VECTOR BUNDLES OVER QUATERNIONIC KÄHLER MANIFOLDS

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Introduction. On vector bundles over oriented 4-dimensional Riemannian manifolds, the notion of self-dual and anti-self-dual connections plays an important role in the geometry of 4-dimensional Yang-Mills theory (see Atiyah, Hitchin and Singer [A-H-S]).

On the other hand, in his differential-geometric study of stable holomorphic vector bundles, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds. Let $E$ be a vector bundle over a quaternionic Kähler manifold $M$, and $p: Z \rightarrow M$ the corresponding twistor space defined by Salamon [SI]. Now the purpose of the present paper is to give a quaternionic Kähler analogue of self-dual and anti-self-dual connections, and then to construct a natural correspondence between $E$'s with such connections and the set of Einstein-Hermitian vector bundles over $Z$.

Let $H$ be the skew field of quaternions. Then the $Sp(n) \cdot Sp(1)$-module $\wedge^2 H^n$ is a direct sum $N'_x \oplus N''_x \oplus L_x$ of its irreducible submodules $N'_x$, $N''_x$, $L_x$, where $N'_x$ (resp. $L_x$) is the submodule of the elements fixed by $Sp(n)$ (resp. $Sp(1)$) and for $n = 1$, we have $N''_x = \{0\}$. Hence, the vector bundle $\wedge^2 T^*M$ is written as a direct sum $A'_x \oplus A''_x \oplus B_x$ of its holonomy-invariant subbundles in such a way that $A'_x$, $A''_x$, $B_x$ correspond respectively to $N'_x$, $N''_x$, $L_x$. Now, a connection for $E$ is called an $A'_x$-connection (resp. $B_x$-connection) if the corresponding curvature is an $\text{End}(E)$-valued $A'_x$-form (resp. $B_x$-form). Then we have:

**Theorem (0.1).** All $A'_x$-connections and also all $B_x$-connections are Yang-Mills connections.

Furthermore, for $E$ with a $B_x$-connection we can associate an $E$-valued elliptic complex (cf. (3.2)) similar to those of Salamon [S2]. Such complexes allow us to analyze the space of infinitesimal deformations of $B_x$-connections (see Theorem (3.5)).

For our quaternionic Kähler manifold $M$, a pair $(E, D_E)$ of a vector bundle $E$ over $M$ and a $B_x$-connection $D_E$ on $E$ is called a Hermitian pair on $M$ if $D_E$ is a Hermitian connection on $E$. On the other hand, a pair $(F, D_F)$ of a holomorphic vector bundle over $Z$ and a Hermitian $(1, 0)$-
connection $D_F$ on $F$ is called an **excellent pair** on $Z$ if the following conditions are satisfied:

(a) $F$ with the corresponding Hermitian metric $h_F$ restricts to a flat bundle on each fibre of $p: Z \to M$. (Hence the real structure $\tau: Z \to Z$ (cf. Nitta and Takeuchi [N-T]) naturally lifts to a bundle automorphism $\tau': F \to F$.)

(b) Let $\sigma: F \to F^*$ be the bundle map defined by $F_z \ni f \mapsto \sigma(f) \in F_{\tau(z)}^*$, $(z \in Z)$, where $\sigma(f)(g) := h_F(g, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then $\sigma$ is an anti-holomorphic bundle automorphism. We then have the following generalization of a result of Penrose's type (cf. Atiyah, Hitchin and Singer [A-H-S]; see also Salamon [S2], Berard-Bergery and Ochiai [B-O]):

**Theorem (0.2).** Let $\mathcal{H}$ (resp. $\mathcal{H}'$) be the set of all Hermitian pairs (resp. all excellent pairs) on $M$ (resp. $Z$). Then

$$\mathcal{H} \ni (E, D_E) \mapsto (p^*E, p^*D_E) \in \mathcal{H}'$$

defines a bijective correspondence between $\mathcal{H}$ and $\mathcal{H}'$.

In particular, if $M$ has positive scalar curvature, then every excellent pair $(F, D_F)$ on $Z$ is a Ricci-flat Einstein-Hermitian vector bundle.

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1. **Notation, convention and preliminaries.** In this section, we give a quick review of the basic facts on quaternionic Kähler manifolds (for more details see Salamon [S1], Nitta and Takeuchi [N-T]).

(1.1) Let $H^{(m)}$ denote the standard $Sp(m)$-module $H^m (= C^{2m})$ of complex dimension $2m$, where $H = R + iR + jR + kR (= C + jC)$. $Sp(m) = \{ S \in GL(m, H) | S^{-1}S = I \}$ is imbedded in $GL(2m, C)$ by

$$Sp(m) \ni A + jB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in GL(2m, C)$$

where $A, B \in GL(m, C)$. Then the multiplication on $H^m$ by $j$ from the right naturally induces a $Sp(m)$-equivariant anti-linear map $j^{(m)}: H^{(m)} \to H^{(m)}$ with $(j^{(m)})^2 = -\text{id}$. We now define a non-degenerate skew-symmetric bilinear form $\omega^{(m)}$ on $H^m$ by

$$\omega^{(m)}(h, h') := -\langle h, j^{(m)}h' \rangle \ (h, h' \in H^m),$$

where $\langle \ , \ \rangle$ is the standard Hermitian inner product on $C^{2m} (= H^m)$. This $\omega^{(m)}$ can be regarded as an $Sp(m)$-invariant bilinear form on $H^{(m)}$ such
that

\[(1.1.1) \quad \omega^{(m)}(j^{(m)}h, j^{(m)}h') = (\omega^{(m)}(h, h'))^\star \quad (h, h' \in H^{(m)}).\]

Let \(Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1)/\mathbb{Z}\). Then \(H^{(n)} \otimes_c H^{(1)}\) is naturally a \(Sp(n) \cdot Sp(1)\)-module of complex dimension \(4n\) with a real structure \(H^{(n)} \otimes_c H^{(1)} \ni a \mapsto \bar{a} \in H^{(n)} \otimes_c H^{(1)}\) defined by

\[(1.1.2) \quad (h \otimes h')^\star := j^{(m)}h \otimes j^{(n)}h' \quad (h \in H^{(m)}, h' \in H^{(n)}).\]

We consider the corresponding real form \((H^{(n)} \otimes_c H^{(1)})_R\) of \(H^{(n)} \otimes_c H^{(1)}\). Then the symmetric bilinear form \(\omega^{(n)} \otimes \omega^{(1)} \in S^2((H^{(n)})^* \otimes (H^{(1)})^*)\) induces an inner product \(\langle , \rangle\) on \((H^{(n)} \otimes_c H^{(1)})_R\).

(1.2) Recall that a \(4n\)-dimensional Riemannian manifold \((M, g_M)\) is called a quaternionic Kähler manifold, if its linear holonomy group is contained in \(Sp(n) \cdot Sp(1) (\subset SO(4n))\) with the additional condition for \(n = 1\) that \(g_M\) is a self-dual Einstein metric. Throughout this paper, we fix once for all a quaternionic Kähler manifold \((M, g_M)\). By the well-known reduction theorem (see, for instance, Kobayashi and Nomizu [K-N]), the frame bundle of the tangent bundle \(TM\) is reduced to a principal \(Sp(n) \cdot Sp(1)\)-bundle \(P\). Then \(TM\) can be regarded as the vector bundle

\[(1.2.1) \quad P \times_{Sp(n) \cdot Sp(1)} (H^{(n)} \otimes_c H^{(1)})_R\]

associated to the \(Sp(n) \cdot Sp(1)\)-module \((H^{(n)} \otimes_c H^{(1)})_R\). The inner product \(\langle , \rangle\) on \((H^{(n)} \otimes_c H^{(1)})_R\) induces a Riemannian metric \(g\) on \(TM\), which coincides with \(g_M\) up to constant multiple. Without loss of generality, we may assume \(g = g_M\).

