AN ANALOGUE OF THE HOLONOMY BUNDLE FOR A FOLIATED MANIFOLD

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1. Introduction. Let $(M, \mathscr{F})$ be a smooth foliated manifold with M connected. Let $E \subset T(M)$ be the tangent bundle of $\mathscr{F}$, and let $D \subset T(M)$ be a subbundle satisfying $T(M) = E \oplus D$.

A horizontal curve is a piecewise smooth curve $\sigma: [0, 1] \to M$ whose tangent vector field lies in D. For $x \in M$, let $P(x)$ be the set of points in M that can be joined to x by a horizontal curve. Clearly the sets $P(x)$ partition M. The purpose of this paper is to investigate the structure of these sets. We show that under certain geometric conditions the sets $P(x)$ are immersed submanifolds of M.

As a special case we consider the situation in which M is a Riemannian manifold, and D is the distribution orthogonal to the leaves. We show that the geometric conditions implying that the sets $P(x)$ are immersed submanifolds are satisfied in the following cases.

(1) $\mathscr{F}$ is totally geodesic and the induced metrics on the leaves are complete (cf. [2]).

(2) $\mathscr{F}$ is totally umbilic with $\dim(\mathscr{F}) \geqq 3$, and the induced conformal structures on the leaves are complete.

(3) $\mathscr{F}$ is totally umbilic with $\dim(\mathscr{F}) \geqq 3$ and the metric on M is complete and bundle-like.

(4) The second fundamental form of the leaves is Bott parallel and the metric on M is complete and bundle-like.

(5) A certain tensor defined in terms of the second fundamental form of the leaves and the Bott connection vanishes, and the induced metrics on the leaves are complete.

(6) The above tensor has a specific form, $\dim(\mathscr{F}) \geqq 2$, and the induced projective structures on the leaves are complete.

REMARK. If $M \to N$ is a principal bundle, $\mathscr{F}$ is the foliation of M by the fibers, and D is a connection in M, then the sets $P(x)$ are just the holonomy bundles of the connection.

2. Definitions. For each horizontal curve $\sigma: [0, 1] \to M$ there exists a family of diffeomorphisms $\phi_t: V_0 \to V_t$ ($0 \leqq t \leqq 1$) such that

(1) V_t is a neighborhood of $\sigma(t)$ in the leaf of $\mathscr{F}$ through $\sigma(t)$ for all $0 \leqq t \leqq 1$,

(2) $\phi_t(\sigma(0)) = \sigma(t)$ for all $0 \leqq t \leqq 1$,

(3) for $x \in V_0$, the curve $t \to \phi_t(x)$ is horizontal, and

(4) ϕ_0 is the identity map of V_0.

We call this family of diffeomorphisms an element of holonomy along σ. The element of holonomy along σ is unique in the sense that two such families must agree in a neighborhood of $\sigma(0)$ (cf. [1]).

A vertical curve is a piecewise smooth curve $\tau \colon [0, 1] \to M$ which lies entirely in one leaf of $\mathscr{F}$. A rectangle is a piecewise smooth map $\delta \colon [0, 1] \times [0, 1] \to M$ such that for each fixed $s \in [0, 1]$, the curve $\delta(-, s)$ is horizontal, and for each fixed $t \in [0, 1]$, the curve $\delta(t, -)$ is vertical. The curves $\delta(-, 0)$, $\delta(-, 1)$, $\delta(0, -)$, and $\delta(1, -)$ will be referred to as the initial horizontal edge, the terminal horizontal edge, the initial vertical edge, and the terminal vertical edge respectively.

We say D is an Ehresmann connection for $\mathscr{F}$ if for every horizontal curve σ and vertical curve τ with the same initial points there exists a rectangle (necessarily unique) whose initial edges are σ and τ.

For the remainder of this section assume D is an Ehresmann connection. Recall the following lemma from [2].

LEMMA 2.1. *Let* $\mu \colon [0, 1] \to M$ *be a piecewise smooth curve. Then there exists a unique rectangle* $\delta \colon [0, 1] \times [0, 1] \to M$ *such that* $\mu(t) = \delta(t, t)$ *for all* $0 \leqq t \leqq 1$.

According to Lemma 2.1 every piecewise smooth curve in M is the diagonal of a unique rectangle. The initial horizontal edge of this rectangle is called the horizontal projection of the curve.

