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THE K-ENERGY MAP, ALMOST EINSTEIN KÄHLER METRICS AND AN INEQUALITY OF THE MIYAOKA-YAU TYPE

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The purpose of this paper is to give some addenda to the paper [2], in which we showed the boundedness as well as a consequence of the K-energy map under the assumption of the existence of Einstein Kähler metrics. We here show how these results can be refined. The author would like to thank Professor H. Urakawa for valuable suggestions.

Let us first fix our notation. Throughout this paper X is an n-dimensional compact complex manifold with positive first Chern class $c_1(X) > 0$. Let K be the set of all Kähler forms on X representing $2\pi c_1(X)$. To each $\omega \in K$ we associate its Ricci curvature form γ_ω, scalar curvature σ_ω, Laplacian Δ_ω acting on the space of real-valued C^∞-functions $C^\infty(X)$, and $f_\omega \in C^\infty(X)$ such that $\gamma_\omega - \omega = \sqrt{-1}\partial\bar{\partial} f_\omega$, which is determined up to a suitably chosen constant.

DEFINITION. For ω_0, $\omega_1 \in K$, we choose a smooth path $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial} u_t \in K$ connecting ω_0 and ω_1, where $u_t \in C^\infty(X)$ and $t \in [0, 1]$. Hereafter when we encounter notation like γ_{ω_t}, σ_{ω_t}, Δ_{ω_t}, f_{ω_t}, ∇_{ω_t}, we simplify their subscripts in the form γ_t, σ_t, Δ_t, f_t, ∇_t. Using this convention, define three functions I, J, M on $K \times K$ as follows:

$$I(\omega_0, \omega_1) = \int_X u_1(\omega_0^n - \omega_1^n)/V ,$$

$$J(\omega_0, \omega_1) = \int_0^1 dt \int_X \frac{du_t}{dt}(\omega_0^n - \omega_t^n)/V ,$$

$$M(\omega_0, \omega_1) = -\int_0^1 dt \int_X \frac{du_t}{dt}(\sigma_t - n)\omega_t^n/V ,$$

where $V = \int_X \omega_0^n$. For ω_0 in K, we define the corresponding K-energy map $\mu = \mu_{\omega_0}$ from K to R by $\mu(\omega) = M(\omega_0, \omega)$ for $\omega \in K$. Set

$$K^+ = \{\omega \in K \mid \gamma_\omega > 0, \text{ i.e. } \gamma_\omega \text{ is positive definite on } X\} .$$

I, J, M are all well-defined and have nice properties (see [1], [7]).

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The boundedness theorem proved in [2] is:

THEOREM A. *If X admits an Einstein Kähler metric, then the restriction of the* K*-energy map to K^+ attains its minimum exactly on the set of Einstein Kähler metrics. In particular, the* K*-energy map is bounded from below on K^+.*

Our refined version of the above theorem is as follows:

THEOREM 1. *If X admits an Einstein Kähler metric, then* K*-energy map on K attains its minimum exactly on the set of Einstein Kähler metrics, and is in particular bounded below on K.*

This is a consequence of the following:

THEOREM 2. *For $\omega \in K^+$, we have*

$$M(\gamma_\omega, \omega) \leqq 0 ,$$

and the equality holds if and only if $\gamma_\omega = \omega$.

PROOF. Regarding $\omega_0 = \gamma_\omega \in K$ as a Kähler metric, we consider the following equation in $\omega_t \in K$ solved by Yau:

$$\omega_t^n = \exp(tf_0 + c_t)\omega_0^n , \qquad t \in [1, 0] ,$$

where $c_t := \log V - \log\left(\int_X \exp(tf_0)\omega_0^n\right)$. Express ω_t as $\omega_0 + \sqrt{-1}\partial\bar{\partial}u_t$, with a smooth path $\{u_t; 0 \leqq t \leqq 1\}$. Differentiating the above equation with respect to t, we have

$$\Delta_t \frac{du_t}{dt} = f_0 + \text{const} .$$

Also we easily see that f_t can be taken as $(1-t)f_0 - u_t$. Thus,

$$\frac{d}{dt}M(\omega_0, \omega_t) = -\int_X \frac{du_t}{dt}\Delta_t f_t \omega_t^n / V = -(1-t)\int_X \left(\Delta_t \frac{du_t}{dt}\right)^2 \omega_t^n / V$$
$$+ \int_X \frac{du_t}{dt}\Delta_t u_t \omega_t^n / V \leqq -\frac{d}{dt}(I - J)(\omega_0, \omega_t) .$$

Since $\omega_1 = \omega$, we conclude that

$$M(\gamma_\omega, \omega) \leqq -(I - J)(\gamma_\omega, \omega) \leqq 0 .$$

q.e.d.

