [10, Appendix]). (1.16) follows from (1.19) and (1.20).

q.e.d.

We put for $f' \in \text{Exp}'(\mathcal{S})$ and $z \in M$.

$$Ff'(z) := \langle f', e^{t \cdot z} \rangle.$$  \quad (1.21)

$Ff'$ is the restriction of $P_{n,d}f'$ to $M$.

Then we have:

**Lemma 1.4** (cf. Ii [3]). *The transformation $F': f' \rightarrow Ff'$ is a one-to-one linear mapping of $H_{n,d}$ onto $P_n(M)$ and we have*

$$\langle f, g \rangle_s = C_n \langle Ff, Fg \rangle_N$$

for $f, g \in H_{n,d}$,

where

$$\langle f, g \rangle_s = \int_S f(s) \overline{g(s)} ds$$

and

(1.22)  \quad C_n = \frac{n! \Gamma(n + (d + 1)/2)N(n, d)}{\Gamma((d + 1)/2)}.

**Proof.** It is known that there exists a system of $N(n, d)$ points $\alpha_1, \alpha_2, \ldots, \alpha_{N(n,d)} \in S$ such that $P_{n,d}(\alpha_k \cdot \ )$, $k = 1, 2, \ldots, N(n, d)$, is a basis
of $H_{n,d}$. Therefore for every $f \in H_{n,d}$, there exist $a_1, a_2, \ldots, a_{N(n,d)} \in \mathbb{C}$ such that

$$f(s) = \sum_{k=1}^{N(n,d)} a_k P_{n,d}(\alpha_k \cdot s) \quad s \in S$$

(see, for example, Müller [8, Theorem 3]). If $z$ belongs to $M$, then

$$Ff(z) = \sum_{k=1}^{N(n,d)} \frac{a_k}{n!} P_{n,d}(\alpha_k \cdot z)^n .$$

Thus $Ff$ belongs to $P_n(M)$. For $f(s) = \sum_{k=1}^{N(n,d)} a_k P_{n,d}(\alpha_k \cdot s)$ and $g(s) = \sum_{k=1}^{N(n,d)} b_k P_{n,d}(\alpha_k \cdot s) \in H_{n,d}$ we have

$$\langle f, g \rangle_S = \sum_{1 \leq k, l \leq N(n,d)} a_k \overline{b_l} \int_S P_{n,d}(\alpha_k \cdot s) P_{n,d}(\alpha_l \cdot s) ds$$

$$= \sum_{1 \leq k, l \leq N(n,d)} \frac{a_k \overline{b_l}}{N(n, d)} P_{n,d}(\alpha_k \cdot \alpha_l) .$$

On the other hand we have from (1.24) and (1.16)

$$\langle Ff, Fg \rangle_N = \sum_{1 \leq k, l \leq N(n,d)} \frac{a_k \overline{b_l}}{n! \frac{N(n, d)}{N(n, d)^2}} \frac{n! \Gamma((d + 1)/2)}{\Gamma(n + (d + 1)/2)} P_{n,d}(\alpha_k \cdot \alpha_l)$$

$$= \frac{\Gamma((d + 1)/2)}{n! N(n, d) \Gamma(n + (d + 1)/2)} \sum_{k=1}^{N(n,d)} \frac{a_k \overline{b_l}}{N(n, d)} P_{n,d}(\alpha_k \cdot \alpha_l) .$$

(1.25) and (1.26) give (1.21) and (1.22). (1.21) shows that $F$ is injective. Since $\dim P_n(M) = N(n, d)$, we can prove the surjectivity of $F$. q.e.d.

2. Integral transformation $F$. Now we define the following subspaces of $\text{Holo}(M)$:

$$\text{Exp}(M, r) := \cap \{ \psi \in \text{Holo}(M); \sup_{z \in M} |\psi(z)| \exp(-r||z||) < \infty \} ,$$

$$\text{Exp}[M, r] := \cup \{ \psi \in \text{Holo}(M); \sup_{z \in M} |\psi(z)| \exp(-r'||z||) < \infty \} ,$$

$$\text{Exp}(M) = \text{Exp}[M, \infty] ,$$

where $||z|| = ||x + iy|| = (||x||^2 + ||y||^2)^{1/2}$ for $x, y \in \mathbb{R}^{d+1}$. 
Our first main theorem in this paper is the following:

