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CLASS NUMBERS OF POSITIVE DEFINITE BINARY AND TERNARY UNIMODULAR HERMITIAN FORMS

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CONTENTS

0. Introduction.

0.1. This paper is a continuation of [32] . Let (V, H) be a positive definite Hermitian space over an imaginary quadratic field K, and let $\mathscr{L}$ be a genus of $\mathcal{O}$-lattices in V with respect to the unitary group $\mathbb{G}:=\mathbb{U}(V, H)$, where $\mathcal{O}$ is the ring of integers of K. As we saw in [32], the class number $\boldsymbol{h}(\mathscr{L})$ of $\mathscr{L}$ is expressed as a finite sum:

$$\boldsymbol{h}(\mathscr{L})=\sum_{f\in F}\sum_{[g]_{\boldsymbol{Q}}}\boldsymbol{h}([g]_{\boldsymbol{Q}};\mathscr{L}), \tag{0.1}$$

where in the first sum f runs through the set of characteristic polynomials of the torsion elements of $\mathbb{G}$; and the second sum is taken over the *locally integral* G-conjugacy classes $[g]_{\boldsymbol{Q}}=[g]_G$ which belong to f, and the invariants $\boldsymbol{h}([g]_{\boldsymbol{Q}};\mathscr{L})$ are given by

$$\boldsymbol{h}([g)_{\boldsymbol{Q}};\mathscr{L})=\sum_{\mathscr{L}(V)}\mathbb{M}(V)\prod_p c_p(g, U_p, V_p), \tag{0.2}$$

with $\mathbb{M}(V)$ the mass of an idélic arithmetic subgroup V of the centralizer $G(g)_{\mathbb{A}}$ of g in $G_{\mathbb{A}}$. See [32] for a more precise definition. We note among others that the masses were evaluated there.

In the present paper, we shall carry out the computations of the local factors $c_p(g, U_p, V_p)$, and derive from (0.1), (0.2) explicit formulas for the class numbers of genera consisting of *unimodular* Hermitian lattices, of ranks **two** and **three**.

0.2. To state our main results, let $K=\boldsymbol{Q}(\sqrt{-m})$ ($m>0$, square free) be an imaginary quadratic field, and let $\mathcal{O}$ be its ring of integers. Let V be a vector space of dimension n over K, which is equipped with a positive definite Hermitian form H:

$V \times V \to K$. An $\mathcal{O}$-lattice in V is said to be *unimodular*, if it coincides with its dual lattice. We assume that (V, H) contains a genus of unimodular lattices. Denote by t the number of distinct prime divisors of the discriminant $d(K)$ of K. It is known that there exist exactly 2^{t-1} mutually nonisometric classes of such (V, H), if $n>1$. They are parameterized by the local norm residues $\varepsilon := (\varepsilon_p = (d(V), K/\boldsymbol{Q})_p;\ p \mid d(K))$ of the discriminant $d(V)$ of (V, H) at the places dividing $d(K)$, which are subject to the condition:

$$\prod_{p \mid d(K)} \varepsilon_p = 1 \qquad \text{(cf. [32, Proposition 6.4])}. \tag{0.3}$$

Also, from a result of Jacobowitz [19], we know that the set of unimodular $\boldsymbol{Q}$-lattices in (V, H) is divided into *at most two* genera with respect to the unitary group $\mathbb{G} := \mathbb{U}(V, H)$. One, which always exists, is said to be *normal* and denoted by $\mathcal{L}_o = \mathcal{L}_o(\varepsilon)$; and the other, which occurs only if n is even and $2 \mid d(K)$, is *subnormal* or *even*, and denoted by $\mathcal{L}_e = \mathcal{L}_e(\varepsilon)$. On the other hand, with respect to the *special unitary group* $\mathbb{G}^{(1)} := \mathbb{SU}(V, H)$, $\mathcal{L}_o$ and $\mathcal{L}_e$ are divided into an *infinite* number of genera.

0.3. Suppose first that n $(=\dim V)=2$. We shall prove:

THEOREM 0.1 $(n=2)$. *The class number $\boldsymbol{h}^{(1)}$ of binary unimodular genus $\mathcal{L}^{(1)}$ of Hermitian $\mathcal{O}$-lattices in (V, H), $V=V(\varepsilon)$, with respect to the special unitary group, depends only on the $\mathbb{G}$-genus $\mathcal{L}_o$ or $\mathcal{L}_e$ which contains $\mathcal{L}^{(1)}$. Moreover, $\boldsymbol{h}^{(1)}$ is given by*

$$\boldsymbol{h}^{(1)} = T_1 + T_2 + T_3 ,$$

with

$$T_1 = (A/12) \prod_p (p + (-1/p))$$

$$T_2 = (B/4) \prod_p (1 + \varepsilon_p)$$

$$T_3 = (C/3) \prod_p (1 + \varepsilon_p(-1/p)(-3/p)) ,$$

where the products are all taken over the prime divisors p of $d(K)$ such that $p \neq 2$, and the constants A, B, C are given in the following table: $(d=d(K))$

	ε_2		$d=$odd	$4\|d$	$d\equiv 8 \pmod{32}$	$d\equiv -8 \pmod{32}$
	± 1	A	1	3	6	6
$\mathcal{L}_o(\varepsilon)$	± 1	B	1	3	2	2
	± 1	C	1	0	0	0

