Tôhoku Math. Journ.
38 (1986), 501-512.

THE WALSH SERIES OF A DYADIC STATIONARY PROCESS

YASUSHI ENDOW

(Received October 3, 1984)

1. Introduction. Let $X(t) = X(t, \omega)$, $t \in R_+ = [0, \infty)$ be a dyadic stationary (DS) process with $EX(t) = 0$, $t \in R_+$ (cf. [2]). Suppose that $X(t)$ is W-harmonizable, namely it is expressible as

$$X(t) = \int_0^\infty \psi_t(\lambda) d\zeta(\lambda) \quad (t \in R_+) , \tag{1.1}$$

where $\psi_t(\lambda)$ is the (generalized) Walsh function [4] and $\zeta(\lambda)$, $\lambda \in R_+$ is a second order process with orthogonal increments. The covariance function of $X(t)$ is expressed by

$$r(t, s) = EX(t)\overline{X(s)} = \int_0^\infty \psi_t(\lambda)\psi_s(\lambda) dF(\lambda) , \tag{1.2}$$

where $F(\lambda)$ is the spectral distribution function with

$$dF(\lambda) = E|d\zeta(\lambda)|^2 . \tag{1.3}$$

A necessary and sufficient condition for the W-harmonizability of a DS process was given by the present author [2].

We shall now define the Walsh series of $X(t)$. Since the integral $\int_0^a X(t)dt$ exists for any $a \in R_+$ in quadratic mean, we define the Walsh coefficients of $X(t)$ over $(0, 2^p)$ as

$$C_n = C_n(p) = 2^{-p} \int_0^{2^p} X(t)\psi_n(2^{-p}t) dt , \tag{1.4}$$

where p is a positive integer. The Walsh series of $X(t)$ is written as

$$X(t) \sim \sum_{n=0}^\infty C_n \psi_n(2^{-p}t) . \tag{1.5}$$

Here we introduce the known properties of the Walsh functions, which are frequently used afterwards (cf. [1], [3], [4], [9]).

LEMMA 1.1.

(1.6) (i) $\psi_t(x) = \psi_{[x]}(t)\psi_{[t]}(x)$.

(1.7) (ii) $\psi_{2^{-p}t}(x) = \psi_t(2^{-p}x)$ *for* t, x *in* R_+.

(1.8) (iii) $\psi_x(t)\psi_y(t) = \psi_{x\oplus y}(t)$ *if* $\mu(x) + \mu(y) \in \mathfrak{G}$.

(1.9) (iv) $D_{2^p}(x) = \sum_{k=0}^{2^p-1} \psi_k(x) = \begin{cases} 2^p, & (x - [x] < 2^{-p}), \\ 0, & (\textit{otherwise}). \end{cases}$

(1.10) (v) $D_n(t) = D_{2^m}(t) + \psi_{2^m}(t) D_{n'}(t)$, *where* $n = 2^m + n'$, $n' < 2^m$.

As for the notations μ and $\mathfrak{G}$ see [4]. We remark that (i) implies the symmetry; $\psi_t(x) = \psi_x(t)$.

LEMMA 1.2.

(i) $EC_n = 0$.

(ii) $EC_m\bar{C}_n = \delta_{m,n}(F(2^{-p}(n+1)) - F(2^{-p}n))$,

where $\delta_{m,n}$ *is the Kronecker delta.*

PROOF. By (1.2),

$$EC_m\bar{C}_n = 2^{-2p}\int_0^\infty dF(x)\int_0^{2^p}\psi_t(x)\psi_m(2^{-p}t)dt\int_0^{2^p}\psi_s(x)\psi_n(2^{-p}s)ds\ ,$$

which is equal by Lemma 1.1, (1.6)–(1.8) to

$$2^{-2p}\int_0^\infty dF(x)\int_0^{2^p}\psi_t(x\oplus 2^{-p}m)dt\int_0^{2^p}\psi_s(x\oplus 2^{-p}n)ds\ .$$

Since it is easily verified by Lemma 1.1, (i) that

$$\int_0^{2^p}\psi_x(t)dt = \begin{cases} D_{2^p}(x)\ , & x < 1 \\ 0\ , & \text{otherwise} \end{cases}$$

we obtain

$$EC_m\bar{C}_n = 2^{-2p}\int_0^\infty D_{2^p}(x\oplus 2^{-p}m)D_{2^p}(x\oplus 2^{-p}n)dF(x) = \int_{\{[2^px]=m\}\cap\{[2^px]=n\}} dF(x)\ .$$

This completes the proof.