Let $\mu \colon [0, 1] \to M$ be a piecewise smooth curve. For every vertical curve $\tau \colon [0, 1] \to M$ with $\tau(0) = \mu(0)$, the transport of τ along μ is the vertical curve $\mu_* \tau$ obtained by the following construction. Let δ be the rectangle constructed in Lemma 2.1 and let σ be the horizontal projection of μ. Let δ^* be the rectangle associated to σ and τ. Take $\mu_* \tau$ to be the terminal vertical edge of δ traversed from $\mu(1)$ to $\sigma(1)$ followed by the terminal vertical edge of δ^*. If τ_0 and τ_1 are two vertical curves with $\tau_0(0) = \tau_1(0) = \mu(0)$ which are homotopic keeping end-points fixed in the leaf of $\mathscr{F}$ through $\mu(0)$, then $\mu_* \tau_0$ and $\mu_* \tau_1$ are two vertical curves which are homotopic keeping endpoints fixed in the leaf of $\mathscr{F}$ through $\mu(1)$. Therefore, recalling that the universal covering space of a connected, locally simply connected space may be identified with the homotopy classes keeping endpoints fixed of curves beginning at a fixed point, we see that μ_* induces

a diffeomorphism of the universal covering space of the leaf of $\mathscr{F}$ through $\mu(0)$ with the universal covering space of the leaf of $\mathscr{F}$ through $\mu(1)$. (μ_*^{-1} is induced by traversing μ in the reverse direction.) Letting $(\tilde{L}_p, p)$ denote the universal cover of the leaf L_p through p identified with the homotopy classes of curves in L_p beginning at p, we will denote the diffeomorphism induced by transport along μ by $\mu_*: (\tilde{L}_{\mu(0)}, \mu(0)) \to (\tilde{L}_{\mu(1)}, \mu(1))$.

Let $C(x)$ be the loop space of M at x. Let $\Phi(x) = \{\lambda_*: \lambda \in C(x)\}$. Note that $\Phi(x)$ is a subgroup of the group of diffeomorphisms of the universal cover $\tilde{L}_x$ of the leaf through x. If $x_0, x_1 \in M$ there exist isomorphisms of $\Phi(x_0)$ onto $\Phi(x_1)$ given by $\lambda_* \to \mu_* \lambda_* \mu_*^{-1}$ for any curve μ joining x_0 to x_1.

REMARK. If $M \to N$ is a principal bundle, $\mathscr{F}$ is the foliation of M by the fibers, and D is a connection in M, then D is an Ehresmann connection for $\mathscr{F}[1]$ and $\Phi(x)$ is just the holonomy group of the connection.

3. Structure of $P(x)$.

Throughout this section we assume D is an Ehresmann connection.

The proof of the following lemma is elementary.

LEMMA 3.1. *Let $\mu: [0, 1] \to M$ be a curve with $\mu(0) = x$, $\mu(1) = y$.*

(1) *If μ is horizontal, then $\mu_*[x] = [y]$ where $[x]$, respectively $[y]$, is the homotopy class of the constant path at x, respectively y.*

(2) *If μ is vertical, then $\mu_*[x] = [\mu^{-1}]$.*

LEMMA 3.2. *Let $x \in M$. Let L_x be the leaf through x. Let X be the orbit of $[x]$ under $\Phi(x)$. Then $\pi(X) = P(x) \cap L_x$ where $\pi: \tilde{L}_x \to L_x$ is the covering projection.*

PROOF. Let $y \in \pi(X)$. Then for some $\lambda \in C(x)$, $y = \pi(\lambda_*[x]) = \sigma(1)$ where σ is the horizontal projection of λ. Thus $y \in P(x) \cap L_x$.

Let $z \in P(x) \cap L_x$, and let σ be a horizontal path from x to z. Let τ be a vertical path from z to x. Set $\lambda = \tau \cdot \sigma \in C(x)$. Then $\lambda_*[x] = \tau_* \sigma_*[x] = [\tau^{-1}]$ by Lemma 3.1. Hence $z = \pi[\tau^{-1}] = \pi(\lambda_*[x]) \in \pi(X)$.

LEMMA 3.3. *The action of $\pi_1(L_x, x)$ on $\tilde{L}_x$ sends orbits of $\Phi(x)$ to orbits. Thus $N = \pi^{-1}(P(x) \cap L_x)$ is a countable union of orbits.*

PROOF. For $[\alpha] \in \pi_1(L_x, x)$ let $\rho([\alpha])$ denote the deck transformation corresponding to $[\alpha]$. If α is a vertical loop at x, and τ is a vertical path starting at x, then from the definition of transport of τ along α, $\alpha_*[\tau] = [\tau \cdot \alpha^{-1}] = \rho([\alpha^{-1}])[\tau]$.

