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EPOCHS OF REGULARITY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN UNBOUNDED DOMAINS

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1. **Introduction.** Leray, in his famous paper [6] of 1934, proved the existence of weak solutions of the Navier-Stokes equations in the spatial domain $\Omega = \boldsymbol{R}^3$. Hopf, in his famous paper [4] of 1951, proved the existence of weak solutions in arbitrary open subsets Ω of $\boldsymbol{R}^n$, $n \geqq 2$. Leray, however, proved for his solutions two important properties that Hopf did not, namely the *strong energy inequality* (i.e., that the energy inequality (4) below should hold for almost all $s_1 > 0$) and the *epochs of regularity property* (Definition 3, below). Ladyzhenskaya, in her book [5], reformulated Hopf's theorem so as to include the strong energy inequality (slightly modified). She proved it for bounded domains, and claimed that the proof carries over without change to unbounded domains. This claim was evidently based on an oversight; one which has been shared since by several other authors. In recent years, as it has become realized that an oversight was made, there have been many efforts to prove the strong energy inequality in unbounded domains other than the whole space $\boldsymbol{R}^3$, considered by Leray. But to date, these efforts have failed.

The epochs of regularity property was proved in the case of bounded three dimensional domains by Shinbrot and Kaniel, who included it in their 1966 paper [11]. Their proof follows the same line of argument as Leray's. It consists in pointing out that, for almost every time $t \geqq 0$, a weak solution has enough regularity to serve as the initial value for a smooth solution, and that the two solutions can be identified over the time interval during which the smooth solution remains smooth. That the weak and smooth solutions are the same on this interval follows from a uniqueness theorem of Leray, which has been generalized by others, notably by Serrin in [10]. Its application in proving the epochs of regularity property requires that the weak solution satisfy the strong energy inequality. Thus, in [6] and [11], the epochs of regularity property

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is obtained as a consequence of the strong energy inequality. In my paper [2], on the existence, regularity and decay of solutions, I constructed smooth solutions in unbounded domains which, among other things, are suitable for use as the smooth solutions in the argument of Leray and of Shinbrot and Kaniel. As an application of the main results of [2], I claimed in Theorem 8 of that paper, that the epochs of regularity property holds for weak solutions in unbounded three-dimensional domains. I claimed this believing that the strong energy inequality had been established and was well known for arbitrary domains, unaware of the error in [5]. That this is not true has been very kindly brought to my attention by Professor K. Masuda. It is particularly unfortunate that I did not realize this when writing [2], because the special manner in which solutions were constructed there makes possible a direct proof of the epochs of regularity property, in either bounded or unbounded domains, without appealing to the strong energy inequality. Our purpose here is to present this alternative argument, proving for the first time the existence of weak solutions possessing the epochs of regularity property in unbounded domains other than the whole space $\boldsymbol{R}^3$, considered by Leray.

Leray proved for his solutions that the Dirichlet norm tends to infinity at the right end point of each epoch of regularity (except the last which is a semi-infinite interval). Using this, he obtained a bound for the sum of the square roots of the lengths of the epochs (excluding the last). We have been unable to show for our solutions that the Dirichlet norm must necessarily tend to infinity at singularities; it will be explained how there might be other types of singularities. However, by carefully examining our construction, we do prove a result like Leray's on the lengths of the epochs. Also, what is closely related, we obtain a bound for the one-half dimensional Hausdorff measure of the set of singular points in time.

Hopf's original construction of solutions by Galerkin approximation applies in a single step to any domain, bounded or unbounded. In Ladyzhenskaya's modification of the argument, solutions in unbounded domains are obtained from a sequence of solutions defined in an expanding sequence of bounded subdomains. We adopted the latter procedure in [2], in order to use the eigenfunctions of the Stokes operator as basis functions. This permitted further estimates for the Galerkin approximations, ultimately yielding the local existence of a smooth classical solution. Here, we use the same basic construction, but with two innovations. The first is a new estimate for the Galerkin approximations, given as (30) below. The other is a new condition in the selection of a subsequence of the Galerkin approximations, and subsequently of the weak solutions defined in bounded subdomains, so as to preserve (30) in the final result, i.e., for a globally existing weak solution. The experienced reader will probably understand these main points by simply reading Theorems 3 and 4. However, to make the proof solid, we need to lay out, in just the right form, the construction and selection procedure of Hopf/Ladyzhenskaya. This is done in Theorems 1 and 2. The proofs of these theorems are given somewhat briefly, though hopefully clarifying several points which were not addressed in [4] and [5].

In the papers just mentioned, the solution u is obtained as the limit of Galerkin approximations. But a choice is not made between introducing it as a limit in L^2 of space-time, or as a limit in L^2 of space, at every time. It is simply regarded as the limit in both senses simultaneously, without any mention of justification. This identification of the two limits, early in the proof, greatly facilitates matters, particularly in proving the solution's weak continuity in $\boldsymbol{L}^2(\Omega)$, as a function of time, and in proving the strong energy inequality for bounded domains. These properties then follow directly from corresponding properties of the Galerkin approximations. Here, in the appendix to this paper, we provide a lemma which justifies the identification of the two limits. Although this lemma was originally proven in my thesis [3], it has not appeared in my previous papers because they have all dealt with stronger solutions possessing a time derivative in L^2 of space-time. In that context, I have regarded it as preferable to take limits only in L^2 of space-time, and to introduce the values of u on time-cuts as traces. In the context of weak solutions, some authors have treated the issue carefully by taking limits only in space-time, and then late in the proof justified a redefinition of the solution on a set of t-measure zero, so as to make it weakly L^2-continuous in time; see [8] and [10]. The justifications given at that point are essentially based on having established the existence of a weak distributional time derivative, something not needed for our argument. It seems much easier to follow the line of proof given in [5], coupled with the justification provided here, in the appendix.

2. Definitions. Let $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, be open.

Let $\boldsymbol{D}(\Omega) = \{\phi \in \boldsymbol{C}_0^\infty(\Omega) \colon \nabla \cdot \phi = 0\}$,

$\boldsymbol{J}(\Omega) =$ Completion of $\boldsymbol{D}(\Omega)$ in $\boldsymbol{L}^2(\Omega)$,

$\boldsymbol{J}_1(\Omega) =$ Completion of $\boldsymbol{D}(\Omega)$ in $\boldsymbol{W}_2^1(\Omega)$.

$\boldsymbol{J}_1^*(\Omega) = \{\phi \in \overset{\circ}{\boldsymbol{W}}{}_2^1(\Omega) \colon \nabla \cdot \phi = 0\}$,

$P: \boldsymbol{L}^2(\Omega) \rightarrow \boldsymbol{J}(\Omega)$ be defined by orthogonal projection in $\boldsymbol{L}^2(\Omega)$,
$\tilde{\Delta} \equiv P\Delta$ be the Stokes operator.

Let $a \in \boldsymbol{J}(\Omega)$, and let $f \in L^1(0, T; \boldsymbol{L}^2(\Omega))$, for every $T > 0$.

We distinguish between two types of weak solutions of the Navier-Stokes problem

$$(1) \qquad \begin{aligned} &u_t + u\cdot\nabla u = -\nabla p + \Delta u + f \quad \text{and} \quad \nabla\cdot u = 0\,, \quad \text{in} \quad \Omega\times(0, \infty)\,, \\ &u_{|t=0} = a\,, \qquad u_{|\partial\Omega} = 0\,, \qquad u(t) \in \boldsymbol{J}_1(\Omega) \quad \text{for} \quad t > 0\,. \end{aligned}$$

The first definition below is Leray's original definition, adopted also by Ladyzhenskaya, making the strong energy inequality one of the basic properties of a weak solution. When a Leray solution exists in a three-dimensional domain with smooth boundary, it possesses the epochs of regularity property. But its existence is not yet known in unbounded three-dimensional domains other than the whole space $\boldsymbol{R}^3$. Dropping the strong energy inequality from the other conditions, we have what is essentially Hopf's definition. The existence of a Hopf solution is known in any open set $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$. We shall modify Hopf's construction so as to obtain the epochs of regularity property in any three-dimensional domain whose boundary, if nonempty, is uniformly twice continuously differentiable.

