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ON A MOMENT PROBLEM

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Abstract. Let n_0 be any fixed non-negative integer, $-\infty \leqq a < b \leqq \infty$ and $f(x) \geqq 0$ an absolutely continuous function with $f'(x) \neq 0$, a.e. on (a, b). Then the sequence of functions $\{(f(x))^n e^{-f(x)}\}_{n=n_0}^{\infty}$ is complete in $L(a, b)$ if and only if the function $f(x)$ is strictly monotone on (a, b).

1. Introduction. Let $-\infty \leqq a < b \leqq \infty$, and let $L(a, b)$ be the space of all summable functions defined on the interval (a, b). Then a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ is said to be complete in $L(a, b)$ if for every $g \in L(a, b)$, the equalities

$$\int_a^b g(x) f_n(x) dx = 0 , \qquad \text{for all} \quad n = 1, 2, \cdots ,$$

imply $g(x) = 0$, a.e. (almost everywhere) on (a, b). The well-known Müntz-Szász theorem (Boas [1, p. 235]) is concerned with a complete sequence of functions in $L(a, b)$, where (a, b) is a *bounded* interval, and is stated as follows:

THEOREM A. *Let* $0 \leqq a < b < \infty$ *and* $0 < n_1 < n_2 < \cdots$. *Then* $\{x^{n_i}\}_{i=1}^{\infty}$ *is complete in* $L(a, b)$ *if and only if* $\sum_{i=1}^{\infty} 1/n_i = \infty$.

In this paper, we shall first consider the completeness of a sequence of functions $\{x^n e^{-x}\}$ in $L(a, \infty)$, where $a \geqq 0$ (Theorem 1), then use a theorem of Zarecki to extend the result just obtained to the sequence of functions $\{(f(x))^n e^{-f(x)}\}$ (Theorem 2). Finally, we give some remarks on Laguerre and Hermite functions.

THEOREM 1. *For any fixed integer* $n_0 \geqq 0$ *and for any fixed real number* $a \geqq 0$, *the sequence of functions* $\{x^n e^{-x}\}_{n=n_0}^{\infty}$ *is complete in* $L(a, \infty)$.

THEOREM 2. *Let* n_0 *be any fixed non-negative integer,* $-\infty \leqq a < b \leqq \infty$ *and* $f(x) \geqq 0$ *an absolutely continuous function with* $f'(x) \neq 0$, *a.e. on* (a, b). *Then the sequence of functions* $\{(f(x))^n e^{-f(x)}\}_{n=n_0}^{\infty}$ *is complete in*

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$L(a, b)$ if and only if the function $f(x)$ is strictly monotone on (a, b).

When we compare Theorem 1 with Theorem A, it is natural to ask whether the subsequence $\{x^{n_i}e^{-x}\}_{i=1}^{\infty}$ is still complete in $L(0, \infty)$ if $\sum_{i=1}^{\infty} 1/n_i = \infty$ and $0 < n_1 < n_2 < \cdots$. The answer is negative due to the following example.

EXAMPLE. The sequence of functions $\{x^{4m-1}e^{-x}\}_{m=1}^{\infty}$ is incomplete in $L(0, \infty)$, since the function $g(x) = e^{-x} \sin 2x$ is summable on $(0, \infty)$ and $g(x) \neq 0$, a.e. on $(0, \infty)$, but (see Gradshteyn and Ryzhik [3, p. 490])

$$\int_0^\infty x^{4m-1}e^{-x}g(x)dx = \int_0^\infty x^{4m-1}e^{-2x}\sin 2x dx = \frac{\Gamma(4m)}{8^{2m}}\sin m\pi = 0 , \quad \text{for} \quad m = 1, 2, \cdots .$$

2. Proofs.

PROOF OF THEOREM 1. Assume that $g \in L(a, \infty)$ satisfies

$$(1) \qquad \int_a^\infty g(x)x^n e^{-x}dx = 0 , \qquad \text{for all} \quad n = n_0, n_0 + 1, \cdots .$$

We want to show that $g(x) = 0$, a.e. on $(a, \infty) \equiv I$. For any $\lambda > 0$, the function $h(x) = \chi_I(x)g(x)x^{n_0}e^{-\lambda x}$ is summable on $(0, \infty)$, since $g \in L(a, \infty)$ and $\chi_I(x)x^{n_0}e^{-\lambda x}$ is a bounded measurable function on $(0, \infty)$, where χ_I denotes the characteristic function of I. Hence the function

$$\Phi(s) = \int_0^\infty \chi_I(x)g(x)x^{n_0}e^{-sx}dx$$

is analytic on the half-plane $K = \{s: s = \alpha + i\beta, \alpha > 0\}$.

The condition (1) implies that the m-th derivative $\Phi^{(m)}(1) = 0$ for every $m = 0, 1, 2, \cdots$. This means that Φ is a zero function in some neighborhood of $s = 1$ and hence $\Phi(s) = 0$ for every $s \in K$, by the uniqueness theorem for analytic function. Finally, the uniqueness of the Laplace transform implies that $\chi_I(x)g(x)x^{n_0} = 0$, a.e. on $(0, \infty)$, and hence $g(x) = 0$, a.e. on (a, ∞), which is the desired result.