The following theorem is an immediate consequence of Lemma 1.2.

THEOREM 1.1. *Let* $C_n = C_n(p)$ *be the Walsh coefficient of a* DS *process over* $[0, 2^p)$. *Then*

(1.11) (i) $\sum_{n\geqq N} E|C_n|^2 = F(\infty) - F(2^{-p}N)$.

(ii) *If*

(1.12) $$\int_0^\infty x^\alpha dF(x) < \infty \quad (0 \leqq \alpha)\ ,$$

then

$$(1.13) \qquad \sum_{n \geqq N} E|C_n|^2 = o(N^{-\alpha}) .$$

PROOF. It is clear by (i) above that

$$\sum_{n \geqq N} E|C_n|^2 \leqq (2^{-p}N)^{-\alpha} \int_{2^{-p}N}^{\infty} x^\alpha dF(x) .$$

Hence the proof is completed by (1.12).

2. The mean convergence and the absolute convergence of the Walsh series. Let $S_n(t)$ be the partial sum of the Walsh series of $X(t)$;

$$S_n(t) = \sum_{k=0}^{n-1} C_k \psi_n(2^{-k}t) .$$

THEOREM 2.1. *The Walsh series of $X(t)$ converges in the mean to the original process at every t in the interval* $[0, 2^p)$;

$$(2.1) \qquad \lim_{n \to \infty} S_n(t) = X(t) \quad \textit{for } t \in [0, 2^p) .$$

Before proving the theorem we show the following:

LEMMA 2.1.

$$(2.2) \qquad \lim_{n \to \infty} \int_0^1 (\psi_t(x) - 1) D_n(t) dt = 0 .$$

PROOF. First consider the case $n = 2^m$. By (1.9) and $\psi_t(x) = \psi_{[x]}(t)$ $(0 \leqq t < 2^{-m})$, which is verified by (1.6),

$$(2.3) \qquad \int_0^1 (\psi_t(x) - 1) D_{2^m}(t) dt = 2^m \int_0^{2^{-m}} (\psi_{[x]}(t) - 1) dt .$$

For fixed x there is an $N > 0$ such that $[x] < 2^N$. Then for $m \geqq N$, $\psi_{[x]}(t) = 1$ $(t < 2^{-m})$. Hence for sufficiently large m the right hand side of (2.3) is equal to zero.

For any positive integer n there is an integer m such that $2^m \leqq n < 2^{m+1}$. Putting $n = 2^m + n'$ $(n' < 2^m)$, and using (1.10), we obtain

$$\begin{aligned} \int_0^1 (\psi_t(x) - 1) D_n(t) dt &= \int_0^1 (\psi_t(x) - 1) D_{2^m}(t) dt + \int_0^1 (\psi_t(x) - 1) \psi_{2^m}(t) D_{n'}(t) dt \\ &= I_1 + I_2 , \end{aligned}$$

say. Then I_1, as was shown above, will vanish for n sufficiently large. We recall that $\psi_k(2^{-(m+1)}) = 1$ for $k < 2^m$, so that $D_{n'}(t \oplus 2^{-(m+1)}) = D_{n'}(t)$ and $\psi_{2^m}(t \oplus 2^{-(m+1)}) = -\psi_{2^m}(t)$. Hence, using the invariance of integration, we may write