PROOF OF THEOREM 1. For any $\omega \in K$ there exists $\tilde{\omega} \in K^+$ such that $\gamma_{\tilde{\omega}} = \omega$ by Yau's result. Then we have

$$\mu(\omega) - \mu(\tilde{\omega}) = -M(\gamma_{\tilde{\omega}}, \tilde{\omega}) \geqq 0 .$$

q.e.d.

Next, fixing ω_0 in K, we consider the following theorem proved in [2].

THEOREM B. *Suppose that the* K-*energy map is bounded below. Then we can solve Aubin's equation on* [0, 1). *In particular, for arbitarary* $\varepsilon > 0$, *we get Kähler metric* $\omega \in K$ *such that* $\gamma_\omega > (1-\varepsilon)\omega$, *hence* $\sigma_\omega - n > -n\varepsilon$.

Here by Aubin's equation we mean

$$\gamma_{\omega_t} = t\omega_t + (1-t)\omega\,, \qquad t \in [0, 1]\,,$$

with given ω in K, where for each t, the solution ω_t is required to sit in K and is written in the form $\omega + \sqrt{-1}\partial\bar{\partial}u_t$, with a smooth path $\{u_t;\, 0 \leqq t \leqq 1\}$ in $C^\infty(X)$. Our theorem is:

THEOREM 3. *For the Kähler metrics* ω_t, $0 \leqq t < 1$, *obtained just above, we moreover have*

$$\operatorname{Osc} f_t := \operatorname{Max} f_t - \operatorname{Min} f_t \leqq \varepsilon\,,$$

for t sufficiently near 1.

PROOF. As we showed in [2], $\operatorname{Osc} u_t$ can be estimated by $(I-J)(u_t)$, and we can take f_t as $-(1-t)u_t$, it suffices to show that $j=(1-t)(I-J)(u_t)$ tends to zero as $t \to 0$. This is easily seen by

$$\frac{d}{dt}j + (1-t)^{-1}j = -\frac{d}{dt}\mu\,.$$

q.e.d.

Furthermore under the same assumption we can prove the existence of an almost Einstein Kähler metric in the following sense:

THEOREM 4. *If the* K-*energy map is bounded below, then for arbitrary* $\varepsilon > 0$, *X admits a Kähler metric* $\omega \in K$ *such that* $|\sigma_\omega - n| \leqq \varepsilon$.

PROOF. The strategy for the proof is as follows. First we use Aubin's equation to make $|f_\omega|_{C^0(X)}$ as small as possible. Then using Hamilton's equation, we regularlize f_ω to get an estimate on $\sigma_\omega - n = \Delta_\omega f_\omega$. For simplicity, we denote ω_t, f_t, Δ_t, ∇_t by ω, f, Δ, ∇ below, if there is no fear of confusion.

First we note that Hamilton's equation

$$\frac{d}{dt}\omega = -(\gamma_\omega - \omega)$$

always has a unique solution $\omega = \omega_t$ in K for $0 \leqq t < +\infty$ (see, for instance, [4]). In terms of f the equation can be written as

(a) $\frac{d}{dt}f = \Delta f + f$.

We then have the following equalities:

(b) $\frac{d}{dt}f^2 = \Delta f^2 - 2|\nabla f|^2 + 2f^2$,

(c) $\frac{d}{dt}|\nabla f|^2 = \Delta|\nabla f|^2 - |\nabla\bar{\nabla} f|^2 - |\nabla\nabla f|^2 + |\nabla f|^2$,

(d) $\frac{d}{dt}\Delta f = \Delta\Delta f + \Delta f + |\nabla\bar{\nabla} f|^2$.

Let $0 < \varepsilon \ll 1$. We devide the proof into the four steps.

(1) If the initial f_0 satisfies $|f_0| < \varepsilon$, then $|f| < e^2\varepsilon$ for $t \in [0, 2]$.

(2) For $t \in [1, 2]$, we have $|\nabla f|^2 < e^4\varepsilon^2$ by combining (1) with (cf. (b), (c))

$$\frac{d}{dt}(f^2 + t|\nabla f|^2) \leqq \Delta(f^2 + t|\nabla f|^2) + 2(f^2 + t|\nabla f|^2),$$

for $0 \leqq t \leqq 2$.

(3) If the initial f_0 satisfies $\Delta_0 f_0 \geqq -\varepsilon$, then $\Delta f \geqq -e^2\varepsilon$ holds for $t \in [0, 2]$ by (d).