	ε_2		d=odd	$4\|d$	$d\equiv 8 \pmod{32}$	$d\equiv -8 \pmod{32}$
$\mathscr{L}_e(\varepsilon)$	+	A	*	*	3	1
	−			1	1	3
	+	B	*	*	1	1
	−			1	1	1
	+	C	*	*	0	2
	−			1	2	0

0.4. Next we study the ternary case: $n=3$, where we assume that our Hermitian space (V, H) is the standard one. Namely $H(x, y)=\sum_{i=1}^{3} x_i y_i^\rho$ $(x, y \in V:=K^3)$. Then the (unique) genus $\mathscr{L}=\mathscr{L}_o$ of unimodular lattices in (V, H) is represented by the standard $\mathcal{O}$-lattice $\mathcal{O}^3$, and it is called the *principal genus.*

THEOREM 0.2 ($n=3$). *The class number $\boldsymbol{h}^{(1)}$ of a genus $\mathscr{L}^{(1)}$ with respect to the special unitary group $\mathbb{SU}(V,H)$, which is contained in the principal genus $\mathscr{L}_o$, is uniquely determined by $\mathscr{L}_o$ and it is given as follows*:

$\boldsymbol{h}^{(1)}=1$ (*resp.* 2) *if* $K=\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\sqrt{-3})$ (*resp.* $\boldsymbol{Q}(\sqrt{-2})$, $\boldsymbol{Q}(\sqrt{-7})$), *and otherwise,*

$$\boldsymbol{h}^{(1)}=T_1+T_2+T_3+T_{41}+T_{42}\,,$$

with

$$T_1 =B_{3,\chi}/144$$

$$T_2 =(h(K)/48)[4|d(K)|-1-3\chi(2)]$$

$$T_3 =(h(K)/8)[3+\chi(2)+\{1+(2, K/\boldsymbol{Q})_2\}\{1+(5, K/\boldsymbol{Q})_2\}]$$

$$T_{41}=(h(K)/12)[7-\chi(3)]$$

$$T_{42}=(h(K)/12)[1+\chi(3)]\,.$$

Here χ denotes the Dirichlet character attached to K, and $B_{m,\chi}$ denotes the m-th generalized Bernoulli number attached to χ, $h(K)$ is the class number of K, and $(c, K/\boldsymbol{Q})_p$ is the local norm residue symbol at p.

0.5. Here we recall briefly the known results on class number formulas of definite Hermitian forms. In [15], Hayashida gave a formula for the class numbers of positive unimodular Hermitian matrices of rank two, with coefficients in $\mathcal{O}$, in connection with his study of curves of genus two in the product of two elliptic curves having $\mathcal{O}$ as the ring of complex multiplications. From [32, Theorem 2.14], we see that his result is the same as our Theorem 0.1, in the case (V, H) is the standard Hermitian space so that $\varepsilon=(1, \cdots, 1)$, and $\mathscr{L}=\mathscr{L}_o$ is the principal genus. As was shown in [15], the class number in the binary case is reduced to that of certain orders in a quaternion algebra over $\boldsymbol{Q}$, which are in general *not* maximal, but the calculation is much easier than that in higher rank cases. Also we refer simply to an item [6] in Math. Reviews, where F. T. Chu announced a result on class numbers of positive Hermitian unimodular matrices of ranks two and three over some rings of imaginary quadratic fields of class number *one*, which contains an error at $K=\boldsymbol{Q}(\sqrt{-7})$.

0.6. This paper is organized as follows. In § 1, we give, without proofs, the lists of characteristic polynomials of all torsion elements of our group $\mathbb{G}=\mathbb{U}(V, H)$, in the cases where the ranks are two and three. In § 2, we shall give a proof of Theorem 0.1, for class numbers of binary Hermitian forms. In § 3 and § 4, which are the most difficult and laborious parts of this paper, we shall compute the contributions of each conjugacy classes belonging to all possible characteristic polynomials $f(X)$, in the case $n=3$. Here the calculations are reduced to the classification of the conjugacy classes in the local unit group U_p of the given lattice L_p over $\mathcal{O}_p$, and the calculation of the masses of the centralizers of each representatives of 'locally integral' conjugacy classes. In § 5, we resume the results obtained before, and state our first main result for $n=3$, in a slightly more convenient way than Theorem 0.2. We also give a result on a relation between the class numbers for $\mathbb{U}(V, H)$ and $\mathbb{SU}(V, H)$ in the ternary case, which is rather remarkable compared with a general results which was given in [32]. In § 6, we give another application of our computation, which gives an explicit formula for the dimension of automorphic forms on our special unitary groups ($n=3$).