(1.3) Let \(Sp(n)\) act trivially on \(C^2\). Then the standard \(Sp(1)\)-action on \(C^2\) naturally induces an \(Sp(n) \times Sp(1)\)-action (resp. \(Sp(n) \cdot Sp(1)\)-action) on \(C^n\) (resp. \(P^1C\)). Associated to these actions, we have:

\[\hat{\varphi}: V := P \times_{Sp(n) \times Sp(1)} C^2 \to M\]

(resp. \(\varphi: Z := P \times_{Sp(n) \cdot Sp(1)} P^1C \to M\)),

which is a "locally defined" vector bundle (resp. a globally defined fibre bundle). Here, the bundle \(Z\) is nothing but \(P(V) := V - \{\text{zero section}\}/C^*,\) and is called the twistor space of \(M\) (see Salamon [S1; p. 147]). Then \(Z\) is a complex manifold with a natural real structure \(\tau\) as follows:

(1.3.1) By the connection on \(V\) induced from that of \(P\), we have a decomposition of \(T(V - \{\text{zero section}\})\) into the subbundles \(S^h\) and \(S^v\) corresponding respectively to horizontal and vertical distributions. Let \(y\) be an arbitrary point of \(V - \{\text{zero section}\}\), and put \(x := \hat{\varphi}(y)\). Via the projection \(\hat{\varphi}\), the fibre \((S^h)_y\) of \(S^h\) over \(y\) is regarded as the tangent
space $T_xM$ at $x$. Then by the identification of $H^{(n)} \otimes_c H^{(1)}$ with $(T_xM)^c$ (cf. (1.2.1)), the space $H^{(n)} \otimes C_y$ defines a $C$-linear subspace of $(T_xM)^c$, denoted also by $H^{(n)} \otimes C_y$. Furthermore, let $(H^{(n)} \otimes C_y)^\prime$ be the subspace of $(T_xM)^c$ corresponding to $H^{(n)} \otimes C_y$ via the natural isomorphism $(T_xM)^c \cong (T_xM)^c$ induced by $g_M$. Now we define the complex structure of $T_xV$ by specifying the subspace $\Lambda^{1,0}$ of $(1, 0)$-forms in $(T_xV)^c$ as follows:

$$\Lambda^{1,0} = (\Lambda^{1,0})^b \oplus (\Lambda^{1,0})^y,$$

where $(\Lambda^{1,0})^y := \tilde{p}^*((H^{(n)} \otimes C_y)^\prime)$, and $(\Lambda^{1,0})^y$ is the subspace of $(1, 0)$-forms in $T_xC^c$ by the identification of $V_x$ with $C^c$. Then this induces a complex structure on $Z$.

(1.3.2) The map $j^{(1)}: H^{(1)} \to H^{(1)}$ naturally defines an antilinear bundle automorphism $\tilde{e}: V \to V$, which induces a real structure $\tau$ on $Z$.

(1.3.3) Recall that $M$ always has a constant scalar curvature (denoted by $t$). Let $g_F$ be the Fubini-Study metric for $P^1C (= (C + jC - [0])/C^*)$. If $t \neq 0$, then for some nonzero real constant $c_t$,

$$g_z := p^*g_m + c_t g_F$$

defines a pseudo-Kählerian metric on $Z$, i.e., the corresponding $(1, 1)$-form on $Z$ is a nondegenerate $d$-closed $(1, 1)$-form.

2. $A'_c$-connections and $B'_c$-connections. We shall here give fundamental properties of the $A'_c$-connections and $B'_c$-connections defined in the Introduction.

(2.1) Let $(H^{(m)})^*$ be the dual $Sp(m)$-module of $H^{(m)}$. Then in view of $\wedge^3((H^{(1)})^*)^* = C\omega^{(1)}$, we have

$$\wedge^3((H^{(n)})^* \otimes_c (H^{(1)})^*) = (\wedge^3(H^{(n)})^* \otimes_c S^2(H^{(1)})^*) \oplus (S^2(H^{(n)})^* \otimes_c C\omega^{(1)}).$$

Furthermore, the $Sp(n)$-module $\wedge^3(H^{(n)})^*$ is written as a direct sum $C\omega^{(n)} + \wedge^3(H^{(n)})^*$ of its submodules, where $\wedge^3(H^{(n)})^*$ is the orthogonal complement of $C\omega^{(n)}$ in $\wedge^3(H^{(n)})^*$. Hence,

$$(2.1.1) \wedge^3((H^{(n)})^* \otimes_c (H^{(1)})^*) = N'^c \oplus N''^c \oplus L^c,$$

where $N'^c := C\omega^{(n)} \otimes_c S^3(H^{(1)})^*$, $N''^c := \wedge^3(H^{(n)})^* \otimes_c S^1(H^{(1)})^*$ and $L^c := S^5(H^{(n)})^* \otimes_c C\omega^{(1)}$. Note that the $Sp(n) \cdot Sp(1)$-modules $N'^c$, $N''^c$, $L^c$ respectively admit real forms $N'^r$, $N''^r$, $L^r$ fixed by the real structure induced from the one in (1.1.2). We have the identification $H^{(n)} \otimes_c H^{(1)} \cong (H^{(n)})^* \otimes_c (H^{(1)})^*$ by the metric $\langle , \rangle$ (cf. (1.1)). Together with $H^{(n)} \otimes_c H^{(1)} \cong H^* \otimes_r C$, the above (2.1.1) induces the decomposition of its real form:
\[ \wedge^2 H^s = N'_s \bigoplus N''_s \bigoplus L_z , \]

which is nothing but the decomposition in the Introduction now for our principal \( Sp(n) \cdot Sp(1) \)-bundle \( P \), the bundle \( T^*M \) is regarded as the vector bundle associated to the \( Sp(n) \cdot Sp(1) \)-module \( ((H^{(n)})^* \otimes_c (H^{(1)})^*)_R = H^s \). Hence, \( \wedge^2 T^*M \) is a direct sum \( A'_s \bigoplus A''_s \bigoplus B'_s \) of its subbundles \( A'_s, A''_s, B'_s \) corresponding respectively to the \( Sp(n) \cdot Sp(1) \)-modules \( N'_s, N''_s, L_z \) (cf. Introduction).

(2.2) Fix an arbitrary point \( x \) of \( M \). Note that each point \( z \) on the fibre \( Z_x \) defines an almost complex structure \( J_z \) on \( T^*M \) (cf. (1.3.1)). We then have the corresponding space \( \wedge^{1,1}(T^*_x M, J_z) \) of \( (1,1) \)-forms of \( (T^*_x M, J_z) \). Choose a point \( y(\neq 0) \) of \( V \) such that its natural image (denoted by \( [y] \)) is \( z \). In view of (1.3.1), the space \( \wedge^{1,1}(T^*_x M, J_z) \) in \( \wedge^{1,1}(T^*_x M)^c \) is associated to the \( C \)-linear subspace \( (H^{(n)} \otimes_c C y)' \cap ((H^{(n)} \otimes_c C y)' \otimes (H^{(n)} \otimes_c C y)' \otimes (H^{(n)} \otimes_c C y)' \otimes (H^{(n)} \otimes_c C y)' \otimes (H^{(n)} \otimes_c C y)' ) \).