**Theorem 2.1.**

1. \( F \) is a one-to-one linear mapping of \( \text{Exp}'(\mathcal{S}) \) onto \( \text{Holo}(M) \).
2. \( F \) is a one-to-one linear mapping of \( \mathcal{O}'(\mathcal{S}) \) onto \( \text{Exp}(M) \).
3. \( F \) is a one-to-one linear mapping of \( \mathcal{O}'(\mathcal{S}[r]) \) onto \( \text{Exp}(M, r/\sqrt{2}) \) for \( 1 \leq r < \infty \).
4. \( F \) is a one-to-one linear mapping of \( \mathcal{O}'(\mathcal{S}(r)) \) onto \( \text{Exp}(M, r/\sqrt{2}) \) for \( 1 < r \leq \infty \).
5. \( F \) is a one-to-one linear mapping of \( \mathcal{O}(\mathcal{S}[r]) \) onto \( \text{Exp}(M, 1/(\sqrt{2}r)) \) for \( 1 \leq r < \infty \).
6. \( F \) is a one-to-one linear mapping of \( \mathcal{O}(\mathcal{S}(r)) \) onto \( \text{Exp}(M, 1/(\sqrt{2}r)) \) for \( 1 < r \leq \infty \).
7. \( F \) is a one-to-one linear mapping of \( \mathcal{O}(\mathcal{S}) \) onto \( \text{Exp}(M, 0) \).

**Proof.** By (1.14) \( F \) is a linear mapping of \( \text{Exp}'(\mathcal{S}) \) into \( \text{Holo}(M) \). Conversely, if \( \psi \) belongs to \( \text{Holo}(M) \) there exist \( \tilde{\psi} \in \mathcal{O}(C^{d+1}) \) and \( \psi_n \in P_n(C^{d+1}) \) \( (n = 0, 1, \ldots, ) \) such that

\[ \tilde{\psi}|_M = \psi \quad \text{and} \quad \tilde{\psi}(z) = \sum_{n=0}^{\infty} \psi_n(z) \]

for any \( z \in C^{d+1} \). It is known that

\[ \tilde{\psi}_n(z) = \frac{1}{2i\pi} \int_{|t|=\rho} \frac{\tilde{\psi}(tz)}{t^{n+1}} dt \]

for any \( \rho > 0 \). We put \( ||\tilde{\psi}||_{\infty, \sqrt{2}\rho} = \sup_{|z|=\sqrt{2}\rho} |\tilde{\psi}(z)| \) and \( \psi_n = \tilde{\psi}_n|_M \). If \( z \) belongs to \( N \) then \( ||z|| = \sqrt{2} \). Hence we get from (2.11)

\[ \sup_{z \in N} |\psi_n(z)| = \sup_{z \in N} \left| \frac{1}{2i\pi} \int_{|t|=\rho} \frac{\tilde{\psi}(tz)}{t^{n+1}} dt \right| \leq \rho^{-n} ||\tilde{\psi}||_{\infty, \sqrt{2}\rho} \cdot \]

Put \( K_n := \sup_{z \in N} |\psi_n(z)| \). (2.12) implies that \( \limsup_{n \to \infty} K_n^{1/n} \leq 1/\rho \) for any \( \rho > 0 \). Hence we see

\[ \lim_{n \to \infty} \sup K_n^{1/n} = 0. \]

From Lemma 1.4 there exist \( f_n \in H_{n,d} \) \( (n = 0, 1, \ldots) \) such that

\[ Ff_n = \psi_n \]

and
(2.15) \[ \|f_n\|_2 = \sqrt{C_n} \|\psi_n\|_N. \]

Since \( \sqrt{C_n} = \{(n! \Gamma(n + (d + 1)/2)N(n, d))/\Gamma((d + 1)/2)\}^{1/2} \leq a\Gamma(n + d) \), where \( a \) is a constant independent of \( n \), (2.13) and (2.15) give

(2.16) \[ \|f_n\|_2 \leq a\Gamma(n + d)K_n \]

and

(2.17) \[ \limsup_{n \to \infty} \left( \frac{1}{n^a} \|f_n\|_2 \right)^{1/n} = 0. \]

\( f' := \sum_{n=0}^{\infty} f_n \) belongs to \( \text{Exp}'(\bar{S}) \) by (1.6) and (2.17). Moreover, (2.14) implies that

\[ Ff'(z) = \langle f', e^{tx} \rangle = \sum_{n=0}^{\infty} \int_{S} f_n(s) e^{tx} ds = \sum_{n=0}^{\infty} Ff_n(z) = \psi(z). \]

Therefore, we get \( F(\text{Exp}'(\bar{S})) = \text{Holo}(M) \).