To prove Theorem 2, we need the following theorem of Zarecki (see Natanson [5, Vol. I, p. 271] or Saks [6, p. 128]) which was used to extend Müntz-Szász theorem by Hwang and Lin [4].

THEOREM B. *Let $f(x)$ be a continuous and strictly increasing function on a closed and bounded interval $[c, d]$. Then $f'(x) \neq 0$, a.e. on $[c, d]$ if and only if the inverse function $f^{-1}(t)$ is absolutely continuous on the interval $[f(c), f(d)]$.*

PROOF OF THEOREM 2. (Necessity) Assume that the sequence of functions $\{(f(x))^n e^{-f(x)}\}_{n=n_0}^{\infty}$ is complete in $L(a, b)$. Then we want to show that the function $f(x)$ is strictly monotone on (a, b). Suppose that $f(x)$ is not strictly monotone on (a, b). Then there exist two points $x_1 < x_2$ such that $f(x_1) = f(x_2) = y_0$, say. Let us define a function $g\colon (a, b) \to (-\infty, \infty)$ by

$$g(x) = \begin{cases} f'(x) , & \text{if } x \in [x_1, x_2] \text{ and } f'(x) \text{ exists} , \\ 0 , & \text{otherwise} . \end{cases}$$

Then $g \in L(a, b)$ and $g(x) \neq 0$, a.e. on $[x_1, x_2] \subset (a, b)$, but (Natanson [5, I, p. 265 and II, p. 236])

$$\int_a^b g(x)(f(x))^n e^{-f(x)} dx = \int_{x_1}^{x_2} (f(x))^n e^{-f(x)} df(x) = \int_{y_0}^{y_0} y^n e^{-y} dy = 0 , \quad \text{for every} \quad n = n_0, n_0 + 1, \cdots ,$$

a contradiction to the assumption.

(Sufficiency) Assume that the function $f(x)$ is strictly monotone on (a, b). Then we want to show that the sequence of functions $\{(f(x))^n e^{-f(x)}\}_{n=n_0}^{\infty}$ is complete in $L(a, b)$. Without loss of generality, we may assume that $f(x)$ is strictly increasing on (a, b). Suppose $h \in L(a, b)$ satisfy

$$(2) \qquad \int_a^b h(x)(f(x))^n e^{-f(x)} dx = 0 , \qquad \text{for all} \quad n = n_0, n_0 + 1, \cdots .$$

Then we want to prove that $h(x) = 0$, a.e. on (a, b). Taking the transformation $t = f(x)$ and using Theorem B, we have, from (2),

$$(3) \qquad \int_{f(a^+)}^{f(b^-)} h(f^{-1}(t)) t^n e^{-t} (f^{-1})'(t) dt = 0 , \qquad \text{for all} \quad n = n_0, n_0 + 1, \cdots ,$$

where $f(a^+) \equiv \lim_{x \to a^+} f(x)$, and analogously for $f(b^-)$. Note that (3) is equivalent to

$$(4) \qquad \int_0^{\infty} \chi_I(t) h(f^{-1}(t)) t^n e^{-t} (f^{-1})'(t) dt = 0 , \qquad \text{for all} \quad n = n_0, n_0 + 1, \cdots ,$$

where $I = (f(a^+), f(b^-))$. Since $h \in L(a, b)$, we know that $k(t) \equiv \chi_I(t) \cdot h(f^{-1}(t))(f^{-1})'(t)$ is summable on $(0, \infty)$ and hence both (4) and Theorem 1 imply that $k(t) = 0$, a.e. on $(0, \infty)$. Therefore

$$h(f^{-1}(t)) = 0 , \quad \text{a.e. on} \quad (f(a^+), f(b^-)) ,$$

and hence $h(x) = 0$, a.e. on (a, b). The proof of this theorem is complete.

3. Remarks. Define $L_n(x)$ be the coefficient of e^{-x} in the n-th derivative of $x^n e^{-x}$, that is,

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} n(n-1) \cdots (k+1) x^k .$$

Then $L_n(x)$, $n = 0, 1, 2, \cdots$, are called the Laguerre polynomials. It is well-known (see Goffman [2, p. 193] or Shilov [7, p. 403]) that the Laguerre functions,

$$\psi_n(x) = \frac{1}{n!} e^{-x/2} L_n(x) , \quad n = 0, 1, 2, \cdots ,$$

form a complete sequence in $L_2(0, \infty)$, the square summable functions on $(0, \infty)$. As a consequence of Theorem 2, it is also a complete sequence in $L(0, \infty)$.

The functions obtained in orthogonalizing the expressions e^{-x^2}, xe^{-x^2}, $\cdots$, $x^n e^{-x^2}$, $\cdots$ in the space $L_2(-\infty, \infty)$ are called Hermite functions. The completeness of the sequence of Hermite functions is based on the fact that $\{x^n e^{-x^2}\}_{n=0}^{\infty}$ is complete in $L_2(-\infty, \infty)$ (Shilov [7, p. 403]). Applying Theorem 2 again we know that the sequence of functions $\{x^{2n} e^{-x^2}\}_{n=0}^{\infty}$ is complete in $L(0, \infty)$, but incomplete in $L(-\infty, \infty)$.

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