DEFINITION 1. We call u a *Leray solution* of (1) if and only if it

(2) is defined (pointwise) and measurable on $\Omega \times [0, \infty)$; is also measurable in Ω and belongs to $\boldsymbol{J}(\Omega)$ for every $t \in [0, \infty)$; has $\boldsymbol{L}^2(\Omega)$-norm $\|u(\cdot, t)\|$ uniformly bounded over every finite time interval $t \in [0, T]$; has generalized first order spatial derivatives $\nabla u \in L^2(\Omega \times (0, T))$ for every $T > 0$; belongs to $\boldsymbol{J}_1(\Omega)$ for almost all $t > 0$; satisfies

$$(3) \qquad \int_0^s [(u, \phi_t) - (\nabla u, \nabla\phi) - (u\cdot\nabla u, \phi) + (f, \phi)]dt = (u(s), \phi(s)) - (a, \phi(0))\,,$$

for all smooth solenoidal functions $\phi(x, t)$ with compact supports in $\Omega \times [0, \infty)$, and for all $s \geqq 0$; and satisfies

$$(4) \qquad \frac{1}{2}\|u(s)\|^2 + \int_{s_1}^s \|\nabla u\|^2 dt \leqq \frac{1}{2}\|u(s_1)\|^2 + \int_{s_1}^s (f, u)dt\,,$$

for almost all values of $s_1 \geqq 0$ including $s_1 = 0$, and for all values of $s > s_1$.

DEFINITION 2. We call u a *Hopf solution* of (1) if it satisfies the conditions (2), (3), and (4), except that the energy inequality (4) is required to hold only for $s_1 = 0$.

DEFINITION 3. We say that a weak solution u of (1) possesses the *epochs of regularity property* if and only if there exists an open subset $R \subset [0, \infty)$, such that the measure of $[0, \infty) - R$ is zero, and such that for every compact interval $I \subset R$ there holds

$$\sup_{t \in I} \|u\|^2_{W^1_2(\Omega)} + \int_I (\|u\|^2_{W^2_2(\Omega)} + \|u_t\|^2) dt < \infty . \tag{5}$$

REMARK 1. The condition that $u(t) \in \boldsymbol{J}_1(\Omega)$, for a.a. $t > 0$, is in general necessary for the well posing of problem (1) in unbounded domains. Indeed, without it, and even if it is replaced by the weaker condition that $u(t) \in \boldsymbol{J}_1^*(\Omega)$ for a.a. $t > 0$, there exist domains for which problem (1) possesses fully classical nontrivial solutions corresponding to the data $a \equiv 0$ and $f \equiv 0$. This was shown and the matter investigated in [1].

REMARK 2. Since $u(t)$ is bounded in $\boldsymbol{J}(\Omega)$ over any finite interval of time, it follows from (3) that $u(t)$ is weakly continuous in $\boldsymbol{L}^2(\Omega)$ for all $t \geqq 0$. Therefore, (4) ensures that $u(t)$ is strongly right continuous in $\boldsymbol{L}^2(\Omega)$ for almost all $t \geqq 0$, including $t = 0$. In particular, $\|u(t) - a\| \to 0$ as $t \to 0^+$.

3. The preliminary construction. In this section we give Ladyzhenskaya's version of Hopf's theorem. Where we are sketchy with the details, most can be found in [5]. The principal thing which may be new here is our suggestion for clarifying the limits to be taken in introducing the solution u.

THEOREM 1. *If Ω is a bounded open subset of $\boldsymbol{R}^n$, $n \geqq 2$, then there exists a Leray solution of the Navier-Stokes problem* (1).

PROOF. There exists a system of functions $\{a^l(x)\}$ belonging to $\boldsymbol{J}_1(\Omega) \cap \boldsymbol{W}_2^2(\Omega) \cap \boldsymbol{C}(\bar{\Omega})$, which is orthonormal in $\boldsymbol{J}(\Omega)$, whose finite linear combinations can approximate any function from $\boldsymbol{D}(\Omega)$ arbitrarily well in the norm $\sup_\Omega |\cdot| + \|\nabla \cdot\|_{L^2(\Omega)}$. Let $\{a^l(x)\}$ be such a system. Let

$$u^n(x, t) = \sum_{k=1}^n c_{kn}(t) a^k(x) , \quad n = 1, 2, \cdots ,$$

satisfy, for $l = 1, \cdots, n$, the conditions

$$\begin{aligned} &(u_t^n + u^n \cdot \nabla u^n - \Delta u^n - f, a^l) = 0 , \quad \text{for a.a. } t \geqq 0 , \\ &(u^n(0) - a, a^l) = 0 . \end{aligned} \tag{6}$$

From (6) one obtains $(1/2)(d/dt)\|u^n\|^2 + \|\nabla u^n\|^2 = (f, u^n)$, and so

$$(7)\qquad \frac{1}{2}\|u^n(s)\|^2 + \int_{s_1}^{s} \|\nabla u^n\|^2 dt = \frac{1}{2}\|u^n(s_1)\|^2 + \int_{s_1}^{s} (f, u^n) dt ,$$

for all $0 \leqq s_1 < s < \infty$. Since $(d/dt)\|u^n\| \leqq \|f\|$, one has

$$(8)\qquad \|u^n(s)\| \leqq \|a\| + \int_0^s \|f\| dt , \qquad \text{for all} \quad s \geqq 0 ,$$

and

$$(9)\qquad \sup_{[0,T]} \|u^n\|^2 + \int_0^T \|\nabla u^n\|^2 dt \leqq C_T , \qquad \text{for all} \quad T \geqq 0 .$$

It follows that the coefficients $\{c_{kn}(t)\}$ are all uniformly bounded on any finite interval $0 \leqq t \leqq T$. For fixed k, the sequence $\{c_{kn}(t)\}$, $n = 1, 2, \cdots$, is also equicontinuous over $[0, T]$. Indeed, one finds using (6) and (9), that for any $0 \leqq s < s' \leqq T$, there holds

$$(10)\qquad \begin{aligned} |c_{kn}(s') - c_{kn}(s)| &= \left| \int_s^{s'} \{-(u^n \cdot \nabla u^n, a^k) - (\nabla u^n, \nabla a^k) + (f, a^k)\} dt \right| \\ &\leqq C_k C_T^* \sqrt{s' - s} + \int_s^{s'} \|f\| dt , \end{aligned}$$

where C_T^* depends only on C_T in (9), and C_k depends only on a^k.

Hence there exists a subsequence of $\{n\}$, again denoted by $\{n\}$, such that for each k and T, the sequence $\{c_{kn}(t)\}$ is uniformly convergent on $[0, T]$, as $n \to \infty$, to a continuous limit $c_k(t)$. To be as concrete as possible, we will introduce u as a limit of the series

$$u(x, t) \sim \sum_{k=1}^{\infty} c_k(t) a^k(x) .$$

There are several ways this can be done. One can define u as a limit in $\boldsymbol{L}^2(\Omega)$, for each fixed t, or as a limit in $L^2(\Omega \times [0, T))$, for every $T \geqq 0$. Convergence is assured in either sense because the partial sums of the series all satisfy the estimate (8), and because the system $\{a^k\}$ is orthogonal in $\boldsymbol{L}^2(\Omega)$. In fact, we need u to be the limit in both senses. For each fixed $t \geqq 0$, let $\bar{u}(\cdot, t)$ be the limit of the series in $L^2(\Omega)$. More precisely, since an element of $\boldsymbol{L}^2(\Omega)$ is an equivalence class of measurable functions, let $\bar{u}(\cdot, t)$ be a particular measurable function representing the limit. Of course, $\bar{u}(x, t)$ need not be measurable in $\Omega \times [0, \infty)$. Let $\tilde{u}(x, t)$ be a function, measurable in $\Omega \times [0, \infty)$, which is representative of the limit of the series in $L^2(\Omega \times [0, T))$, for every T. Of course, one can not expect $\tilde{u}(x, t)$ to be the limit of the series in $\boldsymbol{L}^2(\Omega)$, for all values of t. However, in the appendix, we show that for almost all $t \geqq 0$,

$$(11)\qquad \bar{u}(x, t) = \tilde{u}(x, t) , \qquad \text{for almost all} \quad x \in \Omega .$$

Therefore, setting

$$u(x, t) = \begin{cases} \tilde{u}(x, t) , & \text{if } t \text{ is such that (11) holds ,} \\ \bar{u}(x, t) , & \text{if } t \text{ is such that (11) does not hold ,} \end{cases}$$

we obtain a function (defined pointwise) which is measurable in Ω, for every t, as well as in $\Omega \times [0, \infty)$, and to which the series converges, both in $\boldsymbol{L}^2(\Omega)$ for every t, and in $L^2(\Omega \times [0, T))$, for every T.