The space \( C(y \wedge j^{(1)} y) \) (where \( y \wedge j^{(1)} y = (y \otimes j^{(1)} y - j^{(1)} y \otimes y)/2 \) in \( H^{(1)} \otimes_c H^{(1)} \)) corresponds to \( C\omega^{(1)} \) in \( (H^{(1)})^* \otimes_c (H^{(1)})^* \) via the natural isomorphism \( H^{(1)} \otimes_c H^{(1)} \cong (H^{(1)})^* \otimes_c (H^{(1)})^* \) induced by the nondegenerate bilinear form \( \omega^{(1)} \). Furthermore,

\[ \cap_y C(y \otimes j^{(1)} y + j^{(1)} y \otimes y) = \{0\} , \]

where \( \cap_y \) always denotes the intersection taken over all \( y \) in \( V_x - \{0\} \). Thus,

\[ \cap_y (H^{(n)} \otimes_c C y)' \cap (H^{(n)} \otimes_c C y)' = S^2(H^{(n)})* \otimes_c C\omega^{(1)} = L_z \] (cf. Introduction),

and we obtain:

**Lemma (2.3).** The fibre \( (B_x)_x \) of \( B_x \) over \( x \) is given by

\[ (B_x)_x = \cap_y \wedge^{1,1}(T^*_x M, J_{[y]}) . \]

We next give a typical example of an \( A'_s \)-connection and also a \( B'_s \)-connection.

**Example (2.4).** If \( n \geq 2 \), the induced connection on the locally defined vector bundle

\[ V := P_{x \in Sp(n) \times Sp(1)}(H^{(1)}) \quad (\text{resp. } W := P_{x \in Sp(n) \times Sp(1)}(H^{(n)})) \]

is an \( A'_s \)-connection (resp. \( B'_s \)-connection). See Salamon [S1; p. 150] for
related computations of curvatures.

Recall that a connection $\nabla$ is called a Yang-Mills connection if the corresponding curvature $R^\nabla$ satisfies $d^\nabla R^\nabla = 0$. We shall finally show:

**Theorem (2.5).** All $A_2$-connections and also all $B_2$-connections are Yang-Mills connections.

**Corollary (2.6).** The Riemannian connection on $TM$ is a Yang-Mills connection.

**Proof of (2.6).** By (1.2), (2.4) and (2.5), we obtain (2.6).

**Proof of (2.5).** Fix an arbitrary point $x_0$ of $M$. It then suffices to show $(d^\nabla R^\nabla)(x_0) = 0$. We may take a local section $s$ to $P$ over a neighbourhood $U$ of $x_0$ such that the corresponding differential at the point $x_0$ transforms the tangent space $T_{x_0}M$ to a horizontal space at $s(x_0)$ in the tangent space $T_{s(x_0)}P$. Let $(u^1, \cdots, u^{4n})$ be the local frame of $T^*_{s(x_0)}M$ associated to $s$. Then all covariant derivatives of $u^i$'s ($1 \leq i \leq 4n$) at the point $x_0$ is zero. Moreover in terms of the frame $(u^1, \cdots, u^{4n})$, we can identify $T^*_xM$ with $U \times \mathbb{R}^{4n}$ ($U \times \mathbb{H}^n$). Note that $\nabla$ on $E$ naturally induces a connection (denoted by the same $\nabla$) on $\text{End}(E)$.

(i) We first assume that $\nabla$ is an $A_2$-connection on $E$. Recall that the rank 3 subbundle $A_2$ of $\wedge^2 T^*M$ corresponds to the $\text{Sp}(n) \cdot \text{Sp}(1)$-submodule $N_2$ of $\wedge^2 \mathbb{H}^n$, where $N_2$ is the irreducible submodule of the elements fixed by $\text{Sp}(n)$ (cf. Introduction). Let $I, J$ and $K$ be

$$
I = \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+2} + u^{4k+3} \wedge u^{4k+4}), \\
J = \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+3} + u^{4k+4} \wedge u^{4k+1}), \\
K = \sum_{k=0}^{n-1} (u^{4k+1} \wedge u^{4k+4} + u^{4k+2} \wedge u^{4k+3}).
$$

Then it is easy to check that $A_2$ is spanned by the sections $I, J$ and $K$. Therefore, the curvature form $R^\nabla$ is written on $U$ as

$$
R^\nabla = a \otimes I + b \otimes J + c \otimes K,
$$

where $a, b$ and $c$ are smooth sections to $\text{End}(E)$ over $U$. Let $(u^1, \cdots, u^{4n})$ be the base for $T^*_xM$ dual to $(u^1, \cdots, u^{4n})$ defined by $u^i(u_j) = \delta_{ij}$. Then by the first Bianchi identity,

$$
0 = d^\nabla(R^\nabla)(x_0)
= \sum_{i=1}^{4n} (\nabla_i u^i)(x_0) \wedge I(x_0) + (\nabla_i u^i)(x_0) \wedge J(x_0) + (\nabla_i u^i)(x_0) \wedge K(x_0),
$$

where $(\nabla_i u^i)(x_0)$ is the covariant derivative of $u^i$ at $x_0$.
where $\nabla_i$ denotes $\nabla_{e_{i}(z_0)}$. Consequently,

$$\nabla_a a = \nabla_b b = \nabla_c c = 0,$$

for $1 \leq i \leq 4n$ if $n \geq 2$.

Therefore, $(d^v * R^v)(x_0) = 0$.

(ii) We next assume that $\nabla$ is a $B_2$-connection on $E$. Since the vector subbundle $B_2$ (of rank $n(2n + 1)$) of $\wedge^2 T^* M$ corresponds to the irreducible $Sp(n) \cdot Sp(1)$-submodule $L_2$ of the elements in $\wedge^2 H^s$ fixed by $Sp(1)$, the subbundle $B_{2|U}$ is spanned by

$$I_s, J_s, K_s, D_{pq}, E_{pq}, F_{pq}, G_{pq}, \quad (0 \leq s \leq n - 1, \ 0 \leq p < q \leq n - 1).$$

where

$$I_s = u^{4s+1} \wedge u^{4s+2} - u^{4s+3} \wedge u^{4s+4},$$

$$J_s = u^{4s+1} \wedge u^{4s+3} - u^{4s+4} \wedge u^{4s+2},$$

$$K_s = u^{4s+1} \wedge u^{4s+4} - u^{4s+2} \wedge u^{4s+3},$$

$$D_{pq} = u^{4p+1} \wedge u^{4q+1} + u^{4p+2} \wedge u^{4q+2} + u^{4p+3} \wedge u^{4q+3} + u^{4p+4} \wedge u^{4q+4},$$

$$E_{pq} = u^{4p+1} \wedge u^{4q+2} - u^{4p+2} \wedge u^{4q+1} - u^{4p+3} \wedge u^{4q+4} + u^{4p+4} \wedge u^{4q+3},$$

$$F_{pq} = u^{4p+1} \wedge u^{4q+3} + u^{4p+2} \wedge u^{4q+4} - u^{4p+3} \wedge u^{4q+1} - u^{4p+4} \wedge u^{4q+2},$$

$$G_{pq} = u^{4p+1} \wedge u^{4q+4} - u^{4p+2} \wedge u^{4q+3} + u^{4p+3} \wedge u^{4q+2} - u^{4p+4} \wedge u^{4q+1}.$$ 