Suppose $[\tau_0]$ and $[\tau_1]$ are in the same orbit of $\Phi(x)$. Then $\lambda_*[\tau_0] = [\tau_1]$ for some $\lambda \in C(x)$. Take $[\alpha] \in \pi_1(L_x, x)$. Then $\alpha^{-1}\lambda\alpha \in C(x)$ and

$$(\alpha^{-1}\lambda\alpha)_{\#}\rho([\alpha])[\tau_0] = \alpha_{\#}^{-1}\lambda_{\#}\alpha_{\#}\alpha_{\#}^{-1}[\tau_0] = \alpha_{\#}^{-1}\lambda_{\#}[\tau_0] = \alpha_{\#}^{-1}[\tau_1] = \rho([\alpha])[\tau_1] \; .$$

That N is a union of orbits now follows from Lemma 3.2.

LEMMA 3.4. *Let $y \in M$ and let L be the leaf through y. There exists a neighborhood U of y in M, a neighborhood W of y in L, and a diffeomorphism $h: W \times V \to U$ where V is a neighborhood of 0 in R^q ($q = \mathrm{codim}\,\mathscr{F}$) such that $h \mid W \times \{0\}$ is the inclusion and $P(x) \cap U = h((W \cap P(x)) \times V)$.*

PROOF. Let U be a neighborhood of y in M such that $\mathscr{F} \mid U$ is defined by a submersion $f: U \to V$ where V is a convex neighborhood of 0 in R^q and $f(y) = 0$. We may assume the level sets of f are connected. Let $W = f^{-1}(0)$. Define $h: W \times V \to U$ by letting $h(w, v)$ be the endpoint of the horizontal lift starting at w of the straight line joining 0 to v in V. It may be necessary to cut down V and W in order that h be defined. By the further cutting down of U, V, and W h will be a diffeomorphism.

Clearly $h \mid W \times \{0\}$ is the inclusion. If $z \in P(x) \cap U$, then $z = h(w, v)$ for some $(w, v) \in W \times V$. By construction of h, $w \in W \cap P(x)$. Thus $z \in h((W \cap P(x)) \times V)$. The other inclusion is obvious.

THEOREM 3.5. *Suppose for some $x_0 \in M$, $\Phi(x_0)$ is contained in G where $G \subset \mathrm{Diff}(\tilde{L}_{x_0})$ is a Lie group acting smoothly on $\tilde{L}_{x_0}$. Then for each $x \in M$, $P(x)$ is an immersed submanifold of M.*

PROOF. Let $C^0(x_0)$ be the subset of $C(x_0)$ consisting of all contractible loops. The subgroup $\Phi^0(x_0)$ of $\Phi(x_0)$ arising from all loops in $C^0(x_0)$ is an arcwise connected subgroup of G. Hence $\Phi^0(x_0)$ is a Lie subgroup of G [4]. Since $\Phi(x_0)/\Phi^0(x_0)$ has at most countably many elements, $\Phi(x_0)$ can be given a Lie group structure in which $\Phi^0(x_0)$ is the connected component of the identity.

Thus the orbits of $\Phi(x_0)$ on $\tilde{L}_{x_0}$ are immersed submanifolds of $\tilde{L}_{x_0}$. By Lemma 3.3, $N = \pi^{-1}(P(x_0) \cap L_{x_0})$ is a countable union of orbits of $\Phi(x_0)$ and hence is an immersed submanifold of $\tilde{L}_{x_0}$. Since $\pi_1(L_{x_0}, x_0)$ acts freely and properly discontinuously on N, it follows that $N/\pi_1(L_{x_0}, x_0)$ is an immersed manifold of L_{x_0} whose image in L_{x_0} is $P(x_0) \cap L_{x_0}$. By Lemma 3.4, $P(x_0)$ can be given a topology and differentiable structure making it into an immersed submanifold of M by requiring as charts on $P(x_0)$ the maps $\psi: \Omega \times V \to P(x_0)$ defined by $\psi(w, v) = h(\phi(w), v)$ where $\phi: \Omega \to W \cap P(x_0)$ is a chart on $P(x_0) \cap L_{x_0}$.