NOTATION. As usual, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ denote the fields of rational, real, and complex numbers, respectively, and $\boldsymbol{Z}$ denotes the ring of rational integers. For an algebraic object B over $\boldsymbol{Q}$ or $\boldsymbol{Z}$, we denote by B_p the p-adic completion of B. Thus $\boldsymbol{Q}_p$ (resp. $\boldsymbol{Z}_p$) is as usual the field (resp. ring) of p-adic numbers (resp. integers). Also we denote by $B_{\mathbb{A}}$ the idélization of B. If G is a group, and H is a subgroup of G, we denote the set of H-conjugacy classes in G by $G // H$, and its element containing g by $[g]_H$. When $H=\boldsymbol{G}$ is a $\boldsymbol{Q}$-group, we put simply $[g]_{\boldsymbol{Q}}:=[g]_G$, $[g]_p:=[g]_{G_p}$, where $G=\mathbb{G}_{\boldsymbol{Q}}$, G_p are the group of $\boldsymbol{Q}$-rational, $\boldsymbol{Q}_p$-rational points of $\mathbb{G}$, respectively. Also, we denote by $\mathbb{G}(g)$ the centralizer of g in $\mathbb{G}$. The cardinality of a finite set S is written as $\sharp(S)$. Throughout this paper, K denotes an imaginary quadratic field, and ρ denotes the non-trivial automorphism of $K/\boldsymbol{Q}$. For $c\in\boldsymbol{Q}^\times$, and a place v of $\boldsymbol{Q}$, we denote by $(c, K/\boldsymbol{Q})_v$ the local norm residue symbol of c, i.e., $(c, K/\boldsymbol{Q})_v=1$ or -1 according as c is a norm of an element of $K_v^\times$ or not. Notice that we have $(c/\boldsymbol{Q})_v=(c, m)_v$ ($:=$Hilbert symbol) if $K=\boldsymbol{Q}(\sqrt{m})$. Also we denote by $\chi(*)=(K/*)=(d(K)/*)$ the Dirichlet character attached to K, where $d(K)$ is the discriminant of K. And we denote by t the number of distinct prime divisors of $d(K)$.

The symbol (V, H) will denote a ρ-Hermitian space over K, i.e., V is a vector space over K which is equipped with a Hermitian form

$$H\colon V\times V\to K, \qquad (x, y)\to H(x, y) \quad (x, y\in V),$$

which we always assume to be nondegenerate. We denote by $\mathbb{U}(V, H)$, $\mathbb{SU}(V, H)$ the unitary group, the special unitary group of (V, H), respectively, which are often abbreviated as $\mathbb{G}$, $\mathbb{G}^{(1)}$, throughout this paper. Also we denote by $G(f)$ the set of semi-simple elements of G whose characteristic polynomials are $f(x)$. We shall use some more standard notation frequently.

1. Characteristic polynomials of torsion elements. Our first task is to make a list of F, the set of all possible polynomials which can make in (0.1) a non-trivial contribution.

LEMMA 1.1. *Suppose $n=2$. Then F consists of the following polynomials*:

(i) *Generic case*: $d(K) \neq -3, -4$

$$f_1(X) := (X-1)^2, \qquad f_1(-X),$$
$$f_2(X) := (X^2+1),$$
$$f_3(X) := (X^2+X+1), \quad f_3(-X),$$
$$f_4(X) := (X-1)(X+1).$$

(ii) *Exceptional case*: $d(K) = -3$

$$f_1((-\omega)^k X), \quad f_4((-\omega)^k X) \quad (k=0, 1, \cdots, 5), \quad f_2(X),$$
$$f_{31}(X) := (X-1)(X-\omega), \quad f_{31}((-\omega)^k X) \quad (k=0, 1, \cdots, 5),$$
$$f_{32}(X) := (X-1)(X+\omega), \quad f_{32}((-\omega)^k X) \quad (k=0, 1, \cdots, 5),$$

(iii) *Exceptional case*: $d(K) = -4$

$$f_1(i^k X), \quad f_4(i^k X) \quad (k=0, 1, 2, 3), \quad f_3(\pm X),$$
$$f_{41}(X) := (X-1)(X-i), \quad f_{41}(i^k X) \quad (k=0, 1, 2, 3).$$

Here, we put $\omega := (-1+\sqrt{-3})/2$, $i := \sqrt{-1}$, and, by abuse of notation, we write $f(cX)$ instead of $c^{-2}f(cX)$.

LEMMA 1.2. *Suppose $n=3$. Then F consists of the following polynomials*:

(i) *Generic case*: $d(K) \neq -3, -4, -7, -8$

$$f_1(X) := (X-1)^3, \quad f_1(-X),$$
$$f_2(X) := (X-1)(X+1)^2, \quad f_2(-X),$$
$$f_3(X) := (X-1)(X^2+1), \quad f_3(-X),$$
$$f_{41}(X) := (X-1)(X^2+X+1), \quad f_{41}(-X),$$
$$f_{42}(X) := (X-1)(X^2-X+1), \quad f_{42}(-X).$$

(ii) *Exceptional case*: $d(K)=-3$

$$f_1((-\omega)^j X),$$
$$f_{21}((-\omega)^j X) \quad \textit{with} \quad f_{21}(X):=f_2(X),$$
$$f_{22}((-\omega)^j X), \quad f_{22}^{\varrho}((-\omega)^j X) \quad \textit{with} \quad f_{22}(X):=(X-1)(X-\omega)^2,$$
$$f_{23}((-\omega)^j X), \quad f_{23}^{\varrho}((-\omega)^j X) \quad \textit{with} \quad f_{23}(X):=(X-1)(X+\omega)^2,$$
$$f_{31}((-\omega)^j X) \quad \textit{with} \quad f_{31}(X):=f_3(X),$$
$$f_{32}((-\omega)^j X), \quad f_{32}^{\varrho}((-\omega)^j X) \quad \textit{with} \quad f_{32}(X):=(X-\omega)(X^2+1),$$
$$f_{41}(\omega^k X), \quad f_{42}((-\omega)^j X),$$
$$f_{43}((-\omega)^j X), \quad f_{43}^{\varrho}((-\omega)^j X) \quad \textit{with} \quad f_{43}(X):=(X-1)(X+1)(X-\omega),$$
$$f_7(\pm X), \quad f_7^{\varrho}(\pm X) \quad \textit{with} \quad f_7(X):=(X^3-\omega),$$

where $\omega:=(-1+\sqrt{-3})/2$, $j=0, 1, \cdots, 5$, $k=0, 1, 2$.