Let $\nabla$ be a $B_2$-connection on $E$. Then over $U$, the curvature form $R^v$ is written in the form

$$R^v = \sum_{0 \leq s \leq n-1} (i_s \otimes I_s + j_s \otimes J_s + k_s \otimes K_s)$$

$$+ \sum_{0 \leq p < q \leq n-1} (d_{pq} \otimes D_{pq} + e_{pq} \otimes E_{pq} + f_{pq} \otimes F_{pq} + g_{pq} \otimes G_{pq}),$$

where $i_s$, $j_s$, $k_s$, $d_{pq}$, $e_{pq}$, $f_{pq}$ and $g_{pq}$ are smooth sections to End$(E)$ over $U$. In view of the first Bianchi identity $d^v R^v = 0$, we have

$$- \nabla_{4s+1} i_s + \nabla_{4s+2} j_s + \nabla_{4s+3} k_s = 0,$$

$$\nabla_{4s+1} i_s - \nabla_{4s+2} j_s + \nabla_{4s+3} k_s = 0,$$

$$\nabla_{4s+1} i_s + \nabla_{4s+2} j_s - \nabla_{4s+3} k_s = 0,$$

$$\nabla_{4s+1} i_s + \nabla_{4s+2} j_s + \nabla_{4s+3} k_s = 0,$$

for $s$ with $0 \leq s \leq n - 1$. Furthermore, if $l$ is either $p$ or $q$, the identity $d^v R^v = 0$ implies

$$(-1)^l (i_s) \nabla_{l+1} d_{pq} - \nabla_{l+1} e_{pq} - \nabla_{l+1} f_{pq} - \nabla_{l+1} g_{pq} = 0,$$

$$(-1)^l (i_s) \nabla_{l+1} d_{pq} - \nabla_{l+1} e_{pq} + \nabla_{l+1} f_{pq} + \nabla_{l+1} g_{pq} = 0,$$

$$(-1)^l (i_s) \nabla_{l+1} d_{pq} + \nabla_{l+1} e_{pq} - \nabla_{l+1} f_{pq} + \nabla_{l+1} g_{pq} = 0,$$

$$(-1)^l (i_s) \nabla_{l+1} d_{pq} + \nabla_{l+1} e_{pq} + \nabla_{l+1} f_{pq} - \nabla_{l+1} g_{pq} = 0,$$
for all $p, q$ with $0 \leq p < q \leq n - 1$, where $\varepsilon(p) := 0$ and $\varepsilon(q) := 1$.

Then a straightforward computation shows that $(d^* R^v)(x_v) = 0$, as required.

3. Deformations of $B_\tau$-connections. In this section, we shall give an elliptic complex whose first cohomology group canonically contains the space of infinitesimal deformations of $B_\tau$-connections on $M$ (see Salamon [S2] for a similar complex).

(3.1) Let $r$ be an integer with $r \geq 2$. By setting $N^\tau_r := \Lambda^*(H^{(n)})^* \otimes_c S^r(H^{(1)})^*$ (cf. (2.1)), we can express the $Sp(n) \cdot Sp(1)$-module $\Lambda^*(H^{(n)})^*$ as a direct sum $N^\tau_r \oplus L^\tau_r$, where $L^\tau_r$ is the orthogonal complement of $N^\tau_r$ in $\Lambda^*(H^{(1)})^*$. As in (2.1), the $Sp(n) \cdot Sp(1)$-modules $N^\tau_r$ and $L^\tau_r$ respectively admit real forms $N_r$ and $L_r$ fixed by the natural real structure (cf. (1.1.2)). Since $T^*M$ is associated to the $Sp(n) \cdot Sp(1)$-module $(H^{(n)})^* \otimes_c H^{(1)}$ (see (1.2.1)), the vector bundle $\Lambda^r T^*M$ is a direct sum $A_r \oplus B_r$ of its subbundles $A_r, B_r$ corresponding respectively to $N_r, L_r$. Let $\pi^r : \Lambda^r T^*M \to A_r \oplus B_r$ be the projection to the first factor. Then we have:

**Theorem (3.2).** For a $B_\tau$-connection $\nabla$ on $E$, the following is an elliptic complex:

\[
0 \to \mathcal{G}(E) \xrightarrow{\nabla} \mathcal{G}(E \otimes T^*M) \xrightarrow{d_1} \mathcal{G}(E \otimes A_2) \\
d_2 \mathcal{G}(E \otimes A_2) \xrightarrow{d_3} \cdots \mathcal{G}(E \otimes A_{2n}) \to 0 ,
\]

where $d_i := (id \otimes \pi^i \circ d^m)$ and for every vector bundle $E'$ on $M$, we denote by $\mathcal{G}(E')$ the sheaf of germs of $C^\infty$-sections of $E'$.

**Proof.** (i) Fix a section $s \in \Gamma(M, E \otimes A_i)$ $(i \geq 1)$ and define a section $t \in \Gamma(M, E \otimes B_{i+1})$ by

\[
d^m s = d_i s + t .
\]

Then from $(d^m \circ d^m) s = (d^m \circ d_i) s + d^m t$, we obtain

\[
((id \otimes \pi_{i+2} \circ d^m) \circ d^m) s = (d_{i+1} \circ d_i) s + ((id \otimes \pi_{i+2} \circ d^m) \circ d^m) t .
\]

Since $\nabla$ is a $B_\tau$-connection, the $A_{i+2}$-component of $(d^m \circ d^m) s$ is zero, i.e.,

\[
0 = (d_{i+1} \circ d_i) s + ((id \otimes \pi_{i+2} \circ d^m) \circ d^m) t .
\]

Write $t$ as $t = \sum_k v_k \otimes b_k$ locally, where $v_k, b_k$ is a local section of $E, B_{i+1}$, respectively. The $S^{i+2}(V^*)$-component of $b_k$ is zero, and hence the $S^{i+2}(V^*)$-component of $\nabla(v_k) \wedge b_k$ is zero. Therefore,

\[
((id \otimes \pi_{i+2} \circ d^m) \circ d^m) t = \sum_k v_k \otimes db_k .
\]
Since $d$ is the composite of the Riemannian connection and the alternation operator, the $S^{i+2}(V^*)$-component of $db_s$ is zero. Thus, $(d_{i+1}d_i)s = 0$, as required.