Let \( f' = \sum_{n=0}^{\infty} f_n' \in \text{Exp}'(\bar{S}) \) and \( Ff' = 0 \). From the proof of Lemma 1.4, \( \{(z \cdot \alpha)^n; \alpha \in S\} \) spans \( P_n(M) \). From this fact and (1.16) we see that \( P_n(M) \bot P_m(M) \) with respect to \( \langle \cdot, \cdot \rangle \) if \( m \neq n \). Hence \( Ff_n' = 0 \) on \( N \), because \( Ff_n' \) is in \( P_n(M) \). Thus \( Ff_n' = 0 \) on \( M \), since \( Ff_n' \) is a homogeneous polynomial. Therefore, we obtain \( f_n' = 0 \) on \( M \) and \( f' = 0 \) by Lemma 1.4. Hence we have (2.4).

\( F \) is a one-to-one linear mapping of \( \mathcal{O}'(\bar{S}) \) into \( \text{Exp}(M) \) from (1.15) and (2.4). Conversely, if \( \psi \) belongs to \( \text{Exp}(M) \), there exists \( \tilde{\psi} \in \mathcal{O}(C^{d+1}) \) such that \( \tilde{\psi}|_M = \psi \) and that for some positive constants \( C \) and \( A \)

(2.18) \[ |\tilde{\psi}(z)| \leq C e^{A|z|} \quad \text{for any } z \in M. \]

We put \( \tilde{\psi} = \sum_{n=0}^{\infty} \tilde{\psi}_n \) and \( \tilde{\psi}_n = \psi_n \), where \( \tilde{\psi}_n \) is given by (2.11). (2.11) and (2.18) imply

\[
K_n = \sup_{z \in N} |\psi_n(z)| \leq \sup_{z \in N, |t|=\rho} \rho^{-n} |\psi(tz)| \leq \sup_{z \in N, |t|=\rho} \rho^{-n} C e^{A|tz|} \leq \sup_{|z|=\rho} \rho^{-n} C e^{A|z|}.
\]

since \( tN \subseteq M \) for any \( t \in C \setminus \{0\} \). Hence we have

(2.19) \[ K_n \leq \rho^{-n} C e^{\frac{A}{\sqrt{2}}} \quad \text{for any } \rho > 0. \]

Since \( \inf \{\rho^{-n} e^{\frac{A}{\sqrt{2}}}; \rho > 0\} = (\sqrt{2} A e/n)^n \) we get

(2.20) \[ K_n \leq C (\sqrt{2} A e/n)^n. \]

There exist \( f_n \in H_n, (n = 0, 1, 2, \cdots) \) which satisfy (2.14) and (2.15). By (2.16) and (2.20) we have

\[ \|f_n\|_2 \leq a\Gamma(n + d)(\sqrt{2} A e/n)^n. \]

Since \( \limsup_{n \to \infty} (n e^{-n} \sqrt{2} \pi n^n/n!)^{1/n} = 1 \) by Stirling's formula, we have
(2.21) \[ \limsup_{n \to \infty} \| f_n \|_{1/n} \leq \limsup_{n \to \infty} \{ aC(n + d)(\sqrt{2} A e/n)^{n^2} e^{-n\sqrt{2} \pi n/n!} \}^{1/n} = \sqrt{2} A < \infty. \]

(2.21) and (1.7) show that \( f' = \sum_{n=0}^{\infty} f_n n \in C^v(S) \) and we have (2.5).