(iii) *Exceptional case*: $d(K)=-4$

$$f_1(i^k X), \quad f_{21}(i^k X),$$
$$f_{24}(i^k X), \quad f_{24}^{\varrho}(i^k X) \quad \textit{with} \quad f_{24}(X):=(X-1)(X-i)^2,$$
$$f_3(i^k X), \quad f_{41}(i^k X), \quad f_{42}(i^k X),$$
$$f_{44}(i^k X), \quad f_{44}^{\varrho}(i^k X) \quad \textit{with} \quad f_{44}(X):=(X-i)(X^2+X+1),$$

where $i:=\sqrt{-1}$, $k=0, 1, 2, 3$.

(iv) *Exceptional case*: $d(K)=-7$

$$f_1(\pm X), \quad f_2(\pm X), \quad f_3(\pm X), \quad f_{41}(\pm X), \quad f_{42}(\pm X),$$
$$f_6(\pm X), \quad f_6^{\varrho}(\pm X) \quad \textit{with} \quad f_6(X):=X^3+\bar{\eta}X^2-\eta X-1,$$

where $\eta:=(-1+\sqrt{-7})/2$, $\bar{\eta}:=(1-\sqrt{-7})/2$.

(v) *Exceptional case*: $d(K)=-8$

$$f_1(\pm X), \quad f_2(\pm X), \quad f_3(\pm X), \quad f_{41}(\pm X), \quad f_{42}(\pm X),$$
$$f_5(\pm X), \quad f_5^{\varrho}(\pm X), \quad \textit{with} \quad f_5(X):=(X-1)(X^2-\sqrt{-2}X+1).$$

Here again, we write $f(cX)$ *instead of* $c^{-3}f(cX)$.

We omit the proofs of these lemmas, since they are proved by quite elementary computations.

2. Quaternion algebras and nonmaximal orders. Here we shall give a proof of Theorem 0.1. Thus we assume, throughout this section, that $n=\dim_K(V)=2$.

2.1. Let (V, H) be a non-degenerate Hermitian space of rank two over K. Fixing an orthogonal basis, we identify V with K^2, and write

$$H=\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \qquad (D=d(V, H)).$$

Since any positive rational number can be represented by H, one can always find such a basis. Then a direct computation shows that

$$\mathbb{SU}(V, H)=B\cap SL_2(K) \quad (=B^{(1)}, \text{say}) \tag{2.1}$$

where

$$B:=\left\{\begin{pmatrix} a & b \\ -D\bar{b} & \bar{a} \end{pmatrix}; a, b\in K\right\}$$

is a (definite) quaternion algebra over $\boldsymbol{Q}$. The determinant det: $B\to\boldsymbol{Q}(\subset K)$ coincides with the reduced norm Nr of B. It follows that any element $\xi\in B^\times$ defines a similitude transformation on (V, H) such that $H(x\xi, y\xi)=\mathrm{Nr}(\xi)H(x, y)$ for all $x, y\in V$.

LEMMA 2.1. *The following conditions are equivalent*:

(i) *B is ramified at p (i.e., B_p is a division algebra).*

(ii) *(V_p, H) is anisotropic.*

(iii) $(-d(V), K/\boldsymbol{Q})_p=-1$.

The proof is immediate.

Let L be an $\mathcal{O}$-lattice in (V, H). We put

$$R=R(L):=\{g\in B; Lg\subset L\}. \tag{2.2}$$

It is clear that R is a $\boldsymbol{Z}$-order of B which contains $\mathcal{O}$. From the above remark we have the following:

LEMMA 2.2. (i) *The class number of the genus of L with respect to the group $\mathbb{HU}(V, H)\,(=B^\times)$ of direct similitudes is equal to the class number of R.* (ii) *If $\mathrm{Nr}(R_{\mathbb{A}}^\times)=\boldsymbol{Z}_{\mathbb{A}}^\times$, then the class number of the genus of L with respect to the group $\mathbb{SU}(V, H)\,(=B^{(1)})$ is also equal to that of R.*

It is known by Shimura [26] that R is a maximal order if L is a maximal lattice in the sense of [26]. However, this is not necessarily the case if L belongs to a genus $\mathscr{L}$ of unimodular lattices. So we shall first study the structure of R. According to [32, Proposition 6.4], the isometry classes of positive Hermitian spaces (V, H) containing a unimodular lattice are parametrized by the discriminant $d(V)\in\boldsymbol{Q}^\times/N_{K/\boldsymbol{Q}}(K^\times)$ which satisfies the condition

$$(d(V), K/\boldsymbol{Q})_p=1 \qquad \text{for all} \quad p \quad \text{with} \quad (K/p)\neq 0.$$