(ii) Secondly, we shall show that (3.1.1) is an elliptic complex. Then we need to calculate the symbol $\sigma(d_i, u)$ ($u \in T^*_0 M - \{0\}$). Fix a point of $M$ and an element $s$ of $E_x \otimes A_x$. All computations below are taken at the point $x$.

\[
\sigma(d_i, u)s := \frac{1}{(d/dt)(e^{-tq}d_i(e^{qt}s))}|_{t=0} = (id \otimes \pi_{i+1})(u \wedge s),
\]

where $q$ is a locally defined function such that $dq_x = u$. We next show that the following sequence is exact for every $u$:

\[
\begin{array}{ccc}
E \otimes A_{i-1} & \xrightarrow{\sigma(d_{i-1}, u)} & E \otimes A_i \\
& \xrightarrow{\sigma(d_i, u)} & E \otimes A_{i+1} \end{array}
\]

Without loss of generality, we may assume

\[
u = e_1 \otimes h_1 + (e_1 \otimes h_1)(-e_1 \otimes h_1 + e_2 \otimes h_2),
\]

where $\langle e_1, \cdots, e_n \rangle$ (resp. $\langle h_1, h_2 \rangle$) is a symplectic basis of $W^* \cong W$ (resp. $V^* \cong V$), i.e., an orthonormal basis and $j^{(n)}e_{2j+1} = e_{2j+2}$ (resp. $j^{(n)}h_1 = h_2$). Let $s \in E \otimes A_i$ be such that $\sigma(d_{i-1}, u)s = 0$. Note that $S^i V^* = \text{Span}(h^k \cdot h^{i-k}_1, 0 \leq k \leq i)$, where $h^k \cdot h^{i-k}$ denotes the symmetric component of $h^k \otimes h^{i-k}$. Hence, there are local sections $s_0, \cdots, s_i$ of $E \otimes \wedge^i W^*$ such that

\[
s = \sum_{k=0}^i s_k \otimes h^k \cdot h^{i-k}_1.
\]

We can now write $\sigma(d_{i-1}, s) = 0$ as follows:

\[
0 = (id \otimes \pi_{i+1})(u \wedge s) = (id \otimes \pi_{i+1})(e_1 \otimes h_1 + e_2 \otimes h_2) \wedge \sum s_k \otimes h^k \cdot h^{i-k}_1
\]

\[
= \sum_{k=0}^i (e_1 \wedge s_k) \otimes h^k_{i-1} \cdot h^{i-k}_1 + (e_2 \wedge s_k) \otimes h^k \cdot h^{i-k}_1.
\]

Since the coefficient of the right-hand side in $h^k_{i-1} \cdot h^{i-k}_1$ is zero, we have:

\[
\begin{array}{cc}
(0) & e_2 \wedge s_0 = 0 \\
(1) & e_1 \wedge s_0 + e_2 \wedge s_1 = 0 \\
& \vdots \\
(i) & e_1 \wedge s_{i-1} + e_2 \wedge s_i = 0 \\
(i+1) & e_1 \wedge s_i = 0
\end{array}
\]

By (0), there exists $r_0 \in \wedge^{i-1} W^*$ such that $s_i = e_2 \wedge r_0$. Plugging this into (1), we obtain $e_2 \wedge (-e_1 \wedge r_0 + s_1) = 0$. Hence there exists $r_1 \in \wedge^{i-1} W^*$ such that $s_1 = e_1 \wedge r_0 + e_2 \wedge r_1$. Repeating this process inductively, we obtain $r_k \in \wedge^{i-1} W^*$ such that $s_k = e_1 \wedge r_{k-1} + e_2 \wedge r_k$, $1 \leq k \leq i$. Now by
(i + 1), the identity \( e_1 \wedge e_2 \wedge r_i = 0 \) holds. It then follows that there exists \( r'_i \in \wedge^{i-2} W^* \) such that \( e_2 \wedge r_i = e_1 \wedge e_2 \wedge r'_i \). Since \( e_2 \wedge (r_{i-1} + e_2 \wedge r'_i) = e_2 \wedge r_{i-1} \), we may replace \( r_{i-1} \) by \( r_{i-1} + e_2 \wedge r'_i \). Therefore,

\[
\begin{align*}
s_0 &= e_2 \wedge r_0, \\
s_1 &= e_1 \wedge r_0 + e_2 \wedge r_1, \\
&\vdots \\
s_t &= e_1 \wedge r_{i-1}
\end{align*}
\]

Thus,

\[
s = \sum_{k=0}^t s_k \otimes h_k^t \cdot h_{t-k}^t = \sigma(d_{i-1}, u) \left( \sum_{k=0}^{i-1} r_k \otimes h_k^t \cdot h_{t-1-k}^t \right),
\]

i.e., the sequence (3.2.2) is exact, as required.

**Definition (3.3).** Let \( \mathcal{C} \) be the set of all \( B_z \)-connections on \( E \) with holonomy groups contained in a compact semisimple Lie group \( G \). Assume that \( \mathcal{C} \neq \emptyset \) and let \( \nabla \in \mathcal{C} \). Then the frame bundle \( Q \) of \( E \) can be regarded as a principal \( G \)-bundle. Put \( G_Q := Q \times_G G \) and \( g_Q := Q \times_{Ad} g \), where \( \theta \) is the group conjugation and \( Ad: G \to GL(g) \) is the adjoint representation of \( G \). Now, a \( C^\infty \)-section to \( G_Q \) over \( M \) is called a **gauge transformation** of \( Q \). Let \( \mathcal{C} \) be the set of all gauge transformations of \( Q \). Then \( \mathcal{C} \) naturally acts on \( \mathcal{C} \) (see Atiyah-Hitchin-Singer [A-H-S]). We call \( \mathcal{M}(:= \mathcal{C} / \mathcal{G}) \) the **moduli space** of the \( B_z \)-connections on \( E \) with holonomy groups in \( G \).

(3.4) Let \( \nabla \in \mathcal{C} \) be irreducible in the sense that \( g_\theta \) admits no non-zero parallel section over \( M \). Fix a smooth one-parameter family \( \nabla^t (|t| < \varepsilon) \) of connections in \( \mathcal{C} \) such that \( \nabla^0 = \nabla \). Put \( S = (d/dt)\nabla^t|_{t=0} \). We write the curvature form \( R^{\nabla^t} \) of \( \nabla^t \) as

\[
R^{\nabla^t} = R^\nabla + \pi^2 S + \text{higher order terms in } t,
\]

where \( \nabla' \) is the connection on \( g_Q \) naturally induced by \( \nabla \). Since \( R^{\nabla^t} \) is a \( g_Q \)-valued \( B_z \)-form, the corresponding derivative \( d^{\nabla^t} S \) at \( t = 0 \) also satisfies

\[
((id \otimes \pi^2) \circ d^{\nabla}) S = 0.
\]

Let \( f^t (|t| < \varepsilon) \) be a one-parameter family of gauge transformations such that \( f^0 = \text{id} \). Then,

\[
\frac{d}{dt}(f^t(\nabla))_{t=0} = \nabla'(f^t),
\]

where \( f^t := (d/dt)(f^t)|_{t=0} \). Since \( f^t(\nabla) \in \mathcal{C} \) for all \( t \), the same argument as above shows that the \( g_Q \)-valued 1-form \( \nabla'(f^t) \) satisfies
Quadratic forms 

((id ⊗ \tau^*) \circ d^{\tau'}) (\nabla'(\hat{f})) = 0.