Let \( f' = \sum_{n=0}^{\infty} f_n \) be in \( C^v(S[r]) \) \((1 \leq r < \infty)\) and put \( \psi = \sum_{n=0}^{\infty} \psi_n = Ff' \). Then we have for \( z \in M \)

\[
\psi(z) = \langle f', \exp(\xi \cdot z) \rangle = \sum_{n=0}^{\infty} \int_{S} f_n'(s)(s \cdot z)^{m} ds
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{m!} \int_{S} f_n'(s)(s \cdot z)^{m} ds
\]

since \((s \cdot z)^{m} \in H_{n,d} \) and \( H_{m,d} \perp H_{n,d} \) if \( n \neq m \). (2.22) implies that

\[
\psi_n(z) = \frac{1}{n!} \int_{S} f_n'(s)(s \cdot z)^{m} ds.
\]

For \( z = x + iy \in M \) we get

\[
\sup_{z \in \mathbb{S}} |s \cdot z|^2 = \sup_{z \in \mathbb{S}} |x|^2 |s \cdot (x/|x|) + is \cdot (y/|y|)|^2 \leq |x|^2 \leq |z|^2/2.
\]

From (2.23) and (2.24) we see that

\[
|\psi_n(z)| \leq \frac{1}{n!} \| f_n' \|_{1} (||z||/\sqrt{2})^n.
\]

If we put \( \rho := \limsup_{n \to \infty} \| f_n' \|_{1/n} \), then \( \rho \leq r \) by (1.8) and for any \( \epsilon > 0 \) there exists \( k_\epsilon > 0 \) such that

\[
\sup_{k_\epsilon \leq k_{z}} \| f_k' \|_{1/k} < \rho + \epsilon \leq r + \epsilon.
\]

By (2.25) and (2.26) we have

\[
|\psi(z)| \leq \sum_{n=0}^{\infty} |\psi_n(z)| \leq \sum_{n=0}^{k_{z}-1} (1/n!)(\|z\|/\sqrt{2})^n
\]

\[
+ \sum_{n=k_{z}}^{\infty} (1/n!)(r + \epsilon)^n(\|z\|/\sqrt{2})^n \leq C \exp((r + \epsilon)||z||/\sqrt{2})
\]

for all \( z \in M \), where \( C \) is a constant. From (2.27) we see that \( \psi \in \text{Exp}(M, r/\sqrt{2}) \). Therefore, \( F \) is a one-to-one linear mapping of \( C^v(S[r]) \) into \( \text{Exp}(M, r/\sqrt{2}) \). Conversely, if \( \psi = \sum_{n=0}^{\infty} \psi_n \) belongs to \( \text{Exp}(M, r/\sqrt{2}) \), then there exists \( \psi \in \text{Exp}(C^{d+1}) \) such that \( \psi|_M = \psi \), \( \psi_n|_M = \psi_n \) and that

\[
\sup_{z \in \mathbb{N}} |\psi(z)| \exp(-r'||z||/\sqrt{2}) < \infty \quad \text{for any} \quad r' > r.
\]
(2.18), (2.20) and (2.28) imply

\[ (2.29) \quad K_n \leq C_{r'} (r' e/n)^n \]

for any \( r' > r \) and a constant \( C_{r'} \). If \( Ff_n = \psi_n \) for \( f_n \in H_{n,d} \) \( (n = 0, 1, 2, \ldots) \), from (2.16) and (2.29) we have

\[ \limsup_{n \to \infty} ||f_n||_{i^n} \leq r' \]

for any \( r' > r \). Hence we get

\[ (2.30) \quad \limsup_{n \to \infty} ||f_n||_{i^n} \leq r \]

and (1.8) and (2.30) imply \( f' = \sum_{n=0}^{\infty} f_n \in \mathcal{O}'(\tilde{S}[r]) \). Thus we have (2.6).