(4) We now show that $\Delta_2 f_2 < 2ne^5\varepsilon$. Let $\tilde{t} := t - 1$, and $\alpha := n^{-1}$. Then for $0 \leqq \tilde{t} \leqq 1$, we have

$$\frac{d}{dt}(|\nabla f|^2 + \varepsilon\alpha\tilde{t}\Delta f) \leqq \Delta(|\nabla f|^2 + \varepsilon\alpha\tilde{t}\Delta f)$$
$$+ (|\nabla f|^2 + \varepsilon\alpha\tilde{t}\Delta f) + \varepsilon\alpha\Delta f - (1 - \varepsilon\alpha\tilde{t})|\nabla\bar{\nabla} f|^2 .$$

Therefore, by the Cauchy-Schwarz inequality $n|\nabla\bar{\nabla} f|^2 \geqq (\Delta f)^2$, we get

(e) $$\frac{d}{dt}e^{-\tilde{t}}(|\nabla f|^2 + \varepsilon\alpha\tilde{t}\Delta f) \leqq \Delta e^{-\tilde{t}}(|\nabla f|^2 + \varepsilon\alpha\tilde{t}\Delta f)$$
$$+ e^{-\tilde{t}}\Delta f\{\varepsilon\alpha - n^{-1}(1 - \varepsilon\alpha)\Delta f\} .$$

We claim that $e^{-\tilde{t}}(|\nabla f|^2 + \varepsilon\alpha\tilde{t}\Delta f) < 2e^4\varepsilon^2$, for $\tilde{t} \in [0, 1]$. Otherwise at the point of $[0, 1]\times X$ where it fails to hold for the first time $\tilde{t} > 0$, we have $e^{-\tilde{t}}\varepsilon\alpha\tilde{t}\Delta f \geqq e^4\varepsilon^2$, which implies $\Delta f \geqq e^4\varepsilon$. But we derive from (e) that $n^{-1}(1 - \varepsilon\alpha)\Delta f \leqq \varepsilon\alpha$ at the point. Thus, $(1 - \varepsilon\alpha)e^4 \leqq 1$, which is a contradiction. Therefore we obtain the desired estimate:

$$\Delta f < 2ne^5\varepsilon , \qquad \text{for} \quad \tilde{t} = 1 .$$

q.e.d.

REMARK. The smoothing theorem in [3] can also be derived in a similar manner without using difficult analysis.

THEOREM 5. *If the* K-*energy map is bounded below, then we have an inequality of the Miyaoka-Yau type*:

$$2(n+1)c_2(X)c_1(X)^{n-2}[X] \geqq nc_1(X)^n[X] .$$

PROOF. For $\omega \in K$, we define a tensor T which measures the deviation of (X, ω) from being of constant holomorphic sectional curvature by

$$T_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \sigma/n(n+1) \quad (g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) ,$$

where R and g are the curvature tensor and the metric tensor, respectively. A calculation shows that

$$n(n-1)(2\pi)^n\{2(n+1)c_2(X)c_1(X)^{n-2} - nc_1(X)^n\}[X] = \int_X \{(n+1)|T|^2 - (n+1-2/n)(\sigma-n)^2\}\omega^n .$$

We then apply Theorem 4, and get the desired result. q.e.d.

REFERENCES

[1] T. AUBIN, Réduction du cas positif de l'équation de Monge-Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité, J. Funct. Anal. 57 (1984), 143–153.

[2] S. BANDO AND T. MABUCHI, Uniqueness of Einstein Kähler metrics modulo connected group actions, to appear in Algebraic Geometry, Sendai, 1985 (T. Oda, ed.), Adv. Studies in pure Math. 10, Kinokuniya, Tokyo and North-Holland, Amsterdam, New York, Oxford.

[3] J. BEMELMANS, MIN-OO AND E. A. RUH, Smoothing Riemannian Metrics, Math. Z. 188 (1984), 69–74.

[4] H.-D. CAO, Deformation of Kähler matrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. Math. 81 (1985), 359–372.

[5] R. HAMILTON, Three-manifolds with positive Ricci curvature, J. Diff. Geom. 17 (1982), 255–306.

[6] T. MABUCHI, A functional integrating Futaki's invariant, Proc. Japan Acad. 61A (1985), 119–120.

[7] T. MABUCHI, K-energy maps integrating Futaki invariants, Tôhoku Math. J. 38 (1986), 575–593.

[8] S. T. YAU, Calabi's conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. USA 74 (1977), 1798–1799.

[9] S. T. YAU, On the Ricci curvature of a compact Kähler manifolds and the complex Monge-Ampère equations I, Comm. Pure Appl. Math. 31 (1978), 339–411.

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