$$(2.4) \qquad I_2 = \int_0^1 (\psi_x(t \oplus 2^{-(m+1)}) - 1) \psi_{2^m}(t \oplus 2^{-(m+1)}) D_{n'}(t \oplus 2^{-(m+1)}) dt$$

$$= -\int_0^1 (\psi_x(t \oplus 2^{-(m+1)}) - 1)\psi_{2^m}(t)D_{n'}(t)dt \, .$$

Adding the left and the right hand sides of (2.4), we obtain by (1.9)

$$2I_2 = \int_0^1 (\psi_x(t) - \psi_x(t \oplus 2^{-(m+1)}))\psi_{2^m}(t)D_{n'}(t)dt$$
$$= (1 - \psi_x(2^{-(m+1)})) \int_0^1 \psi_{x\oplus 2^m}(t)D_{n'(t)}dt \, .$$

Therefore I_2 will vanish for sufficiently large m (which may depend on x), since $\psi_x(2^{-(m+1)}) = 1$ for such a large m.

PROOF OF THEOREM 2.1. Since

$$S_n(t) - X(t) = 2^{-p} \int_0^{2^p} (X(u) - X(t))D_n(2^{-p}(u \oplus t))du \, ,$$

we write by (1.2)

$$E|S_n(t) - X(t)|^2 = \int_0^\infty dF(x)\Big[2^{-p} \int_0^{2^p} (\psi_u(x) - \psi_t(x))D_n(2^{-p}(u \oplus t))du\Big]^2$$
$$= \int_0^\infty dF(x)\Big[2^{-p} \int_0^{2^p} (\psi_{u\oplus t}(x) - 1)D_n(2^{-p}(u \oplus t))du\Big]^2 ,$$

by virtue of (1.8) and $|\psi_t(x)| = 1$. Since the inner Dirichlet integral on the right hand side is bounded for all t and x, and converges to zero, as n goes to infinity by Lemma 2.1, the desired result follows.

Next we show the absolute convergence of the Walsh series of $X(t)$.

THEOREM 2.2. *If*

$$\int_0^\infty x^\alpha dF(x) < \infty \quad \textit{for } \alpha > 1 \, , \tag{2.5}$$

then the Walsh series

$$\sum_{n=0}^\infty C_n\psi_n(2^{-p}t) \quad \textit{for } t \in [0, 2^p) \tag{2.6}$$

converges absolutely with probability one.

PROOF. Applying Hölder's inequality, we have

$$\sum_{n=2}^\infty E|C_n| = \sum_{m=1}^\infty \sum_{n=2^{m-1}+1}^{2^m} E|C_n| \leqq \sum_{m=1}^\infty \Big[\Big(\sum_{n=2^{m-1}+1}^{2^m} E|C_n|^2\Big)(2^m - 2^{m-1})\Big]^{1/2} .$$

Because of Theorem 1.1, (ii) the last expression above is of the order $\sum_{m=1}^\infty o(2^{-\alpha(m-1)/2})O(2^{m/2}) = O(1)$. Hence $\sum_{n=0}^\infty |C_n|$ converges with probability one.

COROLLARY 2.1. *If* (2.5) *is satisfied for* $X(t)$, *then it has a version which is sample* W-*continuous*.

This follows from Theorem 2.2 and the fact that the Walsh functions are W-continuous.

3. The almost everywhere convergence of the Walsh series.

THEOREM 3.1. *If*

$$\int_1^\infty \log x dF(x) < \infty , \tag{3.1}$$

then the Walsh series of $X(t)$ *defined by* (1.4) *converges almost everywhere on* $0 \leqq t < 2^p$ *with probability one*.

Before proving the theorem we need a lemma due to Paley [8] (see also [10]).

LEMMA 3.1. *Let* $f \in L_2[0, 1)$ *and its Walsh series be* $f(t) \sim \sum c_n \psi_n(t)$. *If*

$$\int_0^1 dx \int_0^1 [f(x \oplus t) - f(x)]^2/t dt < \infty ,$$

then the Walsh series of $f(t)$ *converges almost everywhere on* $[0, 1)$.