For each \(A \in \Gamma(G_0)\), there exists a one-parameter family \(f^t = \exp(tA)\) such that \(\frac{d}{dt} f^t|_{t=0} = A\). Then together with (3.2), we immediately obtain the following:

**Theorem (3.5).** Assume that \(\mathcal{G} \neq \emptyset\) and let \(\nabla \in \mathcal{G}\) be irreducible. Then the space of infinitesimal (essential) deformations at \(\nabla\) of connections in \(\mathcal{G}\), that is, the tangent space of \(\mathcal{M}\) at \(\nabla\) is a linear subspace of the first cohomology group of the elliptic complex

\[
0 \rightarrow \mathcal{E}(G_0) \xrightarrow{\nabla'} \mathcal{E}(G_0 \otimes T^*M) \xrightarrow{d_1'} \mathcal{E}(G_0 \otimes A_1) \rightarrow \cdots \rightarrow \mathcal{E}(G_0 \otimes A_n) \rightarrow 0,
\]

where \(d_i' := (id \otimes \tau^{i+1}) \circ d^{\tau'}\).

4. **Einstein-Hermitian connections associated with \(B_\tau\)-connections.**

In this section we shall prove Theorem (0.2) (see the Introduction) which clarifies the relationship between \(B_\tau\)-connections and the corresponding Einstein-Hermitian connections.

**Proof of (0.2).** (i) Let \((E, D_E)\) be a Hermitian pair. Then by the definition of \(B_\tau\)-connections, the curvature form corresponding to the connection \(D_E\) is an \(\text{End}(E)\)-valued \(B_\tau\)-form, and by Lemma (2.3) the curvature form corresponding to the connection \(p^*D_E\) on \(p^*E\) is an \(\text{End}(p^*E)\)-valued \((1, 1)\)-form. Hence the connection \(p^*D_E\) induces naturally an integrable complex structure on \(p^*E\) as follows: Put \(l := \text{rank}(E)\) and denote by \(q: p^*E \rightarrow Z\) the natural projection. Let \((s_1, \ldots, s_l)\) (resp. \((y^1, \ldots, y^l)\)) be a local unitary frame for \(p^*E\) (resp. the dual frame corresponding to \((s_1, \ldots, s_l)\)). Then the vector subbundle \(\wedge^{1,0} T^*(p^*E)\) of type \((1, 0)\) in the complexification \(T^*(p^*E)^C\) of the cotangent bundle \(T^*(p^*E)\) is defined as the direct sum of the pull-back \(q^*(\wedge^{1,0} T^*Z)\) and the space spanned by \(\{dy^j + \sum_{i=1}^l y^i \eta^* \theta_{ji}, 1 \leq j \leq l\}\), where \((\theta_{ji})\) is the connection matrix for \(p^*D_E\) with respect to the frame \((s_1, \ldots, s_l)\) (i.e., \((p^*D_E)s_j = \sum_{i=1}^l s_i \theta_{ij}\)). Now, we may take the frame \((s_1, \ldots, s_l)\) as the pull-back \((p^*t_1, \ldots, p^*t_l)\) of a local unitary frame \((t_1, \ldots, t_l)\) on \(E\). Then the 1-forms \(\theta_{ij}, 1 \leq i, j \leq l\), are written as \(p^*\psi_{ij}\), where \((\psi_{ij})\) denotes the connection matrix for \(D_E\) with respect to the frame \((t_1, \ldots, t_l)\). Let \(q': (p^*E)^* \rightarrow Z\) be the projection naturally induced from \(q: p^*E \rightarrow Z\). Since the real structure \(\tau: Z \rightarrow Z\) is antiholomorphic (cf. Nitta and Takeuchi [N-T]), and since the mapping \(q' \circ \sigma: p^*E \rightarrow Z\) is equal to \(\tau \circ q\), the mapping \(\sigma: p^*E \rightarrow (p^*E)^*\) is clearly an antiholomorphic bundle automorphism by the definition of the complex structures on \(p^*E\) and \((p^*E)^*\).
(ii) We next fix an arbitrary excellent pair \((F, D_F)\) on \(Z\). Then by the condition (a) in the definition of excellent pair (see the Introduction), we can choose an open cover \(\{U_\lambda\}\) of \(M\), and a local unitary frame \(\{f_1^\lambda, \cdots, f_r^\lambda\}\) over \(p^{-1}(x)\) \((x \in U_\lambda)\) forms a holomorphic frame for \(F|_{p^{-1}(x)}\). When \(U_\lambda \cap U_\mu \neq \emptyset\), the transition matrix for \(F\) in terms of the frames \(\{f_1^\lambda, \cdots, f_r^\lambda\}, \{f_1^\mu, \cdots, f_r^\mu\}\) is holomorphic (and hence constant) along each fibre \(p^{-1}(x)\) \((x \in U_\lambda \cap U_\mu)\). Hence there exists a Hermitian vector bundle \(E\) on \(M\) such that, including metrics, we have \(p^*E = F\). In particular, we obtain a local unitary frame \(\{f_1^\lambda, \cdots, f_r^\lambda\}\) for \(E|_{U_\lambda}\) such that \((p^*f_1^\lambda, \cdots, p^*f_r^\lambda)\) coincides with the previous \(\{f_1^\lambda, \cdots, f_r^\lambda\}\) over \(p^{-1}(U_\lambda)\). Fix an arbitrary \(\lambda\). If there is no fear of confusion, we shall omit the suffix \(\lambda\) and denote \(U_\lambda, \{f_1^\lambda, \cdots, f_r^\lambda\}\) simply by \(U, \{f_1, \cdots, f_r\}\), respectively.

Let \((\omega_{ij})\) be the connection matrix of \(D_F\) with respect to the frame \(\{f_1, \cdots, f_r\}\), i.e., \(D_F f_j = \sum r_{i=1} f_i \omega_{ij}\). Furthermore, we choose a local symplectic basis \(\{(e_1, \cdots, e_{2n})\}\) (resp. \(\{(h_1, h_2)\}\) for \(W^*_|U\) (resp. \(V^*_|U\) (see Section 3). Now, since \(D_F\) is a Hermitian connection, we have:

\[
(1) \quad \omega_{ij} + \overline{\omega_{ji}} = 0, \quad \text{for } 1 \leq i, j \leq r.
\]

Then the construction of \(D_E\) is reduced to showing that there exist 1-forms \(\omega'_{ij}\) \((1 \leq i, j \leq r)\) on \(U\) satisfying \(\omega_{ij} = p^* \omega'_{ij}\). In fact, once we can find such 1-forms \(\omega'_{ij}\), they define a Hermitian connection on \(E\), such that the corresponding curvature form is pulled back by \(p\) to an \(\text{End}(F)\)-valued \((1,1)\)-form on \(Z\), which together with Lemma (2.3) implies that our connection on \(E\) is a \(B_g\)-connection. Recall that, for each \(x \in U\), the frame \(\{f_1|_{p^{-1}(x)}, \cdots, f_r|_{p^{-1}(x)}\}\) for \(F|_{p^{-1}(x)}\) is trivial. Hence,

\[
(2) \quad \omega_{ij}(v) = 0, \quad 1 \leq i, j \leq r,
\]

for every vector \(v\) tangent to \(p^{-1}(x)\) \((\equiv p'C)\). Since \((e_1 \otimes h_1, e_1 \otimes h_2, \cdots, e_{2n} \otimes h_1, e_{2n} \otimes h_2)\) is a frame for \(T^*M^c|_U = W^*_|U \otimes V^*|_U\), there exist by (2) \(\mathcal{C}^\infty\)-functions \(a_{ij}, b_{ij}\) \((1 \leq i, j \leq r, 1 \leq k \leq 2n)\) on \(p^{-1}(U)\) such that

\[
(3) \quad \omega_{ij} = \sum_{k=1}^{2n} (a_{ij}^k p^*(e_k \otimes h_1) + b_{ij}^k p^*(e_k \otimes h_2)), \quad 1 \leq i, j \leq r.
\]