To complete the proof it suffices to show that for every $x \in M$, $\Phi(x)$ is a Lie group acting smoothly on $\tilde{L}_x$. If μ is a curve joining x_0 to x

then the isomorphism of $\Phi(x_0)$ onto $\Phi(x)$ given by $\lambda_* \rightarrow \mu_* \lambda_* \mu_*^{-1}$ can be used to put a Lie group structure on $\Phi(x)$. This structure is independent of the choice of μ.

EXAMPLE. Let G be a connected Lie group acting smoothly on a manifold N. Let $P \rightarrow B$ be a principal G-bundle with a connection whose holonomy group is G [4]. Let $M = P \times_G N \rightarrow B$ be the associated bundle with fiber N, and let $\mathscr{F}$ be the foliation of M by the fibers. Let D be the distribution on M arising from the connection in P. Then D is an Ehresmann connection for $\mathscr{F}$. Identifying a leaf L of $\mathscr{F}$ with N, we have that the sets $P(x) \cap L$ are just the orbits of G on N. In general, each $\overline{P(x)}$ is a union of $P(y)$'s but these closures need not partition M even if M is compact since the closures of the orbits of a non-compact group action need not partition. If however the leaves are totally geodesic and M is compact, then the sets $\overline{P(x)}$ do partition M and are submanifolds [2].

4. Bundles of Tangential r-frames. Let $(M, \mathscr{F})$ be a foliated manifold, and let E be the tangent bundle of $\mathscr{F}$. A tangential r-frame at $p \in M$ is an equivalence class of immersions $f: (U, 0) \rightarrow (L, p)$ where U is a neighborhood of 0 in R^k $(k = \dim(\mathscr{F}))$ and L is the leaf of $\mathscr{F}$ through p, where two such immersions are equivalent if they agree up to order r at 0. Let $F^r(E)$ be the collection of all tangential r-frames at all points of M. The projection map $\pi: F^r(E) \rightarrow M$ defined by $\pi[f] = f(0)$ has the structure of a principal $G^r(k)$-bundle where $G^r(k)$ is the group of r-frames at 0 in R^k. The restriction of π to a leaf L is the usual bundle of r-frames over L [3].

Let $\widetilde{\mathscr{F}} = \pi^{-1}(\mathscr{F})$. Given a complementary distribution D on $(M, \mathscr{F})$, there is a natural lift of D to a complementary distribution $\widetilde{D}$ on $(F^r(E), \widetilde{\mathscr{F}})$ defined as follows. Let $u_0 \in F^r(E)$, and $p_0 = \pi(u_0)$. Let U be a neighborhood of p_0 in which $\mathscr{F}$ is a product, and let X be a horizontal vector field on U which is parallel along the leaves. The local flow ϕ_t generated by X sends leaves to leaves, and hence induces a local flow $\phi_t^{(r)}$ in a neighborhood of u_0. Let $\widetilde{X}$ be the vector field induced by $\phi_t^{(r)}$, and let $\widetilde{D}_{u_0}$ be the set of all such $\widetilde{X}_{u_0}$. Then $\widetilde{D}$ is a complementary distribution to $\widetilde{\mathscr{F}}$ since $\pi_* \widetilde{X} = X$ for these vector fields.

Let $\pi: P \rightarrow M$ be a reduction of $F^r(E)$ to a Lie subgroup H of $G^r(k)$.

DEFINITION. D is compatible with P if $\widetilde{D}$ is tangent to P.

The restriction of P to a leaf L of $\mathscr{F}$ is a reduction of the bundle of r-frames on L to H. This gives a collection of distinguished r-frames

on each leaf. Then D is compatible with P if and only if the elements of holonomy along horizontal paths in M send distinguished r-frames to distinguished r-frames. In this case, $\tilde{D}$ is a complementary distribution to $\tilde{\mathscr{F}} = \pi^{-1}(\mathscr{F})$ on P.

LEMMA 4.1. *If $\tilde{D}$ is an Ehresmann connection for $(P, \tilde{\mathscr{F}})$ then D is an Ehresmann connection for $(M, \mathscr{F})$.*

PROOF. Let τ be a vertical curve and σ a horizontal curve in M with $\tau(0) = \sigma(0)$. Let $\tilde{\tau}$ be a lift of τ to P. Then $\tilde{\tau}$ is tangent to $\tilde{\mathscr{F}}$. Let ϕ_t be the element of holonomy along σ. Let $\phi_t^{(r)}$ be the natural lift of ϕ_t to P, and let $\tilde{\sigma}(t) = \phi_t^{(r)}(\tilde{\tau}(0))$. The rectangle associated to $\tilde{\sigma}$ and $\tilde{\tau}$ projects under π to a rectangle associated to σ and τ.