Similarly, we get from (1.9)

\[ F(\mathcal{O}'(\tilde{S}(r))) \subseteq \text{Exp}[M, r/\sqrt{2}] \]

On the other hand, if \( \psi = \sum_{n=0}^{\infty} \psi_n \) belongs to \( \text{Exp}[M, r/\sqrt{2}] \), there exists \( \tilde{\psi} = \sum_{n=0}^{\infty} \tilde{\psi}_n \in \mathcal{O}(C^{d+1}) \) such that \( \tilde{\psi}|_M = \psi, \psi_n|_M = \psi_n \) and that

\[ (2.31) \quad \sup_{z \in \mathcal{M}} |\tilde{\psi}(z)\exp(-r'|z||\sqrt{2})| < \infty \]

for some \( r' < r \). (2.31) implies

\[ (2.32) \quad K_n \leq C(r' e/n)^n \]

where \( C \) is a constant. For \( f_n \in H_{n,d} \) \( (n = 0, 1, \ldots) \) such that \( Ff_n = \psi_n \), (2.16) and (2.32) give

\[ (2.33) \quad \limsup_{n \to \infty} ||f_n||_{i^n} \leq r' < r \]

(1.9) and (2.33) show \( f' = \sum_{n=0}^{\infty} f_n \in \mathcal{O}'(\tilde{S}(r)) \) and we obtain (2.7).

Using (1.11), (1.12) and (1.13) we can prove (2.8)-(2.10) similarly.

q.e.d.

Next we consider the image of \( L^*(S) \) by \( F \).

**Lemma 2.2** (c.f. Ii [3, Lemma 2.1]). We denote the modified Bessel function \( K_v \) by

\[ K_v(r) = \int_{0}^{\infty} \exp(-r \cosh t) \cosh vt \, dt \quad (\text{Re} \, v > -(1/2), \, 0 < r < \infty) \]

\[ K_{-v}(r) = K_v(r) \]

and define the function \( \rho_d(r) \) as follows:

\[ (2.34) \quad \rho_d(r) := \begin{cases} \sum_{i=0}^{k} a_k r^{i+1} K_i(2r) & \text{(if } d \text{ is odd)} \\ \sum_{i=0}^{k} a_k r^{i+1/2} K_{i-(1/2)}(2r) & \text{(if } d \text{ is even)} \end{cases} \]
Then we can uniquely determine $k$ and $a_l$ ($l = 0, 1, \ldots, k$) which satisfy
\begin{equation}
\int_0^\infty r^{2n+d-1}\rho_d(r)dr = C_n \quad \text{for all } n = 0, 1, 2, \ldots.
\end{equation}

**Proof.** It is known that
\begin{equation}
\int_0^\infty r^{n-1}K_x(ar)dr = 2^{a-x}a^{-\mu}\Gamma\left(\frac{\mu - \nu}{2}\right)\Gamma\left(\frac{\mu + \nu}{2}\right),
\end{equation}
where $a > 0$ and $\text{Re} \, \mu > |\text{Re} \, \nu|$.

First we assume that $d$ is odd. From (2.34) and (2.36) we get
\begin{equation}
\int_0^\infty r^{2n+d-1}\rho_d(r)dr = (1/4) \sum_{l=0}^k a_l \Gamma\left(n + \frac{d+1}{2}\right)\Gamma\left(n + l + \frac{d+1}{2}\right),
\end{equation}
If (2.35) is valid, from (1.3), (1.22) and (2.37) we have
\begin{equation}
\sum_{l=0}^k \frac{1}{4} a_l \Gamma\left(n + \frac{d+1}{2}\right)\Gamma\left(n + l + \frac{d+1}{2}\right) = C\Gamma\left(n + \frac{d+1}{2}\right)\Gamma(n + d - 1)(2n + d - 1)
\end{equation}
for any $n = 0, 1, 2, \ldots$, where $C$ is a positive constant. Thus we have
\begin{equation}
\sum_{l=0}^k a_l \Gamma\left(n + \frac{d+1}{2}\right)\Gamma\left(n + l + \frac{d+1}{2}\right) = 4C(2n + d - 1)\Gamma(n + d - 1)/\Gamma\left(n + \frac{d+1}{2}\right).
\end{equation}
Since $d \geq 3$, we have $d - 1 \geq (d + 1)/2$. Hence the right hand side of (2.39) is a polynomial of $n$ of degree $(d - 1)/2$. Thus we obtain
\begin{equation}
k = (d - 1)/2,
\end{equation}
and
\begin{equation}
a_k = 8C > 0,
\end{equation}
and we can determine $a_0, a_1, \ldots, a_{k-1}$ uniquely.