In view of (9), and of the convergence $c_{kn}(t) \to c_k(t)$, for all k, the subsequence of Galerkin approximations $\{u^n\}$ converges to u weakly in $\boldsymbol{L}^2(\Omega)$, for every t, as well as in $L^2(\Omega \times [0, T))$, for every T. We wish to show that the convergence is strong in $L^2(\Omega \times [0, T))$.

To that end, note first that the uniformity of the convergence $c_{kn}(t) \to c_k(t)$, on $[0, T]$, for each fixed k, implies that the weak convergence of $u^n(\cdot, t)$ to $u(\cdot, t)$ in $\boldsymbol{L}^2(\Omega)$, is uniform in t. That is, for every $\phi \in \boldsymbol{L}^2(\Omega)$, and every ε, $T > 0$, there exists an integer N such that

$$(12) \qquad |(u^n(t) - u(t), \phi)| < \varepsilon , \qquad \text{for all} \quad t \in [0, T] , \quad \text{and} \quad n \geqq N .$$

Then recall Friedrich's lemma: For any bounded domain Ω, and any $\varepsilon > 0$, there exist functions $\{\omega_1, \cdots, \omega_N\}$ such that

$$(13) \qquad \|u\|^2 \leqq \sum_{k=1}^{N}(u, \omega_k)^2 + \varepsilon\|\nabla u\|^2 , \qquad \text{for all} \quad u \in \overset{\circ}{W}{}_2^1(\Omega) .$$

Thus, for any two Galerkin approximations u^n and u^m, one has

$$(14) \qquad \int_0^T \|u^n - u^m\|^2 dt \leqq \sum_{j=1}^{N} \int_0^T (u^n - u^m, \omega_k)^2 dt + \varepsilon \int_0^T \|\nabla(u^n - u^m)\|^2 dt .$$

Using (9) and (12), this implies, for the chosen subsequence, that $u^n \to u$ strongly in $L^2(\Omega \times [0, T))$, for every $T > 0$.

In view of (9), u has derivatives $\nabla u \in L^2(\Omega \times (0, T))$, and $\nabla u^n \to \nabla u$ weakly in $L^2(\Omega \times (0, T))$, for every $T > 0$.

Let us regard $L^2(0, T; \boldsymbol{J}_1(\Omega))$ as the completion, in the obvious norm, of all smooth solenoidal functions with compact supports in $\Omega \times [0, T]$. Since $\{u^n\}$ is bounded in $L^2(0, T; \boldsymbol{J}_1(\Omega))$, there exists a further subsequence whose arithmetic means $(u^1 + \cdots + u^n)/n$ converge strongly in $L^2(0, T; \boldsymbol{J}_1(\Omega))$, to some element $\hat{u}$ of $L^2(0, T; \boldsymbol{J}_1(\Omega))$. However, these arithmetic means also converge to u in $L^2(0, T; \boldsymbol{L}^2(\Omega))$, permitting the identification $u = \hat{u}$ almost everywhere. Thus $u \in L^2(0, T; \boldsymbol{J}_1(\Omega))$. Using mollifiers with respect to time, we conclude that $u(t) \in \boldsymbol{J}_1(\Omega)$, for almost all $t > 0$.

To establish (3), note that, for $n \geqq m$, and any $s \geqq 0$, (6) implies

$$\int_0^s [(u^n, \phi_t^m) - (\nabla u^n, \nabla\phi^m) - (u^n\cdot\nabla u^n, \phi^m) + (f, \phi^m)]dt \\ = (u^n(s), \phi^m(s)) - (a, \phi^m(0)) , \tag{15}$$

where ϕ^m is an arbitrary function of the form $\phi^m(x, t) = \sum_{k=1}^m d_k(t)a^k(x)$, with continuously differentiable coefficients d_k on $[0, \infty)$. Since u^n converges to u in every time-cut, as well as in space-time, one can justify letting $n \to \infty$ in (15). This implies (3), because any smooth solenoidal function $\phi(x, t)$, with compact support in $\Omega\times[0, s]$, can be approximated arbitrarily well by functions of the form ϕ^m in the norm

$$\sup_{\Omega\times[0,s]} |\phi| + \sup_{\Omega\times[0,s]} |\phi_t| + \int_0^s \|\nabla\phi\|^2 dt .$$

For this, see Masuda [7, p. 630].

Finally, one can show that the strong energy inequality (4) follows from (7). Indeed, we observe that

$$\lim_{n\to\infty} \int_{s_1}^s (f, u^n)dt = \int_{s_1}^s (f, u)dt ,$$

by (8) and the Lebesgue convergence theorem. Hence (7) easily implies

$$\frac{1}{2}\|u(s)\|^2 + \int_{s_1}^s \|\nabla u\|^2 dt \leqq \frac{1}{2} \liminf_{n\to\infty} \|u^n(s_1)\|^2 + \int_{s_1}^s (f, u)dt , \tag{16}$$

for all $0 \leqq s_1 < s < \infty$. For $s_1 = 0$, one has

$$\|u(s_1)\| = \liminf_{n\to\infty} \|u^n(s_1)\| , \tag{17}$$

because $u^n(0) \to a$ strongly in $\boldsymbol{L}^2(\Omega)$. Another argument is needed for $s_1 > 0$. The strong convergence of u^n to u in $L^2(\Omega\times(0, T))$ implies that $\|u^n(s)\|$ converges to $\|u(s)\|$ in $L^1(0, T)$, and hence in measure over the interval $(0, T)$. It follows that there exists a subsequence of the functions $\{\|u^n(s)\|\}$, which converges to $\|u(s)\|$ for almost all $t \in (0, T)$. Hence, (17) holds for almost all $s_1 > 0$. This completes the proof of Theorem 1.

THEOREM 2. *For any open $\Omega\subset\boldsymbol{R}^n$, $n \geqq 2$, there exists a Hopf solution of the Navier-Stokes problem* (1).

PROOF. Let $\Omega = \cup_{n=1}^\infty \Omega_n$, where $\Omega_1\subset\Omega_2\subset\cdots$, and each Ω_n is open and bounded. Let $a_n \in \boldsymbol{D}(\Omega_n)$ satisfy $\|a_n\| \leqq \|a\|$, and $\|a_n - a\| \to 0$ as $n \to 0$. Let $\bar{u}^n$ be a weak solution, obtained by the construction of Theorem 1, of the Navier-Stokes problem in $\Omega_n\times[0, \infty)$, with initial velocity a_n, and with external force taken to be the restriction of f to $\Omega_n\times[0, \infty)$.

The solutions $\bar{u}^n$ inherit the estimates (8) and (9) from the original Galerkin approximations, i.e.,

$$\|\bar{u}^n(s)\| \leqq \|a\| + \int_0^s \|f\|\,dt\ , \qquad \text{for all} \quad s \geqq 0\ , \tag{18}$$

and

$$\sup_{[0,T]} \|\bar{u}^n\|^2 + \int_0^T \|\nabla \bar{u}^n\|^2 dt \leqq C_T\ , \qquad \text{for all} \quad T \geqq 0\ . \tag{19}$$

Let $\{a^l(x)\}$ be a system of functions belonging to $\boldsymbol{D}(\Omega)$, orthonormal in $\boldsymbol{J}(\Omega)$, whose finite linear combinations can approximate any function from $\boldsymbol{D}(\Omega)$ arbitrarily well in the norm $\sup_\Omega|\cdot| + \|\cdot\|_{W_2^1(\Omega)}$. Such systems exist.