PROOF OF THEOREM 3.1. Because of Lemma 3.1 we shall only show that

$$\int_0^{2^p} dt \int_0^{2^p} E|X(t \oplus h) - X(t)|^2/h dh < \infty , \tag{3.2}$$

which implies that

$$\int_0^{2^p} dt \int_0^{2^p} |X(t \oplus h) - X(t)|^2/h dh < \infty$$

with probability one. Now

$$\int_0^{2^p} dt \int_0^{2^p} E|X(t \oplus h) - X(t)|^2/h dh = \int_0^{2^p} dt \int_0^{2^p} 1/h dh \int_0^\infty (\psi_h(x) - 1)^2 dF(x) ,$$

by virtue of (1.8) and $|\psi_t(x)| = 1$. Put

$$\begin{aligned}\int_0^{2^p} 1/h dh \int_0^\infty (\psi_h(x) - 1)^2 dF(x) \\ = \int_0^\infty dF(x)\left(\int_0^1 + \int_1^{2^p}\right)(\psi_h(x) - 1)^2/h dh \\ =: I_1 + I_2 .\end{aligned}$$

By $|\psi_h(x) - 1| \leqq 2$,

$$I_2 \leqq 4 \int_0^\infty dF(x) \int_0^{2^p} dh < \infty .$$

Since it is easy to see that

$$\psi_t(x) = 1 \quad \text{if} \quad tx < 1/2 , \tag{3.3}$$

we have

$$I_1 = \int_{1/2}^\infty dF(x) \int_{1/2x}^1 (\psi_h(x) - 1)^2/h dh \leqq 4 \int_{1/2}^\infty \log 2x dF(x) .$$

4. The limit joint distribution of Walsh coefficients. It is known [2] that if $X(t)$ has a spectral density, then $X(t)$ is expressed as

$$X(t) = \int_0^\infty \Phi(t \oplus s) d\eta(s) \tag{4.1}$$

where $\Phi \in L_2(R_+)$ and is real-valued, and $\eta(t)$, $t \in R_+$ is a stochastic process with orthogonal increments with

$$E|d\eta(t)|^2 = dt . \tag{4.2}$$

It is also shown that the covariance function is written as

$$r(t, s) = \int_0^\infty |\varphi(x)|^2 \psi_t(x) \psi_s(x) dx , \tag{4.3}$$

where φ is the Walsh transform in L_2 of $\Phi(x)$;

$$\varphi(x) = \int_0^\infty \Phi(t) \psi_x(t) dt .$$

We study the joint limit distribution of the random variables $(C_0, C_1, \cdots, C_n)$ as $p \to \infty$, where $C_k = C_k(p)$ is the k-th Walsh coefficient of $X(t)$.

THEOREM 4.1. *Let $X(t)$ be the* DS *process expressed by* (4.1) *with $\Phi \in L_2(R_+)$ and with $\eta(t)$ having independent increments and satisfying* (4.2) *and*

$$E|d\eta(t)|^3 = O(dt) . \tag{4.4}$$

Moreover, if $\Phi \in L_1 \cap L_3(R_+)$, then the joint distribution of the set of the Walsh coefficients of $X(t)$ over $0 \leqq t < 2^p$,

$$2^{p/2}(C_0, C_1, \cdots, C_n) \tag{4.5}$$

converges to the $(n+1)$-ple direct product $(\prod^*)^{n+1} N(0, \sigma^2)$ of the normal distribution $N(0, \sigma^2)$ with mean 0 and variance $\sigma^2 = \left| \int_0^\infty \Phi(t) dt \right|^2$.