Next we assume that $d$ is even. (2.34) and (2.36) imply
\begin{equation}
\int_0^\infty r^{2n+d-1}\rho_d(r)dr = \frac{1}{4} \sum_{l=0}^k a_l \Gamma\left(n + \frac{d+1}{2}\right)\Gamma\left(n + l + \frac{d}{2}\right)
\end{equation}
and we get similarly
\begin{equation}
\sum_{l=0}^k a_l \Gamma\left(n + l + \frac{d}{2}\right)\Gamma\left(n + \frac{d}{2}\right) = 4C(2n + d - 1)\Gamma(n + d - 1)/\Gamma\left(n + \frac{d}{2}\right).
\end{equation}
for \( n = 0, 1, 2, \cdots \). Therefore we get

\begin{equation}
(2.44) \quad k = d/2
\end{equation}

and

\begin{equation}
(2.45) \quad a_k = 8C > 0 ,
\end{equation}

and \( a_0, a_1, \cdots, a_{k-1} \) are determined uniquely. q.e.d.

**Remark 2.3.** (1) Since it is known that

\[
K_{n+1/2}(r) = (\pi/2r)^{1/2}e^{-r}\sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!} (2r)^j
\]

for \( n = 0, 1, 2, \cdots \), there exists a polynomial \( P_{d/2}(r) \) of degree \( d/2 \) such that \( \rho_d(r) = e^{-r}P_{d/2}(r) \), if \( d \) is even. This fact coincides with a result of \( \text{Ii} \) (\[3, \text{Lemma 2.1}\]). Though \( K_i(r) \) is not defined at \( r = 0 \), \( \rho_d(0) \) is well defined for even \( d \) by this fact.

(2) If \( d \) is odd, we have for \( r > 0 \)

\begin{equation}
(2.46) \quad |\rho_d(r)| \leq \sum_{i=0}^{(d-1)/2} |a_i| r^{i+1}K_i(2r)
\end{equation}

\[
\leq \sum_{i=0}^{(d-1)/2} |a_i| r^{i+1}K_{i+1/2}(2r) = e^{-r}r^{i+1/2}P_{(d-1)/2}(r) ,
\]

where \( P_{(d-1)/2} \) is a polynomial of degree \((d - 1)/2\), since \( 0 \leq K_i(r) \leq K_{i+1/2}(r) \). Hence \( \rho_d(r) \) is well defined at \( r = 0 \).

(3) If \( d \) is odd, by (2.41) \( a_{(d-1)/2} > 0 \). Hence we have for \( r > 0 \)

\[
\rho_d(r) \geq a_k r^{k+1}K_k(2r) - \sum_{i=0}^{k-1} |a_i| r^{i+1}K_i(2r)
\]

\[
\geq K_k(2r) \left( a_k r^{k+1} - \sum_{i=0}^{k-1} |a_i| r^{i+1} \right) ,
\]

where we put \( k := (d - 1)/2 \). Therefore \( \rho_d(r) > 0 \) for \( r \) sufficiently large.

For even \( d \) it is trivial by (1) that \( \rho_d(r) > 0 \) for \( r \) sufficiently large.

Now we define a measure \( \mu_d \) on \( M \) by

\begin{equation}
(2.47) \quad \int_M f(z) d\mu_d(z) = \int_0^\infty r^{d-1} \left( \int_N f(r\zeta) dN(z') \right) \rho_d(r) dr .
\end{equation}

We define a subspace \( P(M) \) of \( \text{Holo}(M) \) by

\begin{equation}
(2.48) \quad P(M) := \{ \psi \in \text{Holo}(M); \langle \psi, \psi \rangle_M < \infty \} ,
\end{equation}

where

\begin{equation}
(2.49) \quad \langle \psi, \varphi \rangle_M = \int_M \psi(z) \overline{\varphi(z)} d\mu_d(z) .
\end{equation}

By Remark 2.3, (3) we can prove the following in the same way as
in the proof of Ii [3, Theorem 2.5].

**Theorem 2.4** (cf. Ii [3, Theorem 2.5]). \( F \) is a unitary isomorphism of \( L^2(S) \) onto \( P(M) \) with respect to \( \langle \cdot , \cdot \rangle_s \) and \( \langle \cdot , \cdot \rangle_M \).

**Remark 2.5.** Similarly, we can prove for odd \( d \) the results in Ii [3, Corollary 2.6-Theorem 2.11] given for even \( d \).