We extend the domain of definition of each $\bar{u}^n$ to all $\Omega \times [0, \infty)$, by setting it equal to zero outside Ω_n. Clearly $\bar{u}^n \in \boldsymbol{J}(\Omega)$, for every t. Let

$$\bar{u}^n(x, t) = \sum_{k=1}^{\infty} \bar{c}_{kn}(t) a^k(x)$$

be the Fourier series of $\bar{u}^n$ with respect to the $\{a^k\}$. The coefficients $\{\bar{c}_{kn}(t)\}$ are all uniformly bounded on any finite interval $[0, T]$, in view of (18). For fixed k, the sequence $\{\bar{c}_{kn}(t)\}$ is equicontinuous in n, at least for all n large enough that the support of a^k lies in Ω_n. This is proved using (3), in which we can set $u = \bar{u}^n$ and $\phi = a^k$. Then exactly as in (10), although now only for n large enough that $\operatorname{supp}(a^k) \subset \Omega_n$, one obtains

$$|\bar{c}_{kn}(s') - \bar{c}_{kn}(s)| \leqq C_k C_T^* \sqrt{s' - s} + \int_s^{s'} \|f\|\,dt\ , \tag{20}$$

for all $0 \leqq s < s' \leqq T$, and every $T > 0$.

Therefore, as before, we can select a subsequence of $\{n\}$, again denoted by $\{n\}$, such that for each k and T, the sequence $\{\bar{c}_{kn}(t)\}$ is uniformly convergent on $[0, T]$, as $n \to \infty$, to a continuous limit $\bar{c}_k(t)$. As before, the partial sums of the series

$$u(x, t) \sim \sum_{k=1}^{\infty} \bar{c}_k(t) a^k(x) \tag{21}$$

converge weakly in $\boldsymbol{L}^2(\Omega)$ for every $t \geqq 0$, as well as in $L^2(\Omega \times [0, T])$ for every $T > 0$. And again, by the lemma in the appendix, there is a function u defined pointwise in $\Omega \times [0, \infty)$, which is the limit of the series in both senses, for every $t \geqq 0$, and every $T > 0$.

In view of (19), and of the convergence $\bar{c}_{kn}(t) \to \bar{c}_k(t)$, for all k, the subsequence of solutions $\{\bar{u}^n\}$ converges to u weakly in $\boldsymbol{L}^2(\Omega)$, for every t, as well as weakly in $L^2(\Omega \times [0, T])$, for every T. We have not found a way to show that the convergence is strong in $L^2(\Omega \times [0, T])$, because Ω is unbounded, and this, of course, is why we fail to obtain the inequality (4), for values of $s_1 > 0$. But to prove that u satisfies (3), it suffices to show that $\bar{u}^n$ converges to u strongly in $L^2(\Omega' \times [0, T])$, for every bounded $\Omega' \subset \Omega$.

The analogue of (12) is proved exactly as before. Thus, for every $\phi \in \boldsymbol{L}^2(\Omega)$, and every ε, $T > 0$, there exists an integer N such that

$$(22) \qquad |(\bar{u}^n(t) - u(t), \phi)| < \varepsilon\,, \qquad \text{for all} \quad t \in [0, T], \quad \text{and} \quad n \geqq N\,.$$

Again we have a version of Friedrich's lemma: For any bounded subset Ω' of Ω, and any $\varepsilon > 0$, there exist functions $\{\omega_1, \cdots, \omega_N\}$ defined in Ω, such that

$$(23) \qquad \|u\|_{\Omega'}^2 \leqq \sum_{k=1}^{N} (u, \omega_k)_\Omega^2 + \varepsilon \|u\|_{W_2^1(\Omega)}^2\,, \qquad \text{for all} \quad u \in \mathring{\boldsymbol{W}}_2^1(\Omega)\,.$$

Hence, for any two of our solutions, $\bar{u}^n$ and $\bar{u}^m$, we have

$$(24) \qquad \int_0^T \|\bar{u}^n - \bar{u}^m\|_{\Omega'}^2 dt \leqq \sum_{k=1}^{N} \int_0^T (\bar{u}^n - \bar{u}^m, \omega_k)_\Omega^2 dt + \varepsilon \int_0^T \|\bar{u}^n - \bar{u}^m\|_{W_2^1(\Omega)}^2 dt\,.$$

Using (19) and (22), this implies, for the chosen subsequence, that $\bar{u}^n \to u$ strongly in $L^2(\Omega' \times [0, T])$, for every bounded $\Omega' \subset \Omega$, and every $T > 0$.

In view of (19), u has derivatives $\nabla u \in L^2(\Omega \times [0, T])$, and $\nabla \bar{u}^n \to \nabla u$ weakly in $L^2(\Omega \times [0, T])$, for every $T > 0$. One shows as before that $u(\cdot, t) \in \boldsymbol{J}_1(\Omega)$, for almost every $t > 0$.

To establish (3), let s and ϕ be a given. Then for all n sufficiently large that $\operatorname{supp}(\phi) \subset \Omega_n \times [0, s]$, we have

$$(25) \qquad \int_0^s [(\bar{u}^n, \phi_t) - (\nabla \bar{u}^n, \nabla \phi) - (\bar{u}^n \cdot \nabla \bar{u}^n, \phi) + (f, \phi)] dt = (\bar{u}^n(s), \phi(s)) - (\bar{u}^n(0), \phi(0))\,,$$

since each $\bar{u}^n$ is a solution. Thus, one obtains (3) by taking the limit as $n \to \infty$.

The proof of the energy inequality (4), for $s_1 = 0$, is proved exactly as in Theorem 1. But the argument given in Theorem 1 for $s_1 > 0$ fails, because we have not shown that $\bar{u}^n \to u$ strongly in $L^2(\Omega \times [0, T])$. This completes the proof of Theorem 2.

4. **The main result.** Below, we refer to the boundary $\partial\Omega$ of Ω as uniformly C^2 if and only if $\Omega = \cup_{n=1}^{\infty} \Omega_n$, where $\Omega_1 \subset \Omega_2 \subset \cdots$, and $\partial\Omega_n \in C^2$ for every n, with the C^2-regularity of $\partial\Omega_n$ bounded independently of n.

THEOREM 3. *Let $\Omega \subset \boldsymbol{R}^3$ be open, with uniformly C^2 boundary $\partial\Omega$. Let $f \in L^2(0, T; \boldsymbol{L}^2(\Omega))$, for every $T > 0$. Then there exists a Hopf solution of problem* (1), *possessing the epochs of regularity property.*

PROOF. We construct the solution u exactly as in Theorem 2, except that we are more specific about two things. First, in writing the union $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, we assume that the subdomains Ω_n are of class C^2, with the C^2-regularity of $\partial\Omega_n$ bounded independently of n. Second, in the construction of the solution $\bar{u}^n$, in $\Omega_n \times [0, \infty)$, we use as basis functions the eigenfunctions of the Stokes operator $\tilde{\Delta}_n$: $\boldsymbol{J}_1(\Omega_n) \cap \boldsymbol{W}_2^2(\Omega_n) \to \boldsymbol{J}(\Omega_n)$. This will allow us to obtain a further a priori estimate for the Galerkin approximations, the solutions $\{\bar{u}^n\}$,a nd ultimately u.