PROOF. The characteristic function of (4.5) is written as

$$(4.6)\qquad f(\tau_0, \tau_1, \cdots, \tau_n) = E\left\{\exp\left(\sqrt{-1}\, 2^{p/2} \sum_{j=0}^{n} \tau_j C_j\right)\right\}$$
$$= E\left\{\exp\left(\sqrt{-1}\, 2^{-p/2} \int_0^{2^p} X(t) \sum_{j=0}^{n} \tau_j \psi_j(2^{-p}t) dt\right)\right\}$$
$$= E\{\exp(\sqrt{-1}\, X_p)\} ,$$

where $X_p = 2^{-p/2} \int_0^{2^p} X(t) g_n(t, p) dt$ and $g_n(t, p) = \sum_{j=0}^{n} \tau_j \psi_j(2^{-p}t)$. Now using (4.1) and changing the variables, we have

$$X_p = 2^{-p/2} \int_0^{\infty} d\eta(s) \int_0^{2^p} g_n(t, p) \Phi(t \oplus s) dt$$
$$= 2^{p/2} \int_0^{\infty} d\eta(2^p v) \int_0^1 g_n(2^p u, p) \Phi(2^p(u \oplus v)) du ,$$

where

$$g_n(2^p u, p) = \sum_{j=0}^{n} \tau_j \psi_j(u) = g_n(u) \quad \text{for } 0 \leqq u < 1$$

is independent of p. Hence

$$X_p = 2^{-p/2} \int_0^{\infty} h(v, p) d\eta_p(v) ,$$

where

$$h(v, p) = \int_0^1 g_n(u) 2^p \Phi(2^p(u \oplus v)) du ,$$

and $\eta_p(v) = \eta(2^p v)$. It follows from the assumptions (4.2) and (4.4) that $E|d\eta_p(v)|^2 = 2^p dv$ and $E|d\eta_p(v)|^3 = O(2^p dv)$. Define

$$h(v) = \begin{cases} g_n(v) \int_0^{\infty} \Phi(w) dw , & 0 \leqq v < 1 , \\ 0 , & \text{otherwise} . \end{cases}$$

Then we see that $h(v) \in L_3(R_+)$, since it belongs to $L_2(R_+)$ and is bounded.

Finally we show that

$$(4.7)\qquad \lim_{p \to \infty} \int_0^{\infty} |h(v, p) - h(v)|^2 dv = 0 ,$$

hence it follows from Lemma 4.1 below that the characteristic function of X_p converges to the characteristic function of $N\left(0, \int_0^{\infty} h^2(v) dv\right)$; actually (4.6) converges to

$$\exp\left(-1/2 \int_0^{\infty} h^2(v) dv\right) = \exp\left(-1/2 \left|\int_0^{\infty} \Phi(t) dt\right|^2 \int_0^1 g_n^2(v) dv\right)$$

$$= \exp\Bigl(-1/2\,\sigma^2 \sum_{j=0}^{n} \tau_j^2\Bigr) = \prod_{j=0}^{n} \exp(-1/2\,\sigma^2\tau_j^2)\ .$$

Now we write

$$h(v,\,p) = \int_{2^p v}^{2^p(1+v)} g_n(2^{-p}w \oplus v)\Phi(w)dw = \int_0^\infty g_n(2^{-p}w \oplus v)\Phi(w)dw\ , \tag{4.8}$$

defining $g_n(v) = 0$ outside $0 \leqq v < 1$. Since $g_n(v)$ is a linear combination of the Walsh functions, and hence is W-continuous and bounded, Lebesgue's convergence theorem applies to show that

$$\lim_{p\to\infty} h(v,\,p) = h(v)\ . \tag{4.9}$$

Moreover, the convergence in (4.9) is bounded because of (4.8), which reveals that $h(v,\,p)$ is uniformly bounded. Therefore in order to show (4.7) it is sufficient to show that

$$\lim_{p\to\infty} \int_A^\infty |h(v,\,p) - h(v)|^2 dv = 0$$

for some $A > 0$. For an arbitrarily fixed $A > 1$

$$\int_A^\infty |h(v,\,p) - h(v)|^2 dv = \int_A^\infty |h(v,\,p)|^2 dv \leqq K \int_0^\infty |h(v,\,p)|\,dv\ ,$$