REMARK. The converse is also true. (cf. Proposition 4.3 of [2])

Recall from [2] that a parallelism for $\tilde{\mathscr{F}}$ is a family $X_1, \cdots, X_m$ ($m = \dim(\tilde{\mathscr{F}})$) of vector fields on P everywhere linearly independent and tangent to $\tilde{\mathscr{F}}$. The parallelism is said to be complete if each vector field X_i is a complete vector field on P. We say $\tilde{D}$ preserves the parallelism if each X_i is invariant under the elements of holonomy along $\tilde{D}$-curves.

PROPOSITION 4.2. *If $\tilde{D}$ preserves a complete parallelism for $\tilde{\mathscr{F}}$, then D is an Ehresmann connection for $\mathscr{F}$.*

PROOF. We first show that each $u_0 \in P$ has a neighborhood V in the leaf of $\tilde{\mathscr{F}}$ through u_0 such that for every horizontal curve $\tilde{\sigma}$ with $\tilde{\sigma} \in V$, the element of holonomy along $\tilde{\sigma}$ can be defined throughout V. Indeed, for each $i = 1, \cdots, m$, let ϕ_t^i be the flow generated by X_i. The function $f(t_1, \cdots, t_m) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_m}^m(u_0)$ defines a diffeomorphism of a neighborhood of 0 in R^m onto a neighborhood V of u_0 in the leaf through u_0. Let $\tilde{\sigma}$ be a horizontal curve with $\tilde{\sigma}(0) = f(s_1, \cdots, s_m) \in V$. For each $v \in V$,

$$\psi_t(v) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_m}^m \circ \phi_{-s_m}^m \circ \cdots \circ \phi_{-s_1}^1(\tilde{\sigma}(t))$$

defines the element of holonomy along $\tilde{\sigma}$ at v where $f^{-1}(v) = (t_1, \cdots, t_m)$.

Let $\tilde{\tau}$ be a vertical curve and $\tilde{\sigma}$ a horizontal curve in P with the same initial points. The preceding argument shows that the element of holonomy determined by $\tilde{\sigma}$ can be continued along $\tilde{\tau}$, and so $\tilde{D}$ is an Ehresmann connection. The proposition now follows from Lemma 4.1.

5. Applications. Let M be a Riemannian manifold, $\mathscr{F}$ a foliation of M, and D the distribution orthogonal to the leaves.

LEMMA 5.1. *The elements of holonomy along horizontal curves are*

isometries (respectively, conformal transformations) if and only if the leaves of $\mathscr{F}$ are totally geodesic (respectively, totally umbilic) submanifolds.

PROOF. Let g be the metric tensor, $\bar{\nabla}$ the Riemannian connection on M, and α the second fundamental form for the leaves. Let $\tilde{X}$ be a horizontal vector field parallel along the leaves in some neighborhood U in M. Let Y, Z be vertical vector fields on U. Then

$$
\begin{aligned}
(L_{\tilde{X}}g)(Y, Z) &= \tilde{X}(g(Y, Z)) - g(L_{\tilde{X}}Y, Z) - g(Y, L_{\tilde{X}}Z) \\
&= g(\bar{\nabla}_{\tilde{X}}Y, Z) + g(Y, \bar{\nabla}_{\tilde{X}}Z) - g([\tilde{X}, Y], Z) - g(Y, [\tilde{X}, Z]) \\
&= g(\bar{\nabla}_{\tilde{X}}Y - [\tilde{X}, Y], Z) + g(Y, \bar{\nabla}_{\tilde{X}}Z - [\tilde{X}, Z]) \\
&= g(\bar{\nabla}_Y\tilde{X}, Z) + g(Y, \bar{\nabla}_Z\tilde{X}) = -g(\tilde{X}, \bar{\nabla}_Y Z) - g(\bar{\nabla}_Z Y, \tilde{X}) \\
&= -g(\tilde{X}, \alpha(Y, Z)) - g(\alpha(Z, Y), \tilde{X}) = -2g(\alpha(Y, Z), \tilde{X}).
\end{aligned}
$$

The elements of holonomy along horizontal curves are isometries (respectively, conformal transformations) if and only if $L_{\tilde{X}}g = 0$ (respectively, $L_{\tilde{X}}g = cg$), and so the result follows from the above calculation.