It was proven in [2, p. 646] that any function $v \in \boldsymbol{J}_1(\Omega_n) \cap \boldsymbol{W}_2^2(\Omega_n)$ satisfies the estimate

$$\|D^2 v\|_{\Omega_n} \leqq c[\|\tilde{\Delta}_n v\|_{\Omega_n} + \|\nabla v\|_{\Omega_n}] , \tag{26}$$

with a constant c which depends only on the C^2-regularity of $\partial\Omega_n$, and not the size of Ω_n. Here $D^2 v$ represents all the second order spatial derivatives of v. Therefore, using also the Sobolev inequalities $\|v\|_6 \leqq c\|\nabla v\|$, valid for $v \in \mathring{\boldsymbol{W}}_2^1(\boldsymbol{R}^3)$, and $\|\phi\|_3 \leqq c(\|\nabla\phi\|^{1/2}\|\phi\|^{1/2} + \|\phi\|)$, valid for $\phi \in \boldsymbol{W}_2^1(\Omega)$ with a constant that depends only on the C^2-regularity of $\partial\Omega$, we have

$$\begin{aligned} |(v\cdot\nabla v, \tilde{\Delta}_n v)| &\leqq \|v\|_6 \|\nabla v\|_3 \|\tilde{\Delta}_n v\| \\ &\leqq c\|\nabla v\|^{3/2}\|D^2 v\|^{1/2}\|\tilde{\Delta}_n v\| + c\|\nabla v\|^2\|\tilde{\Delta}_n v\| \\ &\leqq c\|\nabla v\|^{3/2}\|\tilde{\Delta}_n v\|^{3/2} + c\|\nabla v\|^2\|\tilde{\Delta}_n v\| \\ &\leqq \frac{1}{2}\|\tilde{\Delta}_n v\|^2 + \frac{1}{2}c_1\|\nabla v\|^4 + \frac{1}{2}c_2\|\nabla v\|^6 , \end{aligned} \tag{27}$$

with constants independent of the size of Ω_n.

Without loss of generality, it will be enough to establish the epochs of regularity property on the time interval [0, 1]. According to (9), we know there is a constant D, depending only on $\|a\|$ and $\int_0^1 \|f\| dt$, such that

$$\int_0^1 \|\nabla\tilde{u}\|^2 dt \leqq D , \tag{28}$$

for any of the Galerkin approximations $\tilde{u}$ used in constructing any of the solutions $\bar{u}^n$ in the subdomains Ω_n. Henceforth, $\tilde{u}$ will denote any such Galerkin approximation. Since we are now using eigenfunctions of the Stokes operator as basis functions, an appropriate linear combination of the differential equations (6) yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\tilde{u}\|^2 + \|\tilde{\Delta}_n\tilde{u}\|^2 = (\tilde{u}\cdot\nabla\tilde{u}, \tilde{\Delta}_n\tilde{u}) - (f, \tilde{\Delta}_n\tilde{u}) .$$

Therefore, using (27), we have

$$\frac{d}{dt}\|\nabla\tilde{u}\|^2 \leqq c_1\|\nabla\tilde{u}\|^4 + c_2\|\nabla\tilde{u}\|^6 + \|f\|^2, \qquad \text{for} \quad t \geqq 0, \tag{29}$$

with constants c_1 and c_2 independent of $\tilde{u}$; in particular, independent of the size of Ω_n.

We claim the following. For every $k \in \boldsymbol{N}$, there exists a corresponding $N_k \in \boldsymbol{N}$, such that for every Galerkin approximation $\tilde{u}$, there exists a subset $R_k(\tilde{u})$ of the intervals $[0, N_k^{-1}]$, $[N_k^{-1}, 2N_k^{-1}]$, $\cdots$, $[(N_k - 1)N_k^{-1}, 1]$, such that (identifying $R_k(\tilde{u})$ with the union of its intervals)

$$\begin{aligned} &\text{(i)} \quad \operatorname{meas}([0, 1] - R_k(\tilde{u})) \leqq 1/k, \\ &\text{(ii)} \quad \|\nabla\tilde{u}(t)\|^2 \leqq 4kD, \qquad \text{for} \quad t \in R_k(\tilde{u}). \end{aligned} \tag{30}$$

As a first step in proving this, consider a fixed value of k, and let $\phi_k(t; t_0)$ be the solution of the initial value problem

$$\begin{aligned} \phi_k' &= c_1\phi_k^2 + c_2\phi_k^3 + \|f\|^2, \\ \phi_k(t_0; t_0) &= 3kD, \end{aligned}$$

continued both backwards and forwards in time, form an arbitrary initial time $t_0 \in [0, 1]$. Due to the form of the differential equation, and the integrability of $\|f(t)\|^2$, it is possible to choose N_k, independently of $t_0 \in [0, 1]$, so that

$$\begin{aligned} \phi_k &\leqq 4kD \quad \text{on} \quad [t_0, t_0 + N_k^{-1}], \\ \phi_k &\geqq 2kD \quad \text{on} \quad [t_0 - N_k^{-1}, t_0]. \end{aligned} \tag{31}$$

Clearly, any solution ψ of the differential inequality

$$\psi' \leqq c_1\psi^2 + c_2\psi^3 + \|f\|^2, \tag{32}$$

satisfies

$$\psi \leqq 4kD \quad \text{on} \quad [t_0, t_0 + N_k^{-1}[, \quad \text{if} \quad \psi(t_0) \leqq 3kD, \tag{33}$$

and

$$\psi \geqq 2kD \quad \text{on} \quad [t_0 - N_k^{-1}, t_0], \quad \text{if} \quad \psi(t_0) \geqq 3kD. \tag{34}$$

For a later purpose, we also require that $N_k > 2k$.

Having chosen $N_k > 2k$ so that (31) holds, we now consider an arbitrary Galerkin approximation $\tilde{u}$, and seek to find $R_k(\tilde{u})$ so that (30) holds. Of course, (29) implies that $\psi \equiv \|\nabla\tilde{u}\|^2$ is a solution of (32). Therefore (34) implies that

$$\int_{(l-1)N_k^{-1}}^{lN_k^{-1}} \|\nabla u\|^2 dt \geqq 2kDN_k^{-1}, \quad \text{if} \quad \|\nabla\tilde{u}(lN_k^{-1})\|^2 \geqq 3kD. \tag{35}$$

Let ν be the number of time points from the set $\{N_k^{-1}, 2N_k^{-1}, \cdots, (N_k-1)N_k^{-1}\}$

at which $\|\nabla\tilde{u}\|^2 > 3kD$. In view of (28) and (35), we must have $\nu 2kDN_k^{-1} \leqq D$, i.e., $\nu \leqq N_k/2k$. We choose $R_k(\tilde{u})$ to consist of those subintervals $[lN_k^{-1}, (l+1)N_k^{-1}]$ of $[0, 1]$, such that $\|\nabla\tilde{u}(lN_k^{-1})\|^2 \leqq 3kD$. Clearly, the number of subintervals not included in $R_k(\tilde{u})$ is at most $\nu + 1$. Thus

$$\text{meas}([0, 1] - R_k(\tilde{u})) \leqq (\nu + 1)N_k^{-1} \leqq 1/2k + N_k^{-1} < 1/k \; . \tag{36}$$

This establishes (30, i). Clearly, (30, ii) follows from (33).

The numbers N_k are now determined, and henceforth regarded as fixed. We claim next that for every $k \in \boldsymbol{N}$, and every solution $\bar{u}^n$ (defined in Ω_n), there exists a subcollection $R_k(\bar{u}^n)$ of the intervals $[0, N_k^{-1}]$, $[N_k^{-1}, 2N_k^{-1}], \cdots, [(N_k - 1)N_k^{-1}, 1]$, such that

$$\begin{aligned} &\text{(i)} \quad \text{meas}([0, 1] - R_k(\bar{u}^n)) \leqq 1/k \; , \\ &\text{(ii)} \quad \|\nabla\bar{u}^n(t)\|^2 \leqq 4kD \; , \quad \text{for} \quad t \in R_k(\bar{u}^n) \; . \end{aligned} \tag{37}$$

To see this, we fix a value of k, and of the particular solution $\bar{u}^n$ under consideration. The solution $\bar{u}^n$ is the limit of a subsequence of Galerkin approximations, chosen in Theorem 1. Any estimate, which is satisfied by an infinite subsequence of this subsequence, is inherited by $\bar{u}^n$. This is because any such subsequence must converge to some weak solution, which can only be $\bar{u}^n$. The Galerkin approximations which we used to construct $\bar{u}^n$ each satisfy (30), but with subcollections of intervals $R_k(\tilde{u})$ which may vary from one approximation to another. However, there are only a finite number of ways to choose all but $\nu + 1$ of the N_k subintervals $[lN_k^{-1}, (l+1)N_k^{-1}]$. Therefore, there exists an infinite subsequence of the Galerkin approximations which all satisfy (30) with a common choice of $R_k(\tilde{u})$. We take this common choice as $R_k(\bar{u}^n)$. Clearly, $\bar{u}^n$ inherits the estimate (37, ii), from the corresponding estimate (30, ii), for this particular subsequence of the Galerkin approximations. We are justified in claiming (30, ii) for every $t \in R_k(\bar{u}^n)$, in virtue of the lemma in the appendix.