for some constant $K > 0$, since $h(v,\,p)$ is uniformly bounded. Hence

$$\begin{aligned}\int_A^\infty |h(v,\,p)|\,dv &\leqq 2^p \int_A^\infty |g_n(u)|\,|\Phi(2^p(u \oplus v))|\,du \\ &= \int_0^1 |g_n(u)|\,du \int_A^\infty 2^p\,|\Phi(2^p(u \oplus v))|\,dv \\ &= \int_0^1 |g_n(u)|\,du \int_{A\oplus 2^p u}^\infty |\Phi(w)|\,dw\ ,\end{aligned}$$

which converges to zero as $p \to \infty$.

LEMMA 4.1 (Kawata [5]). *Suppose that a real-valued function* $\gamma_\alpha(v) \in L_2(R_+)$ *satisfies*

$$\lim_{\alpha\to\infty} \int_0^\infty |\gamma_\alpha(v) - \gamma(v)|^2 dv = 0$$

for some $\gamma(v) \in L_2 \cap L_3(R_+)$. *Let* $\xi_\alpha(v)$ *be a stochastic process with independent increments satisfying* $Ed\xi_\alpha(v) = 0$, $E|d\xi_\alpha(v)|^2 = \alpha dv$, *and* $E|d\xi_\alpha(v)|^3 = O(\alpha dv)$. *Then the characteristic function of*

$$Y_\alpha = \alpha^{-1/2} \int_0^\infty \gamma_\alpha(v) d\xi_\alpha(v)$$

converges uniformly in every finite interval as $\alpha \to \infty$ *to the characteristic* *function of* $N\left(0, \int_0^\infty \gamma^2(v)dv\right)$.

5. An approximate Walsh series. We shall study the following Walsh series,

$$\hat{X}(t) = \hat{X}_p(t) = \sum_{n=0}^{\infty} \zeta_n(\omega)\psi_t(2^{-p}n) \quad \text{for} \quad t \in R_+ \,, \tag{5.1}$$

where

$$\zeta_n(\omega) = \zeta_{n,p}(\omega) = \int_{2^{-p}n}^{2^{-p}(n+1)} d\zeta(x) \,. \tag{5.2}$$

It is obvious that

$$E\zeta_m\bar{\zeta}_n = 0 \quad \text{if} \quad m \neq n \,. \tag{5.3}$$

The series (5.1) converges at every t in the L_2 sense, since

$$E\left|\sum_{n=M}^{N} \zeta_n\psi_t(2^{-p}n)\right|^2 = \sum_{n=M}^{N} E|\zeta_n|^2 = \int_{2^{-p}M}^{2^{-p}(N+1)} dF(x) \to 0 \,,$$

as $M, N \to \infty$. The mean and the covariance functions are given by

$$E\hat{X}(t) = 0 \,, \tag{5.4}$$

and

$$\hat{r}(t, s) = \sum_{n=0}^{\infty} \psi_t(2^{-p}n)\psi_s(2^{-p}n) \int_{2^{-p}n}^{2^{-p}(n+1)} dF(x) \,, \tag{5.5}$$

respectively. Hence $\hat{X}(t)$ is a W-harmonizable DS process with the spectral distribution function,

$$\hat{F}(x) = \int_0^{2^{-p}n} dF(x) \quad \text{if} \quad 2^{-p}(n-1) \leqq x < 2^{-p}n \,. \tag{5.6}$$

Now

$$EX(t)(\hat{X}(t))^- = \sum_{n=0}^{\infty} \psi_t(2^{-p}n) \int_{2^{-p}n}^{2^{-p}(n+1)} \psi_t(x)dF(x) = \sum_{n=0}^{\infty} \int_{2^{-p}n}^{2^{-p}(n+1)} \psi_t(x \oplus 2^{-p}n)dF(x)$$

by virtue of (1.8), and so

$$\begin{aligned} E|X(t) - \hat{X}(t)|^2 &= 2\left\{\int_0^\infty dF(x) - EX(t)(\hat{X}(t))^-\right\} \\ &= 2\sum_{n=0}^{\infty} \int_{2^{-p}n}^{2^{-p}(n+1)} (1 - \psi_t(x \oplus 2^{-p}n))dF(x) \,. \end{aligned}$$