3. **The Fourier-Borel transformations of \( \mathcal{O}'(S(r)) \) and \( \mathcal{O}'(S[r]) \).** In this section we consider the images of \( \mathcal{O}'(S(r)) \) and \( \mathcal{O}'(S[r]) \) by the Fourier-Borel transformation \( P_2 \). Our second main theorem in this paper is the following:

**Theorem 3.1.** The transformation \( P_2 \) establishes linear topological isomorphisms

\[
P_2: \mathcal{O}'(S(r)) \xrightarrow{\sim} \text{Exp}_2(C^{d+1}; (|\lambda| r: L^*)) \quad (1 < r \leq \infty),
\]

\[
P_2: \mathcal{O}'(S[r]) \xrightarrow{\sim} \text{Exp}_2(C^{d+1}; (|\lambda| r: L^*)) \quad (1 \leq r < \infty),
\]

where

\[
\text{Exp}_2(C^{d+1}; (|\lambda| r: L^*)) := \mathcal{O}_2(C^{d+1}) \cap \text{Exp}(C^{d+1}; (|\lambda| r: L^*))
\]

and

\[
\text{Exp}_2(C^{d+1}; (|\lambda| r: L^*)) := \mathcal{O}_2(C^{d+1}) \cap \text{Exp}(C^{d+1}; (|\lambda| r: L^*)).
\]

We need the following theorem in order to prove the theorem.

**Theorem 3.2** (Martineau [4]). Suppose \( \lambda \in C, \lambda \neq 0 \). The Fourier-Borel transformation \( P_2 \) establishes the linear topological isomorphisms

\[
P_2: \mathcal{O}'(S[r]) \xrightarrow{\sim} \text{Exp}_2(C^{d+1}; (|\lambda| r: L^*)).
\]

**Proof of Theorem 3.1.** Since \( \mathcal{O}'(S(r)) \subseteq \text{Exp}(\tilde{S}) \cap \mathcal{O}'(\tilde{B}(r)) \) we have

\[
P_2(\mathcal{O}'(S(r))) \subseteq \text{Exp}_2(C^{d+1}; (|\lambda| r: L^*))
\]

by (1.14) and (3.4). Hence \( P_2 \) is a one-to-one linear mapping of \( \mathcal{O}'(S(r)) \) into \( \text{Exp}_2(C^{d+1}; (|\lambda| r: L^*]) \).

Conversely, let \( \tilde{\psi} \) be in \( \text{Exp}_2(C^{d+1}; (|\lambda| r: L^*)) \). If we put \( \tilde{\psi}_M = \psi \), there exist \( r' < r \) and \( C > 0 \) such that

\[
|\psi(z)| \leq C \exp(|\lambda| r'L^*(z)) = C \exp(|\lambda| r'\|z\|/\sqrt{2})
\]

for any \( z \in M \). So we get

\[
|\psi(-iz/\lambda)| \leq C \exp(r'\|z\|/\sqrt{2}) \quad \text{for } \forall z \in M.
\]
Now we put $ψ_{-\nu}(z) := ψ(-iz/\lambda)$. Then $ψ_{-\nu}$ belongs to $\text{Exp}[M, r/\sqrt{2}]$ from (3.5). By (2.7) there exists $f' \in \mathcal{O}'(\tilde{S}(r))$ such that

$$(3.6) \quad Ff' = ψ_{-\nu}.$$ 

Since $\tilde{ψ} \in \mathcal{O}_2(C^{d+1})$, we can find $h' \in \text{Exp}'(\tilde{S})$ such that $\tilde{ψ} = P_{r}h'$ by (1.14).

Since $\tilde{ψ}(-iz/\lambda) = P_{r}h'(-iz/\lambda) = Fh'(z)$ for all $z \in M$, we have from (3.6)

$$(3.7) \quad Fh' = Ff'.$$

By Theorem 2.1 and (3.7) we get $h' = f'$ and $\tilde{ψ} \in P_{r}(\mathcal{O}'(\tilde{S}(r)))$. $P_{r}$ and $P_{r}^{-1}$ are continuous by (3.4) and the closed graph theorem. Therefore, we obtain (3.1). Using (3.3) and (2.6), we can prove (3.2) similarly. q.e.d.