Finally, we claim that for every k, there exists a subcollection $R_k(u)$ of the same intervals $[0, N_k^{-1}], \cdots, [(N_k - 1)N_k^{-1}, 1]$, such that

$$\begin{aligned} &\text{(i)} \quad \text{meas}([0, 1] - R_k(u)) \leqq 1/k \; , \\ &\text{(ii)} \quad \|\nabla u(t)\|^2 \leqq 4kD \; , \quad \text{for} \quad t \in R_k(u) \; . \end{aligned} \tag{38}$$

To see this, we fix k and consider the solutions $\bar{u}^n$ used to construct u. Arguing exactly as before, (37) implies (38).

This essentially completes the proof of Theorem 3. One may now set R_k equal to the interior of the union of the intervals $R_k(u)$, and R equal to the union of the R_k. We have focused only on obtaining an estimate for the Dirichlet norm $\|\nabla u(t)\|$ on R, but when this is possible the rest of the estimate (5) follows easily. In fact, in writing (29), we could have retained a term $\|\tilde{\Delta}_n \tilde{u}\|^2$ on the left side, and thereby obtained also an estimate for $\int \|\tilde{\Delta}_n \tilde{u}\|^2 dt$ over $R_k(\tilde{u})$. Then, (26) gives an estimate for $\int \|\tilde{u}\|^2_{W_2^2(\Omega)} dt$ over $R_k(\tilde{u})$. Finally, replacing a^l by $u_t^n \equiv \tilde{u}_t$, in (6), leads to an estimate for $\int \|\tilde{u}_t\|^2 dt$ over $R_k(\tilde{u})$; see [2]. These estimates are all inherited by the final solution.

REMARK 3. One might think that the epochs of regularity property obtained in Theorem 3 could be of use in proving the strong energy inequality. If u is regular on some interval $[t_1, t_2]$, then we have the energy equality

$$\frac{1}{2}\|u(t_2)\|^2 + \int_{t_1}^{t_2} \|\nabla u\|^2 dt = \frac{1}{2}\|u(t_1)\|^2 + \int_{t_1}^{t_2} (f, u) dt \,, \tag{39}$$

obtained by multiplying the Navier-Stokes equations through by u, integrating over $\Omega \times [t_1, t_2]$, and using the inclusion $u(t) \in J_1(\Omega)$ to eliminate the pressure term. Suppose now, for simplicity, that there is a singularity at just one instant of time t_*. Let $0 < t_1 < t_* < t_2$. Then, adding the energy equality over $[t_1, t]$ for $t \in (t_1, t_*)$, to the energy equality over $[t, t_2]$ for $t \in (t_*, t_2)$, one obtains

$$\begin{aligned} \frac{1}{2}\|u(t_2)\|^2 + \int_{t_1}^{t_2} \|\nabla u\|^2 dt = \frac{1}{2}(\lim_{t \to t_*^+} \|u(t)\|^2 - \lim_{t \to t_*^-} \|u(t)\|^2) \\ + \frac{1}{2}\|u(t_1)\|^2 + \int_{t_1}^{t_2} (f, u) dt \,. \end{aligned}$$

Thus, to prove the energy inequality, we need merely show that

$$\lim_{t \to t_*^+} \|u(t)\|^2 \leqq \lim_{t \to t_*^-} \|u(t)\|^2 \,,$$

i.e., that there is no jump up in the energy at the instant t_*. It seems surprising that this eludes us when one considers that the major problem with the Navier-Stokes equations is really to control the rate of energy decay; i.e., a smooth solution can be continued so long as its rate of energy decay remains bounded. Here, the difficulty is that some of the energy present in the approximations might disappear for a while from the solution, and then reappear. As far as we know, it could happen that

$$\|u(t)\| < \liminf_{n \to \infty} \|\bar{u}^n(t)\| \,, \tag{40}$$

for a whole interval of values of t. Indeed, one can imagine that some portion of the energy in the various approximating solutions $\tilde{u}^n$ might move out with time toward spatial infinity, and more rapidly so, as $n \to \infty$. This could result in the disappearance of this energy from the solution, since it is obtained only as a weak limit, with the result that (40) would hold. If so, this energy in the approximations might come back and reappear in the solution, causing a jump up in its energy.

REMARK 4. Suppose a Hopf solution u, such as we have constructed in Theorem 3, is regular on some interval I, the right end point of which, t_*, is a singularity. It would be very useful to know that $\|\nabla u(t)\| \to \infty$ as $t \to t_*^-$. To try to prove this, let us suppose not; suppose that

$$\liminf_{t \to t_*^-} \|\nabla u\| = \alpha < \infty \ . \tag{41}$$

Then we can choose a point $t_1 < t_*$, arbitrarily close to t_*, at which $\|\nabla u(t_1)\| < 2\alpha$. The local existence theorem we proved in [2] guarantees the existence of a smooth solution $\hat{u}$, on some interval $[t_1, \hat{t}_*)$, satisfying the initial condition $\hat{u}(t_1) = u(t_1)$. This smooth solution can be continued beyond any point at which its Dirichlet norm $\|\nabla \hat{u}\|$ is finite. Moreover, the growth of its Dirichlet norm is restricted by the differential inequality

$$\frac{d}{dt}\|\nabla \hat{u}\|^2 \leqq c_1\|\nabla \hat{u}\|^4 + c_2\|\nabla \hat{u}\|^6 \ . \tag{42}$$

Thus, if t_1 is chosen close enough to t_*, then certainly $\hat{t}_* > t_*$. It is clear that $u(t) = \hat{u}(t)$ for $t \in [t_1, t_*]$, but since u lacks the strong energy inequality, we have no way to identify u with $\hat{u}$, for $t > t_*$. Of course, if u is a Leray solution, or if $I = [0, t_*)$, one can apply the Leray/Serrin uniqueness theorem at this point of the argument, identifying u with the smooth solution $\hat{u}$ on an interval extending beyond t_*; a contradiction.

One might hope that by examining the construction of the solution, one could show that a singularity cannot occur without $\|\nabla u(t)\| \to \infty$ as $t \to t_*^-$. It is true that each Galerkin approximation $\tilde{u}$ satisfies (42), and that for so long as we have estimates for $\|\nabla \tilde{u}(t)\|$, we can obtain estimates for all derivatives of the solution; this was shown in [2]. However, it might be, for a sequence of Galerkin approximations $\{\tilde{u}^n\}$ converging to u, that $\lim_{n\to\infty}\|\nabla \tilde{u}^n(t_*)\| = \infty$, so that these estimates for the derivatives are lost, while yet the weak limit $\nabla u(t)$ could remain bounded near t_*. The nature of the singularity at t_* might be a right discontinuity of u in $J_1(\Omega)$, with

$$\int_{t_*}^{t_*+\varepsilon} \|\tilde{\Delta} u\|^2 dt = \infty \,, \qquad \int_{t_*}^{t_*+\varepsilon} \|u_t\|^2 dt = \infty \,,$$

for every $\varepsilon > 0$. If such a situation occurs, the restriction of u to the left of t_* could certainly be continued as a smooth solution beyond the point t_*, but the continuation would differ from u.