Putting $\varepsilon_i:=(d(V), K/\boldsymbol{Q})_{p_i}$ $(1\leq i\leq t)$ for each prime $p_i\,|\,d(K)$, we then see that they are parametrized by the invariants $\varepsilon=(\varepsilon_i)_{1\le i\le t}$, where $\varepsilon_i=\pm 1$ and they are subject to the condition $\prod_i \varepsilon_i=1$. It follows immediately from Lemma 2.1 that the discriminant $d(B)$ of B is the product of p_i $(1\le i\le t)$ such that $(-1, K/\boldsymbol{Q})_{p_i}=-\varepsilon_i$. If we write $d(K)=d^+(K)\cdot d^-(K)$ with $d^{\pm}(K)=\prod_{\varepsilon_i=\pm 1} p_i$, then we have

$$d(B)=2^{\delta}\Bigg(\prod_{\substack{p\mid d^+(K)\\ p\equiv -1(4)}} p\Bigg)\Bigg(\prod_{\substack{q\mid d^-(K)\\ q\equiv 1(4)}} q\Bigg) \qquad (\delta=0 \text{ or } 1)\,, \tag{2.3}$$

where $\delta=1$ exactly in the cases where $(K/2)=0$ and either

(i) $\varepsilon(2):=(d(V), K/\boldsymbol{Q})_2=1$ and $d(K)\equiv 12, -8 \pmod{32}$, or

(ii) $\varepsilon(2)=-1$ and $d(K)\equiv 8 \pmod{32}$.

2.2. Now let (V,H) be as above, with $V=V(\varepsilon)$. Then from Lemmas 6.2, 6.3 of [32], there are at most two genera of unimodular $\mathcal{O}$-lattices in (V, H) with respect to the unitary group, which are distinguished by the property that one is normal and the other is subnormal.

PROPOSITION 2.3. *Let L be a unimodular $\mathcal{O}$-lattice in (V, H), $V=V(\varepsilon)$. Let B, $R=R(L)$ be as above. Then we have*:

(i) *If $(p, d(K))=1$, then $B_p\simeq M_2(\boldsymbol{Q}_p)$ and $R_p\simeq M_2(\boldsymbol{Z}_p)$.*

(ii) *If $p\mid d(K)$, $(p, d(B))=1$, and $p\neq 2$, then*

$$R_p\simeq\begin{pmatrix}\boldsymbol{Z}_p & \boldsymbol{Z}_p\\ p\boldsymbol{Z}_p & \boldsymbol{Z}_p\end{pmatrix},$$

and we have $[GL_2(\boldsymbol{Z}_p): R_p^{\times}]=p+1$.

(iii) *If $p\mid d(K)$, $p\mid d(B)$, and $p\neq 2$, then R_p is the (unique) maximal order of B_p.*

PROOF. Note that the assertions (i), (iii) are contained in Shimura [26], since L_p is a maximal $\mathcal{O}_p$-lattice in those cases. Also they are easily proved in the same way as in the following proof of (ii). So we omit the detail. Let us prove (ii). By [32, Lemma 6.2], L_p has an orthogonal basis so that we may assume

$$H=\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix},$$

and $L_p=\mathcal{O}_p\oplus\mathcal{O}_p$, where $u=d(V)\in\boldsymbol{Z}_p^{\times}$. Then we have

$$R_p=\left\{\begin{pmatrix}a & b\\ -u\bar{b} & \bar{a}\end{pmatrix};\ a, b\in\mathcal{O}_p\right\}=\mathcal{O}_p+\mathcal{O}_pU\,,\quad U:=\begin{pmatrix}0 & 1\\ -u & 0\end{pmatrix}. \tag{2.4}$$

We see that

$$e_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad e_2=\begin{pmatrix}\omega & 0\\ 0 & \omega^{\rho}\end{pmatrix},\quad e_3=U\,,\quad e_4=e_2U$$

form a $\boldsymbol{Z}_p$-basis of L_p, and that

$$(2.5)\qquad \det(\mathrm{Tr}(e_i\bar{e}_j))=\det\begin{pmatrix} 2 & \omega+\omega^\rho & 0 & 0 \\ \omega+\omega^\rho & 2\omega\omega^\rho & 0 & 0 \\ 0 & 0 & 2u & -(\omega+\omega^\rho) \\ 0 & 0 & -(\omega+\omega^\rho) & 2\omega\omega^\rho u \end{pmatrix}$$

$$=(\omega-\omega^\rho)^4u^2=d(K)^2u^2\,.$$

Moreover, it is easy to see that R_p contains a subring isomorphic to

$$\begin{pmatrix} \boldsymbol{Z}_p & 0 \\ 0 & \boldsymbol{Z}_p \end{pmatrix}.$$

It follows from Hijikata [16, §2.2], that R_p is $GL_2(\boldsymbol{Q}_p)$-conjugate to a split order

$$R(m):=\begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p^m\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix}.$$

A direct computation similar to the above one shows that, for any $\boldsymbol{Z}_p$-basis e_i $(1\le i\le 4)$ of $R(m)$, one has $\det(\mathrm{Tr}(e_i\bar{e}_j))\boldsymbol{Z}_p=p^{2m}\boldsymbol{Z}_p$, from which follows $m=1$, as asserted. q.e.d.