Suppose $\mathscr{F}$ is totally geodesic with complete leaves. By Lemma 5.1 the elements of holonomy preserve the metrics on the leaves. Let P be the reduction of $F^1(E)$ obtained by taking orthonormal frames on the leaves. Then D is compatible with P. The leaves of $\tilde{\mathscr{F}}$ have a complete parallelism (that arises from the Levi-Civita connections on the leaves) which is preserved by $\tilde{D}$. By Proposition 4.2, D is an Ehresmann connection for $\mathscr{F}$. Since $\Phi(x)$ is contained in the Lie group of isometries of $\tilde{L}$, it follows from Theorem 3.5 that the sets $P(x)$ are immersed submanifolds.

Suppose $\mathscr{F}$ is totally umbilic with $\dim(\mathscr{F}) \geq 3$, and the induced conformal structures on the leaves are complete. By Lemma 5.1, the elements of holonomy preserve the conformal structures on the leaves. Let $P \subset F^2(E)$ be the reduction arising from the conformal structures on the leaves of $\mathscr{F}$ [3]. Then D is compatible with P. The leaves of $\tilde{\mathscr{F}}$ have a complete parallelism (that arises from the normal conformal Cartan connections on the leaves of $\mathscr{F}$) which is preserved by $\tilde{D}$. By Proposition 4.2, D is an Ehresmann connection for $\mathscr{F}$. Since $\Phi(x)$ is contained in the Lie group of conformal automorphisms of $\tilde{L}$, it follows from Theorem 3.5 that the sets $P(x)$ are immersed submanifolds.

Suppose $\mathscr{F}$ is totally umbilic with $\dim(\mathscr{F}) \geq 3$ and g is complete and bundle-like. Since g is complete and bundle-like, D is an Ehresmann connection [1]. By Lemma 5.1, $\Phi(x)$ is contained in the Lie group of conformal automorphisms of $\tilde{L}$. Thus the sets $P(x)$ are immersed sub-

manifolds by Theorem 3.5.

In addition to the notation introduced in the proof of Lemma 5.1, let ∇ be the induced connection in the leaves, and $\hat{\nabla}$ the Bott connection along the leaves. For vertical vectors W, Y, Z we define a horizontal-valued tensor $S(Y, Z, W) = (\hat{\nabla}_W \alpha)(Y, Z) + (\hat{\nabla}_W g)^\sharp \alpha(Y, Z)$ where $(\hat{\nabla}_W g)^\sharp$ is the linear operator on horizontal vectors satisfying $g((\hat{\nabla}_W g)^\sharp(X_0), X_1) = (\hat{\nabla}_W g)(X_0, X_1)$ for all horizontal vectors X_0, X_1.

LEMMA 5.2. *Let U be a neighborhood where the foliation is a product. Let $\tilde{X}$ be a vector field on U, Bott parallel along the leaves of $\mathscr{F}$. Let Y, Z, W be vertical vector fields on U such $L_{\tilde{X}} Y = L_{\tilde{X}} Z = L_{\tilde{X}} W = 0$. Then*

$$-2g(S(Z, W, Y), \tilde{X}) = g(L_{\tilde{X}} \nabla_Y Z, W) + g(L_{\tilde{X}} \nabla_Y W, Z)$$

and

$$g(L_{\tilde{X}} \nabla_Y Z, W) = g(S(Y, Z, W) - S(W, Y, Z) - S(Z, W, Y), \tilde{X}) .$$

PROOF. Using the formula for $L_{\tilde{X}} g$ derived in the proof of Lemma 5.1 we have