By (3.3),

$$\int_{2^{-p}n}^{2^{-p}(n+1)} (1 - \psi_t(x \oplus 2^{-p}n))dF(x) = \int_0^{2^{-p}} (1 - \psi_t(x))dF(x \oplus 2^{-p}n)$$

$$= 0 \quad \text{for} \quad p > \log_2^+ t + 1 .$$

Therefore we obtain the following:

THEOREM 5.1. *Let $\hat{X}_p(t)$ be the Walsh series defined by* (5.1) *based on $X(t)$. Then*

$$\hat{X}_p(t) = X(t) \quad \textit{for} \quad p > \log_2^+ t + 1 . \tag{5.7}$$

This implies, as expected easily, that $\hat{X}_p(t)$ converges almost surely to $X(t)$ as $p \to \infty$.

THEOREM 5.2. *Let $F(\lambda)$ be a spectral function of a* DS *process. If*

$$\sum_{n=0}^{\infty} (F(n+1) - F(n))^{1/2} < \infty \tag{5.8}$$

holds, then the Walsh series $\hat{X}_p(t)$ defined by (5.1) *absolutely converges almost surely.*

This is an analog of Lemma 2 in [7] of the weakly stationary case, so the proof is omitted.

LEMMA 5.1 (Kubo [7]). *If there exists a function $g(x)$ defined on R_+ which is non-negative, non-decreasing, and satisfies that*

$$\sum_{n=1}^{\infty} 1/g(n) < \infty \tag{5.9}$$

and

$$\int_0^{\infty} g(\lambda) dF(\lambda) < \infty , \tag{5.10}$$

then (5.8) *holds.*

PROOF. This is clear, since

$$\left\{\sum_{n=0}^{\infty} (F(n+1) - F(n))^{1/2}\right\}^2 \leqq \sum_{n=0}^{\infty} g(n+1)(F(n+1) - F(n)) \sum_{n=0}^{\infty} 1/g(n+1)$$
$$\leqq \int_0^{\infty} g(\lambda) dF(\lambda) \sum_{n=1}^{\infty} 1/g(n) .$$

By Theorem 5.2 and Lemma 5.1 we have the following:

COROLLARY 5.1. *If there exists a function $g(x)$ which satisfies the conditions in Lemma* 5.1, *then the Walsh series $\hat{X}_p(t)$ absolutely converges almost surely.*

This is an analogous result obtained by Kawata in the weakly stationary case [6].

6. The sample W-continuity. It is known that a W-harmonizable DS process is mean W-continuous [2]. We shall give a sufficient condition for the sample W-continuity of the process.

LEMMA 6.1. *Let $\hat{X}_p(t)$ be the Walsh series defined by* (5.1). *If* (5.8) *holds, then $\hat{X}_p(t)$ converges uniformly over every finite interval almost surely as $p \to \infty$.*

PROOF. Since by definition $\zeta_{m,p} = \zeta_{2m,p+1} + \zeta_{2m+1,p+1}$ and $\psi_{2m}(2^{-(p+1)}t) = \psi_m(2^{-p}t)$, we see that

$$\hat{X}_{p+1}(t) - \hat{X}_p(t) = \sum_{m=0}^{\infty} (\psi_{2m+1}(2^{-(p+1)}t) - \psi_{2m}(2^{-(p+1)}t))\zeta_{2m+1,p+1}$$

which is majorized by

$$\sum_{m=0}^{\infty} (1 - \psi_1(2^{-(p+1)}t))|\zeta_{2m+1,p+1}| .$$

Hence, in view of (3.3), for $A > 1$

$$\max_{t \leqq A} |\hat{X}_{p+1}(t) - \hat{X}_p(t)| \leqq C(A, p) \sum_{m=0}^{\infty} |\zeta_{2m+1,p+1}| ,$$