PROPOSITION 2.4. *Suppose $p=2$, $p\,|\,d(K)$, and L_p is a normal unimodular $\mathcal{O}_p$-lattice in V_p. Assume further that $\varepsilon(p)=1$ $(p=2)$.*

(1) *If $d(K)\equiv 8$ (mod 32), then we have $(p, d(B))=1$ and*

$$(2.5)\qquad R_p=\left\{\begin{pmatrix} x+(ay-bz) & w+(by+az) \\ -w+(by+az) & x-(ay-bz) \end{pmatrix};\, x, y, z, w\in\boldsymbol{Z}_p\right\},$$

where $a, b\in\boldsymbol{Z}_p$ are fixed solutions of $a^2+b^2=d(K)/4$, and we have $[R_{0p}^\times : R_0^\times]=3$, where $R_{0p}=M_2(\boldsymbol{Z}_p)$.

(2) *If $d(K)\equiv 12$ (mod 16), then we have $p\,|\,d(B)$ and*

$$(2.6)\qquad R_p=\left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix};\, a, b\in\mathcal{O}_p\right\}=\mathcal{O}_p+\pi R_{0p}\,,$$

$$R_{0p}=\left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix};\, a, b\in\pi^{-1}\mathcal{O}_0,\, a+b\in\mathcal{O}_p\right\},$$

where R_{0p} is the (unique) maximal order of B_p, and π is a prime element of R_{0p}, and we have $[R_{0p}^\times : R_p^\times]=3$.

(3) *If $d(K)\equiv -8$ (mod 32), then we have $p\,|\,d(K)$ and*

$$(2.7)\qquad R_0=\left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix};\, a, b\in\mathcal{O}_p\right\},$$

$$pR_{0p}\subset R_p\subset R_{0p}=\mathcal{O}_p+\mathcal{O}_pU\,,\qquad U=\frac{1}{2}\begin{pmatrix} -1 & a+b\sqrt{m} \\ -a+b\sqrt{m} & -1 \end{pmatrix},$$

where R_{0p} *is the maximal order of* B_p, $m=d(K)/4$, *and* (a, b) *is a fixed solution of* $a^2+b^2m=3$ $(a, b\in 1+p\boldsymbol{Z}_p)$. *Moreover, we have* $[R_{0p}^{\times}: R_p^{\times}]=6$.

PROOF. All these assertions are checked by direct computations, together with a well-known criterion that, R_p is a maximal order of B_p if and only if $\det(\mathrm{Tr}(e_i\bar{e}_j))\boldsymbol{Z}_p=d(B_p)^2\boldsymbol{Z}_p$ for a $\boldsymbol{Z}_p$-basis e_i $(1\leq i\leq 4)$ of R_p. We omit the details. q.e.d.

We remark that the case treated in this proposition is the same as that in Hayashida [15], although we employ different notation.

Next we suppose that $\varepsilon(p)=-1$ $(p=2)$. The explicit form of R_p can be obtained again by direct computations. In this case, however, it will be seen that we need only the index $[R_{0p}^{\times}: R_p^{\times}]$ for our class number calculation (see Remark 2.13).

PROPOSITION 2.5 $(p=2)$. *Let the assumptions be as in Proposition* 2.4, *with* $\varepsilon(p)=-1$.

(1) *If* $d(K)\equiv 8$ (mod 32), *then we have* $p\,|\,d(B)$ *and* $[R_{0p}^{\times}: R_p^{\times}]=6$.

(2) *If* $d(K)\equiv 12$ (mod 16), *then we have* $(p, d(B))=1$ *and* $[R_{0p}^{\times}: R_p^{\times}]=3$.

(3) *If* $d(K)\equiv -8$ (mod 32), *then we have* $(p, d(B))=1$ *and* $[R_{0p}^{\times}: R_p^{\times}]=6$.

Proof is omitted.

Now we study the case where L_p is subnormal. According to [32, Lemma 6.3], we may assume that

$$\text{either} \quad H=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad H=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and $L_p=\mathcal{O}_p\oplus\mathcal{O}_p$.

LEMMA 2.6. *Suppose that*

$$H=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (p=2).$$

Then the group $\mathbb{HU}(V, H)$ *of the direct similitudes is expressed as*

$$B^{\times}=\left\{\begin{pmatrix} a & b\sqrt{m} \\ c/\sqrt{m} & d \end{pmatrix};\ a, b, c, d\in\boldsymbol{Q}_p, ad-bc\neq 0\right\}.$$

The proof is immediate. It follows that $B_p\simeq M_2(\boldsymbol{Q}_p)$. We identify them by the correspondence

$$\begin{pmatrix} a & b\sqrt{m} \\ c/\sqrt{m} & d \end{pmatrix}\leftrightarrow\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

PROPOSITION 2.7 $(p=2)$. *Let the assumptions be as above.*

(1) *If* $d(K)\equiv 12$ (mod 16), *then we have* $\varepsilon(p)=-1$, *and* $R_p=M_2(\boldsymbol{Z}_p)$, *so that*

$[R_{0p}^{\times}: R_p]=1$.

(2) *If* $d(K)\equiv \pm 8 \pmod{32}$, *then we have* $\varepsilon(p)=\pm 1$, *and*

$$R_p=\begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix},$$

so that $[R_{0p}^{\times}: R_p^{\times}]=p+1$.