For every form \(\eta\) on \(Z|_U\), we denote by \(\hat{\eta}\) the pull-back of \(\eta\) to \((V - \text{zero section})|_U\). Then by (3), we have:

\[
\hat{R}_{ij} = d \hat{\omega}_{ij} + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj} = \sum_{k=1}^{2n} d(a_{ij}^k \hat{p}^*(e_k \otimes h_1)) + d(b_{ij}^k \hat{p}^*(e_k \otimes h_2)) + \sum_{t=1}^r \hat{\omega}_{it} \wedge \hat{\omega}_{tj}.
\]

Fix an arbitrary point \(x\) on \(U\). Choosing an appropriate \((e_1, \cdots, e_{2n})\) (resp.
((h_1, h_2)), we may assume that $(\nabla^* e_k)(x) = 0$, \(k = 1, 2, \ldots, 2n\) (resp. $(\nabla^* h_i)(x) = 0\), \(i = 1, 2\), where $\nabla^*$ (resp. $\nabla^*$) denotes the connection of $V^*$ (resp. $W^*$) canonically induced by that of $P$ (cf. Example (2.4)). Then, on $\tilde{p}^{-1}(x)$,

$$\tilde{R}_{ij} = \sum_{k=1}^{2n} \{d(\partial_{t_j^i}) \wedge \tilde{p}^*(e_k \otimes h_i) + d(\tilde{b}_{t_j^i}) \wedge \tilde{p}^*(e_k \otimes h_j)\} + \sum_{i=1}^{r} \tilde{\omega}_{ij} \wedge \tilde{\omega}_{ij}. $$

Recall that the complex structure on the twistor space $Z (= (V - \{\text{zero section}\})/\mathbb{C}^*)$ is induced by the complex structure on $V - \{\text{zero section}\}$ (see Section 1). Since $\tilde{R}_{ij}$ is of type $(1, 1)$, we have:

\begin{align}
(4) & \quad \sum_{k=1}^{2n} \{\partial(\partial_{t_j^i}) \wedge (\tilde{p}^*(e_k \otimes h_i))^{(1,0)} + \partial(\tilde{b}_{t_j^i}) \wedge (\tilde{p}^*(e_k \otimes h_j))^{(1,0)} \} \\
& \quad + \sum_{i=1}^{r} \tilde{\omega}_{ij}^{(1,0)} \wedge \tilde{\omega}_{ij}^{(1,0)} = 0 \quad \text{on } p^{-1}(x); \\
(5) & \quad \sum_{k=1}^{2n} \{\partial(\partial_{t_j^i}) \wedge (\tilde{p}^*(e_k \otimes h_i))^{(0,1)} + \partial(\tilde{b}_{t_j^i}) \wedge (\tilde{p}^*(e_k \otimes h_j))^{(0,1)} \} \\
& \quad + \sum_{i=1}^{r} \tilde{\omega}_{ij}^{(0,1)} \wedge \tilde{\omega}_{ij}^{(0,1)} = 0 \quad \text{on } p^{-1}(x),
\end{align}

where for every 1-forms $\zeta$ on $(V - \{\text{zero section}\})_{lV}$, $\zeta^{(1,0)}$ (resp. $\zeta^{(0,1)}$) always denotes the $(1, 0)$-component (resp. $(0, 1)$-component) of $\zeta$. Let $(z^1, z^2)$ be the local triviality for $V_{lV}$ corresponding to $(h_1, h_2)$. Then, by the definition of the complex structure of $(V - \{\text{zero section}\})$, we obtain from (4) and (5) the following:

\begin{align}
(4') & \quad \sum_{k=1}^{2n} \left\{\left(\frac{\partial}{\partial z_1} \tilde{a}_{t_j^i} dz^1 + \frac{\partial}{\partial z_2} \tilde{a}_{t_j^i} dz^2\right) \wedge \tilde{z}^i((z^1)\tilde{p}^*(e_k \otimes h_i) + z^2\tilde{p}^*(e_k \otimes h_j)) \right. \\
& \quad \quad + \left\{\left(\frac{\partial}{\partial z_2} \tilde{b}_{t_j^i} dz^1 + \frac{\partial}{\partial z_1} \tilde{b}_{t_j^i} dz^2\right) \wedge \tilde{z}^i((z^1)\tilde{p}^*(e_k \otimes h_i) + z^2\tilde{p}^*(e_k \otimes h_j)) \right\} \\
& \quad = 0 \quad \text{on } \tilde{p}^{-1}(x); \\
(5') & \quad \sum_{k=1}^{2n} \left\{\left(\frac{\partial}{\partial z_1} \tilde{a}_{t_j^i} dz^1 + \frac{\partial}{\partial z_2} \tilde{a}_{t_j^i} dz^2\right) \wedge (-z^i(\tilde{z}^1\tilde{p}^*(e_k \otimes h_i) - \tilde{z}^i\tilde{p}^*(e_k \otimes h_i))) \right. \\
& \quad \quad + \left\{\left(\frac{\partial}{\partial z_2} \tilde{b}_{t_j^i} dz^1 + \frac{\partial}{\partial z_1} \tilde{b}_{t_j^i} dz^2\right) \wedge z^i(\tilde{z}^1\tilde{p}^*(e_k \otimes h_i) - \tilde{z}^i\tilde{p}^*(e_k \otimes h_i)) \right\} \\
& \quad = 0 \quad \text{on } \tilde{p}^{-1}(x). \end{align}

Since both $z^1_{\tilde{p}^{-1}(x)}$ and $z^2_{\tilde{p}^{-1}(x)}$ are holomorphic on $\tilde{p}^{-1}(x) \cong \mathbb{C}^2 - \{0\}$, we have

$$\frac{\partial}{\partial z_1}(z^1 \tilde{a}_{t_j^i} + z^2 \tilde{b}_{t_j^i}) = \frac{\partial}{\partial z_2}(-z^1 \tilde{a}_{t_j^i} + z^2 \tilde{b}_{t_j^i}) = 0 \quad (i = 1, 2),$$
on $p^{-1}(x)$, i.e., both $f_1(z^1, z^2) := z^1\alpha_{ij}^k + z^2\beta_{ij}^k$ and $f_2(z^1, z^2) := -z^1\tilde{\alpha}_{ij}^k + z^1\tilde{\beta}_{ij}^k$ are holomorphic on $C^2 - \{0\}$. By Hartogs' theorem, both $f_1$ and $f_2$ extend further to holomorphic functions on $C^2$. Since $f_i(cz^1, cz^2) = cf_i(z^1, z^2)$ for all $z = (z^1, z^2) \in C^2$ and $c \in C^*$ ($i = 1, 2$), there exist constants $\alpha_{ij}^k, \beta_{ij}^k, \gamma_{ij}^k, \delta_{ij}^k \in C$ independent of $z$ such that

\begin{align}
(6) & \quad z^1\alpha_{ij}^k + z^2\beta_{ij}^k = z^1\tilde{\alpha}_{ij}^k + z^2\tilde{\beta}_{ij}^k, \\
(7) & \quad -z^1\tilde{\alpha}_{ij}^k + z^1\tilde{\beta}_{ij}^k = -z^2\gamma_{ij}^k + z^2\delta_{ij}^k, \quad (1 \leq k \leq 2n).
\end{align}

Let $\Gamma(Z, F^*)$ (resp. $\Gamma(Z, F^* \otimes T^*Z^c)$) be the space of global $C^\infty$-sections over $Z$ to $F^*$ (resp. $F^* \otimes T^*Z^c$). Let $\psi: \Gamma(Z, F^*) \to \Gamma(Z, F^* \otimes T^*Z^c)$ be the $C$-linear map sending each $s \in \Gamma(Z, F^*)$ to an element $\psi(s)$ of $\Gamma(Z, F^* \otimes T^*Z^c)$ defined by

$$\psi(s)(X) := \sigma((D_Y)_{\tau(i)}(\sigma^{-1}s)) \in F^*_i,$$

for $X \in T_zZ^c$ ($z \in Z$).