5. Leray's corollary on the Hausdorff dimension of the singular set. In this section, as in Theorem 3, we assume that Ω is a three-dimensional domain whose boundary (if nonempty) is uniformly C^2. In addition, for simplicity, we assume that $f \equiv 0$.

The construction of the regular set R, given in Theorem 3, does not ensure that it is maximal. In what follows, let $\mathscr{R}$ denote the largest open subset of $(0, \infty)$ on which $u(t)$ is regular. Also, let $\mathscr{S}$ denote the set of all nonzero singular points, $\mathscr{S} = (0, \infty) - \mathscr{R}$. One has $u \in C^\infty(\Omega \times \mathscr{R})$, by our estimates in [2].

Being an open subset of real numbers, $\mathscr{R}$ can be written as a union of disjoint open intervals,

$$\mathscr{R} = \bigcup_i I_i \,.$$

Among these intervals there is one, we will denote it by I_0, which is semi-infinite. This can be proved, very briefly, as follows. Let ϕ be the solution of

$$\phi' = c_1\phi^2 + c_2\phi^3 \,, \tag{43}$$

which satisfies $\phi(0) = 1$. Noting that $\int_{-\infty}^0 \phi(t)dt = \infty$, let $\gamma > 0$ be chosen to satisfy $\int_{-\gamma}^0 \phi(t)dt = (1/2)\|a\|^2$. Then it is impossible for any of the Galerkin approximations $\tilde{u}$ to satisfy $\|\nabla\tilde{u}(t)\|^2 > 1$, for any $t \geqq \gamma$. For if it did, a comparison based on (29) implies that

$$\int_{t-\gamma}^t \|\nabla\tilde{u}(\tau)\|^2 d\tau > \int_{t-\gamma}^t \phi(\tau - t)d\tau = \frac{1}{2}\|a\|^2 \,,$$

which is impossible, by (7). It follows that $(\gamma, \infty) \subset \mathscr{R}$, which implies the result. To introduce a notation for the end points of the intervals, we set $I_0 = (\alpha_0, \infty)$, and $I_i = (\alpha_i, \beta_i)$, for $i \neq 0$.

For his solutions in the domain $\Omega = \boldsymbol{R}^3$, Leray proved that

$$\kappa \sum_{i \neq 0} \sqrt{\beta_i - \alpha_i} \leqq \frac{1}{2}\|a\|^2 \,, \tag{44}$$

where κ is a certain suitable constant. To begin our considerations, we shall first prove a similar inequality for a Leray solution defined in a more general domain.

As shown in the final remark of the last section, at each right end point β_i, there holds $\|\nabla u(t)\| \to \infty$, as $t \to \beta_i^-$. Also, on any interval where u is smooth, we know that

$$\frac{d}{dt}\|\nabla u\|^2 \leqq c_1\|\nabla u\|^4 + c_2\|\nabla u\|^6 . \tag{45}$$

Therefore, by a comparison argument, we conclude that

$$\|\nabla u(t)\|^2 \geqq \phi(t) , \quad \text{for} \quad t \in (\alpha_i, \beta_i) , \tag{46}$$

where ϕ is the solution of (43) which blows up at $t = \beta_i$.

Clearly, $\beta_i - \alpha_i \leqq \alpha_0 \leqq \gamma$, where γ is determined as above. Let ϕ_0 be the solution of (43) which blows up at α_0. Then, obviously, the solution ϕ of (43) which blows up at β_i must satisfy $\phi(t) \geqq \phi_0(0)$, for $t \in (\alpha_i, \beta_i)$. Hence, (43) implies that

$$\phi' \leqq c_3\phi^3 , \quad \text{for} \quad t \in (\alpha_i, \beta_i) , \quad \text{with} \quad c_3 \equiv c_1/\phi_0(0) + c_2 . \tag{47}$$

Comparing ϕ with the solution ψ of $\psi' = c_3\psi^3$ which blows up at β_i, we conclude that

$$\int_{\alpha_i}^{\beta_i} \phi(t)dt \geqq \int_{\alpha_i}^{\beta_i} \psi(t)dt = \kappa\sqrt{\beta_i - \alpha_i} , \tag{48}$$

where $\kappa = \sqrt{2/c_3}$. Combined with (46) and the energy inequality, this implies the desired result (44). It should be mentioned that our constant κ is not independent of $\|a\|$, as it depends on $\phi_0(0)$, and hence on γ. However, if the sum in (44) is taken over only those intervals with β_i less than some prescribed bound δ (or with $\beta_i - \alpha_i < \delta$), then κ can be chosen to depend on δ, rather than on $\|a\|$.

Continuing our consideration of Leray solutions, observe that for any singularity $\xi \in \mathscr{S}$ (not only for right end points), there must hold

$$\operatorname*{ess\,inf}_{t \to \xi^-} \|\nabla u(t)\| = \infty . \tag{49}$$

If not, there would be points to the left of ξ, arbitrarily close to ξ, where u is smooth, and from which u could be continued smoothly past ξ. Further, if ϕ is the solution of (43) which blows up at ξ, there holds

$$\operatorname*{ess\,inf}_{s \to t} \|\nabla u(s)\| \geqq \phi(t) , \qquad \text{for all} \quad t \in [0, \xi) . \tag{50}$$

For again, if not, there would be points to the left of ξ from which u could be continued smoothly past ξ. Therefore, arguing as in the proof of (48), we conclude that

$$\int_{\xi-\delta}^{\xi} \|\nabla u\|^2 dt \geqq \kappa\sqrt{\delta}\ , \tag{51}$$

for any number $\delta \leqq \xi$. For numbers $\delta \leqq \min\{1, \xi\}$, the constant κ can be chosen independently of $\|a\|$. Henceforth we assume that $\delta \leqq 1$.

Now, for any given δ, $0 < \delta < 1$, we can cover the set $\mathscr{S}$ of all nonzero singularities in the following way. Let ξ_1 be the greatest singularity. Then let ξ_{i+1}, for $i = 1, 2, \cdots$, be the greatest singularity less than or equal to $\xi_i - \delta$. This process terminates, of course, after a finite number of steps; let us say that there are $N(\delta)$ singularities so chosen. It is evident that the set $\mathscr{S}$ is contained in the union of the closed intervals $[\xi_i - \delta, \xi_i]$, for $i = 1, 2, \cdots, N(\delta)$. Moreover, from (51) it follows that

$$\kappa(N(\delta) - 1)\sqrt{\delta} \leqq \int_0^{\xi_1} \|\nabla u\|^2 dt \leqq \frac{1}{2}\|a\|^2\ . \tag{52}$$

In other words, for any $\delta > 0$, we are able to cover $\mathscr{S}$ by some number $N(\delta)$ of closed intervals, each of length δ, in such a way that

$$\kappa \lim_{\delta\to 0} \sum_{i=1}^{N(\delta)} \sqrt{\delta} \leqq \frac{1}{2}\|a\|^2\ . \tag{53}$$

This means that $(1/2)\kappa^{-1}\|a\|^2$ bounds the one-half dimensional Hausdorff measure of the singular set $\mathscr{S}$. So far, we have assumed u is a Leray solution.

THEOREM 4. *The estimates* (44) *and* (53) *remain valid for the solution u obtained in Theorem* 3.

PROOF. We will show that (44) and (53) hold for the regular set R obtained in Theorem 3, and the corresponding quasi-singular set $S \equiv (0, \infty) - R$. Once shown, this implies the result, since $R \subset \mathscr{R}$ and $\mathscr{S} \subset S$.