$$\begin{aligned}
g(L_{\tilde{X}} &\nabla_Y Z, \ W) + g(Z, L_{\tilde{X}} \nabla_Y W) \\
&= \tilde{X}(g(\nabla_Y Z, W)) - (L_{\tilde{X}} g)(\nabla_Y Z, W) - g(\nabla_Y Z, L_{\tilde{X}} W) \\
&\quad + \tilde{X}(g(Z, \nabla_Y W)) - (L_{\tilde{X}} g)(Z, \nabla_Y W) - g(L_{\tilde{X}} Z, \nabla_Y W) \\
&= \tilde{X}(Y(g(Z, W))) + 2g(\alpha(\nabla_Y Z, W), \tilde{X}) + 2g(\alpha(Z, \nabla_Y W), \tilde{X}) \\
&= (L_{\tilde{X}} Y) g(Z, W) + Y((L_{\tilde{X}} g)(Z, W)) + Y(g(L_{\tilde{X}} Z, W)) \\
&\quad + Y(g(Z, L_{\tilde{X}} W)) + 2g(\alpha(\nabla_Y Z, W), \tilde{X}) + 2g(\alpha(Z, \nabla_Y W), \tilde{X}) \\
&= -2Y(g(\alpha(Z, W), \tilde{X})) + 2g(\alpha(\nabla_Y Z, W), \tilde{X}) + 2g(\alpha(Z, \nabla_Y W), \tilde{X}) \\
&= -2(\hat{\nabla}_Y g)(\alpha(Z, W) \tilde{X}) - 2g((\hat{\nabla}_Y \alpha)(Z, W), \tilde{X}) - 2g(\alpha(Z, W), \hat{\nabla}_Y \tilde{X}) \\
&= -2g((\hat{\nabla}_Y \alpha)(Z, W) + (\hat{\nabla}_Y g)^\sharp(\alpha(Z, W)), \tilde{X}) \\
&= -2g(S(Z, W, Y), \tilde{X}) .
\end{aligned}$$

The second equation follows from the first by a short calculation.

PROPOSITION 5.3. *The elements of holonomy along horizontal curves are affine transformations (respectively, projective transformations) if and only if $S \equiv 0$ (respectively, $-2g(S(Z, W, Y), X) = \eta_X(Z)g(Y, W) + \eta_X(W)g(Y, Z) + 2\eta_X(Y)g(Z, W)$ for some section η of the vector bundle $\mathrm{Hom}(D, E^*)$ for every horizontal vector X and vertical vectors Y, Z, W.)*

PROOF. In the notation of Lemma 5.2, the elements of holonomy along horizontal curves are affine transformations if and only if $L_{\tilde{X}} \nabla_Y Z = 0$ for all such $\tilde{X}, Y, Z$. By Lemma 5.2, this occurs if and only if $S \equiv 0$.

The elements of holonomy are projective transformations if and only if $L_{\tilde{X}}(\nabla_Y Z) = \eta_{\tilde{X}}(Y)Z + \eta_{\tilde{X}}(Z)Y$ for all such $\tilde{X}, Y, Z$ and some section η of $\mathrm{Hom}(D, E^*)$. The result follows by Lemma 5.2.

Suppose g is complete and bundle-like (i.e. $\hat{\nabla}g \equiv 0$). Thus if $\hat{\nabla}\alpha \equiv 0$, then $S = \hat{\nabla}\alpha + (\nabla g)^{\sharp} \equiv 0$, and so by Proposition 5.3, $\Phi(x)$ is contained in the Lie group of affine transformations of $\tilde{L}$. Thus the sets $P(x)$ are immersed submanifolds.

Suppose $S \equiv 0$ and the induced metrics on the leaves are complete. By Proposition 5.3, the elements of holonomy along horizontal curves are affine transformations. Let $P = F^1(E)$. Trivially, D is compatible with P. The leaves of $\widetilde{\mathscr{F}}$ have a complete parallelism (that arises from the affine connections on the leaves of $\mathscr{F}$) which is preserved by $\tilde{D}$. By Proposition 4.2, D is an Ehresmann connection for $\mathscr{F}$. Since $\Phi(x)$ is contained in the Lie group of affine transformations of $\tilde{L}$, it follows that the sets $P(x)$ are immersed submanifolds.

Suppose S has the form in Proposition 5.3, so that the elements of holonomy along horizontal curves are projective transformations. Suppose also that $\dim(\mathscr{F}) \geqq 2$, and that the induced projective structures on the leaves are complete. Let $P \subset F^2(E)$ be the reduction arising from the projective structures on the leaves of $\mathscr{F}$ [3]. Then D is compatible with P. The leaves of $\widetilde{\mathscr{F}}$ have a complete parallelism (that arises from the normal projective Cartan connections on the leaves of $\mathscr{F}$) which is preserved by $\tilde{D}$. By Proposition 4.2, D is an Ehresmann connection for $\mathscr{F}$. Since $\Phi(x)$ is contained in the Lie group of projective automorphisms of $\tilde{L}$, it follows that the sets $P(x)$ are immersed submanifolds.

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