Now we define the topology of $\text{Holo}(M)$ to be the quotient topology $\mathcal{O}^{d+1}_2/ \mathcal{F}(M)$ since $\text{Holo}(M) = \mathcal{O}^{d+1}_2|_{\mathcal{F}}$, where we put $\mathcal{F}(M) := \{f \in \mathcal{O}^{d+1}_2; f = 0 \text{ on } M\}$. We also define the topologies of $\text{Exp}(M)$, $\text{Exp}(M, r/\sqrt{2})$ $(1 \leq r < \infty)$ and $\text{Exp}[M, r/\sqrt{2}]$ $(1 < r \leq \infty)$ similarly since we have $\text{Exp}(M) = \text{Exp}(C^{d+1})|_{\mathcal{F}}$, $\text{Exp}(M, r/\sqrt{2}) = \text{Exp}(C^{d+1}; (r: L^*)|_{\mathcal{F}}$ $(1 \leq r < \infty)$ and $\text{Exp}[M, r/\sqrt{2}] = \text{Exp}(C^{d+1}; [r: L^*]|_{\mathcal{F}}$ $(1 < r \leq \infty)$ by Theorem 2.1.

Then by Theorems 1.2, 2.1 and 3.1 and the closed graph theorem, we have:

**Corollary 3.3.** The transformation $F$ establishes the following linear topological isomorphisms

$$(3.8) \quad F: \text{Exp}'(\tilde{S}) \sim \text{Holo}(M).$$

$$(3.9) \quad F: \mathcal{O}'(\tilde{S}) \sim \text{Exp}(M).$$

$$(3.10) \quad F: \mathcal{O}'(\tilde{S}(r)) \sim \text{Exp}(M, r/\sqrt{2}) \quad \text{for } 1 \leq r < \infty.$$ 

$$(3.11) \quad F: \mathcal{O}'(\tilde{S}(r)) \sim \text{Exp}[M, r/\sqrt{2}] \quad \text{for } 1 < r \leq \infty.$$ 

**Corollary 3.4.** (i) For any $f \in \mathcal{O}(C^{d+1})$ there exists a unique $g \in \mathcal{O}_2(C^{d+1})$ such that $f = g$ on $M$.

(ii) For any $f \in \mathcal{O}(C^{d+1})$ such that $\sup_{x \in M} |f(x)| \exp(-A \|x\|) < \infty$ for an $A > 0$, there exists a unique $g \in \text{Exp}_2(C^{d+1})$ such that $f = g$ on $M$.

(iii) Assume that $1 \leq r < \infty$. For any $f \in \mathcal{O}(C^{d+1})$ such that $\sup_{x \in M} |f(x)| \exp(-\|x\|/r') < \infty$ for $\forall r' > r$, there exists a unique $g \in \text{Exp}_2(C^{d+1}; (|x| r': L^*))$ such that $f = g$ on $M$.

(iv) Assume that $1 < r \leq \infty$. For any $f \in \mathcal{O}(C^{d+1})$ such that $\sup_{x \in M} |f(x)| \exp(-\|x\|/r') < \infty$ for some $r' < r$, there exists a unique $g \in \text{Exp}_2(C^{d+1}; (|x| r: L^*))$ such that $f = g$ on $M$. 


PROOF. (i) If \( f \) belongs to \( \mathcal{O}(\mathbb{C}^{d+1}) \) \( f_{-\lambda} \) also belongs to \( \mathcal{O}(\mathbb{C}^{d+1}) \). Then by Corollary 3.3 there exists \( f' \in \text{Exp}'(\mathcal{S}) \) such that \( Ff' = f_{-\lambda} \) on \( M \). If we put \( g = P_\lambda f' \), \( g \) belongs to \( \mathcal{O}_\lambda(\mathbb{C}^{d+1}) \) and \( f \neq g \) on \( M \) by (1.14). The uniqueness follows from the injectivity of \( F \).

By Theorem 1.2, Theorem 3.1 and Corollary 3.3 we can prove (ii), (iii), (iv) similarly. q.e.d.

REMARK. When \( d = 1 \) (the case of the unit circle), Corollary 3.4 is known (see Morimoto [5]).

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