PROOF. This is an easy consequence from Lemma 2.2 and Lemma 2.6. q.e.d.

Next suppose that

$$H=\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We can find $U\in GL_2(K_p)$ such that

$$H=U\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}{}^tU^{\rho}.$$

Then it is easy to see that we have

$$B=UB_0U^{-1}, \quad \text{with} \quad B_0=\left\{\begin{pmatrix} a & b \\ -D\bar{b} & \bar{a} \end{pmatrix}; a, b\in K\right\}.$$

We identify B and B_0 by the inner automorphisms $\mathrm{Int}(U)$.

PROPOSITION 2.8 ($p=2$). *Let the assumptions be as above. Then we have* $p\,|\,d(B)$, *and* $\varepsilon(p)=(3, K/\boldsymbol{Q})_p=1, -1$ *according as* $d(K)\equiv -8, 8 \pmod{32}$. *In both cases,* R_p *is the maximal order of the division algebra* B_p.

PROOF. Put $m=d(K)/4$, $m_0=m/2$, and $\alpha=\sqrt{m}$. Since the scaling of H by a scalar in $\boldsymbol{Q}_p^{\times}$ does not affect the conclusion, we may take

$$U=\frac{\alpha}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then a direct computation shows that, under the above identification, we have

$$R_p=\left\{\frac{1}{2}\begin{pmatrix} a+b & a-b \\ 3(\bar{a}-\bar{b}) & \bar{a}+\bar{b} \end{pmatrix}; a, b\in \mathcal{O}_p\right\}.$$

Now putting

$$e_1=\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}, \quad e_2=\alpha e_1, \quad e_3=\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}, \quad e_4=\alpha e_3,$$

we have

$$\det(\mathrm{Tr}(e_i\bar{e}_j))=\det\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & -2m & 0 & m \\ -1 & 0 & 2 & 0 \\ 0 & m & 0 & -2m \end{pmatrix}=-9m^2\in 2^2\cdot \boldsymbol{Z}_p^\times .$$

It follows, by the criterion noted above, that R_p is a maximal order of B_p. q.e.d.

From the results for R_p above, we have:

COROLLARY 2.9. *Let (V, H) be a binary positive definite Hermitian space over K, and let L be a unimodular $\mathcal{O}$-lattice in V. Then for the $\boldsymbol{Z}$-order $R=R(L)$ of B, we have* $\mathrm{Nr}(R_{\mathbb{A}}^\times)=\boldsymbol{Z}_{\mathbb{A}}^\times$. *Therefore the class number of the genus of L with respect to $\mathbb{SU}(V, H)$ is equal to the class number of R.*

2.3. Now we can apply Theorem 1.2 to obtain the class number of the genus $\mathscr{L}=\mathscr{L}(L)$ with respect to the group $\mathbb{SU}(V, H)$ or $\mathbb{HU}(V, H)$. As for the group $\mathbb{HU}(V, H)=B^\times$, the problem is reduced, by Corollary 2.9, to the class number calculation of the order R in our definite quaternion algebra B, as has been done for the split orders by Eichler [7] and others. Here we take this latter standpoint, rather than work with $\mathbb{SU}(V, H)$, because we can make use of known results on the arithmetic of quaternion algebras as developed in [7], [16].

By Lemma 1.1, the characteristic polynomials of $\mathbb{HU}(V, H)=B^\times$ are $f_1(X)=(X-1)^2$, $f_2(X)=(X^2+1)$, $f_3(X)=(X^2+X+1)$, and $f_i(-X)$. We denote by T_i the contribution from the elements of $B^\times$ to the formula (0.1), whose characteristic polynomial is $f_i(\pm X)$, where we take $\mathbb{HU}(V, H)=B^\times$ in place of $\mathbb{G}=\mathbb{U}(V, H)$. Since B is a simple division algebra over $\boldsymbol{Q}$, we know that each of these elements are semi-simple and they are conjugate in $B^\times$ if and only if they have the same characteristic polynomials. Hence we have $T_i=\boldsymbol{h}([\pm g]; \mathscr{L})$, if g corresponds to $f_i(X)$. We denote by $R_p(f)$ the set of semi-simple elements of R_p whose characteristic polynomials are $f(X)$.

PROPOSITION 2.10 ($f=f_1$). *Let R_0 be a maximal order of B containing $R=R(L)$. Then we have*

(2.8)
$$\mathbb{M}(R_{0\mathbb{A}}^\times)=(1/24)\cdot\prod_{p\mid d(B)}(p-1),$$
$$\mathbb{M}(R_{\mathbb{A}}^\times)=\mathbb{M}(R_{0\mathbb{A}}^\times)\cdot\prod_p\,[R_{0p}^\times : R_p^\times] .$$

We have $T_1=2\mathbb{M}(R_{\mathbb{A}}^\times)$.

PROPOSITION 2.11 ($f=f_2, f_3$). (i) *Suppose that $B_p=M_2(\boldsymbol{Q}_p)$, where p is any prime number.*

(1) *If* $R_p = M_2(\boldsymbol{Z}_p)$, *then we have* $c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times) = 1$ *for any* $g \in R_p(f)$.