Then by the condition (b) in the Introduction, this $\psi$ defines a Hermitian $(1, 0)$-connection on the holomorphic vector bundle $F^*$. The corresponding connection matrix with respect to the frame $(\sigma f_i, \cdots, \sigma f_r)$ for $F^*_{\tau^{-1}(x)}$ is written as $(\tau^*\omega_{ij})$. By the definition of $\sigma$, it is easy to check that the frame $(\sigma f_i, \cdots, \sigma f_r)$ is dual to our previous $(f_i, \cdots, f_r)$. Hence the uniqueness of the $(1, 0)$-connection on the Hermitian vector bundle $F^*$ implies the equality $(\tau^*\omega_{ij})^{-} = \omega_{ij}^{-}$, where $\omega_{ij}^{-} := -\omega_{ij}$. In view of (1), we have $\tau^*\omega_{ij} = \omega_{ij}$ and $\tilde{\tau}^*\tilde{\omega}_{ij} = \tilde{\omega}_{ij}$. By (3) and $\tilde{p} \circ \tilde{\tau} = \tilde{p}$, we obtain:

\begin{align}
(8) & \quad \tilde{\tau}^*\tilde{\alpha}_{ij}^k = \tilde{\alpha}_{ij}^k \quad \text{and} \quad \tilde{\tau}^*\tilde{\beta}_{ij}^k = \tilde{\beta}_{ij}^k \quad (1 \leq k \leq 2n).
\end{align}

Therefore,

$$-z^1\tilde{\tau}^*\tilde{\alpha}_{ij}^k + z^1\tilde{\tau}^*\tilde{\beta}_{ij}^k = -z^1\tilde{\alpha}_{ij}^k + z^1\tilde{\beta}_{ij}^k \quad (1 \leq k \leq 2n).$$

Moreover by (6),

\begin{align}
(9) & \quad -z^1\alpha_{ij}^k + z^1\beta_{ij}^k = -z^1\tilde{\alpha}_{ij}^k + z^1\tilde{\beta}_{ij}^k \quad (1 \leq k \leq 2n).
\end{align}

Hence by (7) and (9), we obtain:

\begin{align}
(10) & \quad \alpha_{ij}^k = \gamma_{ij}^k \quad \text{and} \quad \beta_{ij}^k = \delta_{ij}^k \quad (1 \leq k \leq 2n).
\end{align}

Now, in view of (6), (7) and (10), we see that

$$\left( \begin{array}{c} z^1, z^2 \end{array} \right) \left( \begin{array}{c} \alpha_{ij}^k - \alpha_{ij}^k \\
-\beta_{ij}^k \end{array} \right) = 0 \quad (1 \leq k \leq 2n),$$

where $(z^1, z^2) \in C^2 - \{0\} (= p^{-1}(x))$. Thus, $\tilde{\alpha}_{ij}^k = \alpha_{ij}^k$ and $\tilde{\beta}_{ij}^k = \beta_{ij}^k$ ($1 \leq k \leq 2n$), i.e., both $\alpha_{ij}^k$ and $\beta_{ij}^k$ are constant along $p^{-1}(x)$, as required.

**Remark (4.1).** In some sense, our Theorem (0.2) completely clarifies
the following result by Salamon [S2] (see Berard Bergery and Ochiai [B-O] for another generalization):

For a Hermitian pair \((E, D_E)\) on \(M\), the pull-back \((p^*E, p^*D_E)\) to \(Z\) is a Hermitian holomorphic vector bundle over \(Z\).

**Corollary (4.2).** Let \((F, D_F)\) be an excellent pair on \(Z\). If the quaternionic Kähler manifold \(M\) has positive scalar curvature, then \(F\) with \(D_F\) is a Ricci-flat Einstein Hermitian vector bundle over \(Z\).

**Proof.** Consider the twistor space \(p: Z \to M\). Then the horizontal component of the Kahler form on \(Z\) is a \(p^*A^2\)-form (cf. (1.2), (1.3)). Recall that the curvature of \(D_F\) is an \(\text{End}(F)\)-valued \(p^*\Omega^2\)-form. Hence the Hermitian vector bundle \(F\) with \(D_F\) is Ricci-flat.

**Remark (4.3).** We have the decomposition of \(TZ = T_h \oplus T^v\), where \(T_h\) (resp. \(T^v\)) is the horizontal (resp. vertical) distribution in terms of the connection on \(Z\) induced by that of \(P\). Since the complex structure on \(TZ\) is a direct sum of complex structures on \(T_h\) and \(T^v\), the holomorphic part \(TZ^{(1,0)}\) admits the corresponding decomposition \(TZ^{(1,0)} = T_h^{(1,0)} \oplus T_v^{(1,0)}\), where \(T_h^{(1,0)}\) (resp. \(T_v^{(1,0)}\)) denotes \(T^{hc} \cap TZ^{(1,0)}\) (resp. \(T^{vc} \cap TZ^{(1,0)}\)). Recently, Zandi [Z] obtained the following:

The vector bundle \((T_h^{(1,0)}, D_h)\) is an Einstein-Hermitian vector bundle, where \(D_h\) is the connection on \(T_h^{(1,0)}\) obtained as the restriction of the Riemannian connection on \(TZ\) to \(T_h^{(1,0)}\).

This result can be regarded as a straightforward consequence of our (4.2). We denote by \(L\) a locally defined (line) subbundle of \(p^*W\) (cf. (2.4)) such that, along each fibre \(p^{-1}(x) = P^1C\) \((x \in M)\), it restricts to a universal bundle over \(P^1C\). Let \(\nabla^v\) (resp. \(\nabla^w\)) denote the connection of \(V\) (resp. \(W\)) canonically induced by that of \(P\) and \(\nabla^v\) the restriction of \(p^*\nabla^{w}\) to \(L\). Then the vector bundle \((T_h^{(1,0)}, D^h)\) is nothing but \((p^*W \otimes L^*, p^*\nabla^w \otimes (\nabla^v)^*)\), where \((L^*, (\nabla^v)^*)\) is dual to \((L, \nabla^v)\) (see Salamon [S1]). Since \(L^*\) is a locally defined line bundle and since \(\nabla^w\) is a \(B\)-connection on \(W\), Corollary (4.2) clearly implies Zandi’s result.

**Added in proof.** After the completion of this paper, the author received a preprint by M. M. Capria and S. M. Salamon entitled “Yang-Mills fields on quaternionic Kähler spaces”, which gives (i) a result slightly stronger than (2.6) and (ii) a statement similar to (3.2).

**References**


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