where $C(A, p) = 0$ if $p > \log_2 A$; $= 2$, otherwise. Take a sequence $\{\varepsilon_p\}$ of positive numbers decreasing to zero. By Tchebychev's inequality we have that

$$Q_p = \Pr\{\max_{p \leqq A} |\hat{X}_{p+1}(t) - \hat{X}_p(t)| \geqq \varepsilon_p\} \leqq (C(A, p)/\varepsilon_p)^2 E\left\{\left(\sum_{m=0}^{\infty} |\zeta_{2m+1,p+1}|\right)^2\right\} .$$

In the same way as (2.5) in [7], we can prove that

$$E\left\{\left(\sum_{m=0}^{\infty} |\zeta_{2m+1,p+1}|\right)^2\right\} < \infty .$$

Hence

$$\sum_{p=1}^{\infty} Q_p \leqq 2 \sum_{p=1}^{\log_2 A} 1/\varepsilon_p^2 \int_0^{\infty} dF(x) < \infty .$$

Therefore Borel-Cantelli's lemma implies that with probability one the series $\sum_{p=1}^{\infty} (\hat{X}_{p+1}(t) - \hat{X}_p(t))$ converges uniformly in $0 \leqq t \leqq A$.

THEOREM 6.1. *If the assumption* (5.8) *in Theorem* 5.2 *is satisfied, then $X(t)$ is equivalent to a W-harmonizable* DS *process which is sample W-continuous.*

The proof is clear by Theorem 5.1 and Lemma 6.1, since the limit of a uniformly convergent sequence of W-continuous functions is W-continuous.

COROLLARY 6.1. *If there exists a function* $g(x)$ *defined on* R_+ *which is non-negative, non-decreasing, and satisfies* (5.9) *and* (5.10), *then* $X(t)$ *is equivalent to a W-harmonizable* DS *process which is sample W-continuous.*

Finally we remark that Corollary 6.1 is a generalization of Corollary 2.1 since $g(x) = x^{\alpha}$ $(\alpha > 1)$ satisfies the conditions (5.9) and (5.10).

ACKNOWLEDGMENTS. The author would like to express his thanks to the referees for their useful comments and suggestions.

REFERENCES

[1] P. L. BUTZER AND H. J. WAGNER, A calculus for Walsh functions defined on R_+, in Applications of Walsh Functions (Proc. Sympos. Naval Res. Lab., Washington D.C., Apr. 18-20, 1973), 75-81. Washington.

[2] Y. ENDOW, Analysis of dyadic stationary processes using the generalized Walsh functions, Tôhoku Math. J. 36 (1984), 485-503.

[3] N. J. FINE, On the Walsh functions, Trans. Amer. Math. Soc. 65 (1949), 372-414.

[4] N. J. FINE, The generalized Walsh functions, Trans. Amer. Math. Soc. 69 (1950), 66-77.

[5] T. KAWATA, On the Fourier series of a stationary stochastic process, Z. Wahrscheinlichkeitstheorie verw. Gebiete 6 (1966), 224-245.

[6] T. KAWATA, On the Fourier series of a stationary stochastic process. II, Z. Wahrscheinlichkeitstheorie verw. Gebiete 13 (1969), 25-38.

[7] I. KUBO, On a necessary condition for the sample continuity of weakly stationary processes, Nogoya Math. J. 38 (1970), 103-111.

[8] R. E. A. C. PALEY, A remarkable series of orthogonal functions (I), (II), Proc. London Math. Soc. 34 (1931), 241-279.

[9] R. G. SELFRIDGE, Generalized Walsh Transform, Pacific J. Math. 5 (1955), 451-480.

[10] S. YANO, On Walsh-Fourier series, Tôhoku Math. J. 3 (1951), 223-242.

FACULTY OF SCIENCE AND ENGINEERING
CHUO UNIVERSITY
1-13-27 KASUGA, TOKYO 112
JAPAN