In fact, the set R found in Theorem 3 is not quite satisfactory. We will return to part of the proof of Theorem 3, and make a somewhat more special choice of the sets $R_k(u)$ and thus R. As we do this, we shall select a special subsequence of the solutions $\{\bar{u}^n\}$, which we again denote by $\{\bar{u}^n\}$, with respect to which the set R is maximal. By this, we mean that for every pair of numbers M, $\varepsilon > 0$, and every finite set $\{\xi_1, \cdots, \xi_k\}$ from S, there should exist at least one solution $\bar{u}^n$, for which there is a corresponding set of points $\{\zeta_1, \cdots, \zeta_k\}$, such that for each $i = 1, \cdots, k$, there holds

$$\xi_i - \varepsilon < \zeta_i < \xi_i \quad \text{and} \quad \operatorname*{ess\,sup}_{t\to\zeta_i} \|\nabla\bar{u}^n(t)\|^2 > M\ . \tag{54}$$

Each $\bar{u}^n$ is a Leray solution. Thus, it is easily seen that (54) implies what is claimed in Theorem 4. For if (44) is violated, it must be violated by some finite sum of the terms $\kappa\sqrt{\beta_i - \alpha_i}$. But, by taking ε small enough, these terms can be arbitrarily closely approximated by terms of the form $\kappa\sqrt{\zeta_i - \alpha_i}$. Then, by taking M large enough, the terms $\kappa\sqrt{\zeta_i - \alpha_i}$ can be arbitrarily closely approximated by the integrals $\int_{\alpha_i}^{\zeta_i} \psi(t)dt$, where ψ is the solution of $\psi' = c_3\psi^3$ which satisfies $\psi(\zeta_i) = M$. Finally,

$$\int_{\alpha_i}^{\zeta_i} \|\nabla \bar{u}^n\|^2 dt \geqq \int_{\alpha_i}^{\zeta_i} \phi(t)dt \geqq \int_{\alpha_i}^{\zeta_i} \psi(t)dt ,$$

where ϕ is the solution of (43) which satisfies $\phi(\zeta_i) = M$. Taken together with the energy inequality, this implies that the amount by which the finite sum of terms $\kappa\sqrt{\beta_i - \alpha_i}$ under consideration can exceed $(1/2)\|a\|^2$ is arbitrarily small, contrary to supposition. The proof that (54) implies (53) is virtually the same. It remains to prove (54).

Unlike the proof of Theorem 3, we must now determine the sets $R_k(u)$ recursively, selecting at each stage a further subsequence $\{\bar{u}_k^n\}$, of the solutions $\{\bar{u}^n\}$ which were chosen to converge to u in Theorem 2. Then the diagonal sequence $\{\bar{u}_n^n\}$, after relabelling as $\{\bar{u}^n\}$ again, will be the sequence referred to in (54).

At the k^{th} stage, the set of intervals $R_k(u)$ which is chosen must be maximal, in that it should contain the greatest possible number of the intervals $I_{k,j} = [(j-1)N_k^{-1}, jN_k^{-1}]$ for which there exists an infinite subsequence $\{\bar{u}_k^n\}$ of $\{\bar{u}_{k-1}^n\}$ satisfying

$$\operatorname*{ess\,sup}_{t \in I_{k,j}} \|\nabla \bar{u}_k^n(t)\|^2 \leqq 4kD , \quad \text{for} \quad I_{k,j} \in R_k(u) . \tag{55}$$

Of course, there may be more than one possible choice of the maximal set $R_k(u)$. Also, of course, the argument of Theorem 3 guarantees that R_k, defined to be the interior of the union of the intervals belonging to $R_k(u)$, will be at least large enough that $\text{meas}([0, 1] - R_k) \leqq 1/k$. But now, since $R_k(u)$ is maximal, there can be at most a finite number of the $\{\bar{u}_k^n\}$ which satisfy the estimate (55), for any $I_{k,j}$ other than those in $R_k(u)$. We discard such elements from the sequence $\{\bar{u}_k^n\}$. Denoting the set of those intervals $I_{k,j}$ not contained in $R_k(u)$ by $S_k(u)$, we then have

$$\operatorname*{ess\,sup}_{t \in I_{k,j}} \|\nabla \bar{u}_k^n(t)\|^2 > 4kD , \quad \text{for} \quad I_{k,j} \in S_k(u) , \tag{56}$$

for all members of the sequence $\{\bar{u}_k^n\}$. For the diagonal sequence $\{\bar{u}^n\} \equiv \{\bar{u}_n^n\}$, we have

$$\operatorname*{ess\,sup}_{t \in I_{k,j}} \|\nabla \bar{u}^n(t)\|^2 > 4kD , \quad \text{for} \quad I_{k,j} \in S_k(u) , \tag{57}$$

for all $n \geqq k$. Setting S_k equal to the union of the intervals belonging to $S_k(u)$, we have $S = \cap_k S_k - \{0\}$, and it is obvious that (54) holds.

6. Appendix.

LEMMA. *Suppse $\{u^n\}$ is a sequence of functions which are measurable in Ω for each fixed $t \in [0, T]$, as well as in $\Omega \times [0, T]$ with respect to the product measure. Suppose there is an integrable function $G(t)$ such that*

$$\|u^n(\cdot, t)\| \leqq G(t)\ , \qquad \textit{for all } n, \textit{ and for all } \ t \in [0, T]\ .$$

Suppose $\bar{u}$ is a function defined in $\Omega \times [0, T]$, such that $u^n(\cdot, t) \to \bar{u}(\cdot, t)$ weakly in $\boldsymbol{L}^2(\Omega)$, for each fixed $t \in [0, T]$. Suppose $\tilde{u}$ is a function defined in $\Omega \times [0, T]$, such that $u^n \to \tilde{u}$ weakly in $L^2(\Omega \times [0, T])$. Then, for almost all $t \in [0, T]$ there holds

$$(1) \qquad \bar{u}(x, t) = \tilde{u}(x, t)\ , \qquad \textit{for almost all } \ x \in \Omega\ .$$

Consequently the function

$$u(x, t) = \begin{cases} \tilde{u}(x, t)\ , & \textit{if } t \textit{ is such that } (1) \textit{ holds}\ , \\ \bar{u}(x, t)\ , & \textit{if } t \textit{ is such that } (1) \textit{ does not hold}\ , \end{cases}$$

is measurable in $\Omega \times [0, T]$, as well as in Ω for every fixed t, and $u^n \to u$ weakly in $L^2(\Omega \times [0, T])$, as well as in $\boldsymbol{L}^2(\Omega)$ for each fixed t.

PROOF. Let $\{g_l(x)\}$ be a countable dense subset of $\boldsymbol{L}^2(\Omega)$, and let $h(t)$ be an arbitrary function in $C^\infty[0, T]$. For fixed $g_l(x)$ and $h(t)$, the integrals $\int_\Omega u^n(x, t)g_l(x)h(t)dx$ are measurable functions of t, which, for each t, converge to $\int_\Omega \bar{u}(x, t)g_l(x)h(t)dx$, as $n \to \infty$. Thus $\int_\Omega \bar{u}(x, t)g_l(x)h(t)dx$ is a measurable function of t. Since in addition

$$\left|\int_\Omega u^n(x, t)g_l(x)h(t)dx\right| \leqq G(t)\|g_l\|\,|h(t)|\ ,$$

we can apply the Lebesgue convergence theorem to obtain

$$\lim_{n\to\infty}\int_0^T\int_\Omega u^n(x, t)g_l(x)h(t)dxdt = \int_0^T\left(\int_\Omega \bar{u}(x, t)g_l(x)h(t)dx\right)dt\ .$$

On the other hand, since $u^n \to \tilde{u}$ weakly in $L^2(\Omega \times [0, T])$, we have

$$\lim_{k\to\infty}\int_0^T\int_\Omega u^n(x, t)g_l(x)h(t)dxdt = \int_0^T\int_\Omega \tilde{u}(x, t)g_l(x)h(t)dxdt\ .$$

It follows that

$$\int_0^T h(t)\left(\int_\Omega(\tilde{u} - \bar{u})g_l(x)dx\right)dt = 0\ .$$

The expression in brackets is measurable in t, hence zero for almost all $t \in [0, T]$, since $h(t) \in C^\infty[0, T]$ is arbitrary.

Let N be the set of all $t \in [0, T]$ such that $\int_\Omega(\tilde{u} - \bar{u})g_l(x)dx \neq 0$, for some l. Being a countable union of null sets, N itself has measure zero. For $t \in [0, T] - N$, we have $\tilde{u} = \bar{u}$ as an element of $L^2(\Omega)$, which implies (1). This completes the proof.

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