(2) *If*

$$R_p = \begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix},$$

then we have $c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times) = 1 + (-1/p)$, *or* $1 + (-3/p)$ *for* $g \in R_p(f)$, *according as* $f = f_2$ *or* f_3.

(ii) *Suppose that* R_p *is the maximal order of the division quaternion algebra* B_p *over* $\boldsymbol{Q}_p$. *Then we have*

$$c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times) = 1 - (-1/p), \qquad or \quad 1 - (-3/p),$$

according as $f = f_2$ *or* f_3.

These results are well-known (cf. [7], [16]).

PROPOSITION 2.12 ($f = f_2, f_3$). *Suppose that* $p = 2$, $p \mid d(K)$ *and let* L_p *be a normal unimodular* $\mathcal{O}_p$*-lattice.*

(1) *If* $d(K) \equiv 8 \pmod{32}$, *then we have* $c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times) = 2$, *or* 0, *according as* $f = f_2$ *or* f_3.

(2) *If* $d(K) \equiv 12 \pmod{16}$, *then we have* $c_p(g, \boldsymbol{R}_p^\times, \boldsymbol{Z}_p[g]^\times) = 3$, *or* 0, *according as* $f = f_2$ *or* f_3.

(3) *If* $d(K) \equiv -8 \pmod{32}$, *then we have* $c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times) = 2$, *or* 0, *according as* $f = f_2$ *or* f_3.

PROOF. (1): By Proposition 2.4, we have

$$R_p = \left\{ \begin{pmatrix} x + (ay - bz) & w + (by + az) \\ -w + (by + az) & x - (ay - bz) \end{pmatrix};\ x, y, z, w \in \boldsymbol{Z}_p \right\},$$

and $[GL_2(\boldsymbol{Z}_p) \colon R_p^\times] = 6$. It is easy to show that as representatives of $GL_2(\boldsymbol{Z}_p)/R_p^\times$, we can choose

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Let

$$g := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

be an element of $R_p(f_2)$, and let x be one of the above representatives. Then an immediate check shows that one has

$$x^{-1} g x \in R_p^\times \quad \text{only for} \quad x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and that they belong to distinct double cosets in $\boldsymbol{Q}_p[g]^\times \backslash GL_2(\boldsymbol{Q}_p)/R_p^\times$. This proves that $c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times)=2$. We note that for any element x of $\mathcal{O}_p$, one has $\mathrm{Tr}(x)\in p\boldsymbol{Z}_p$, so that $R_p(f_3)=\varnothing$. This is also the case for (2) and (3).

(2): We see by Proposition 2.4 that $R_p=\mathcal{O}_p+\pi R_{0p}$ with $\pi=1+\sqrt{(d(K)/4)}$ (=a prime element of R_{0p}), and that $\pi R_{0p}\cap R_p=\pi R_p$. It follows that $R_p^\times$ is a normal subgroup of $R_{0p}^\times$, and the quotient is $R_{0p}^\times/R_p^\times \simeq \mathbb{F}_{p^2}^\times/\mathbb{F}_p^\times$ (=cyclic group of order 3). It follows that

$$c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times)=3 \qquad \text{for} \quad g=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\in R_p(f_2)\,.$$

(3): By Proposition 2.4, we see that $R_{0p}/pR_{0p}\simeq \mathbb{F}_{p^2}+\pi\mathbb{F}_{p^2}$ ($\pi^2\equiv 0$), where

$$\pi=\begin{pmatrix} \sqrt{m} & 0 \\ 0 & -\sqrt{m} \end{pmatrix}.$$

Let g be as above, and put $g=\alpha+\beta U$ ($\alpha, \beta\in\mathcal{O}_p$). Then we see that $\alpha\equiv 1+\sqrt{m}$, $\beta\equiv 0$ (mod p), and it follows that $g\equiv 1+\pi$ (mod p), and that

$$R_p/pR_{0p}=(\mathcal{O}_p/p\,\mathcal{O}_p)+(\mathcal{O}_p/p\,\mathcal{O}_p)\cdot(\alpha \bmod p)\simeq \mathbb{F}_p+\pi\mathbb{F}_p\,.$$

Now a direct computation shows that, for $x\in R_{0p}^\times$, one has $x^{-1}gx\in R_p^\times$ if and only if $(x \bmod p)\in 1+\mathbb{F}_{p^2}\pi$. Thus we get four representatives in $R_{0p}^\times/R_p^\times$:

$$1\,,\quad 1+\pi\,,\quad 1+U\pi\,,\quad 1+U^2\pi \quad (\text{Observe } U^3\equiv 1 \pmod p)\,.$$

Two elements $1+\alpha\pi$, $1+\beta\pi$ are easily checked to belong to the same $\boldsymbol{Q}_p[g]^\times-R_p^\times$ double coset if and only if $\alpha-\beta\in\mathbb{F}_p$. So we have $c_p(g, R_p^\times, \boldsymbol{Z}_p[g]^\times)=2$. q.e.d.

Now we have all data that we need. Substituting them to our general formula (0.1), (0.2), we get an explicit formula as stated in Theorem 0.1. We note, among others, that the orders $R(L)$ of B are all isomorphic when we let L vary in the $\mathbb{G}$-genus $\mathscr{L}$. Therefore the class number $\boldsymbol{h}^{(1)}$ depends only on $\mathscr{L}$.

REMARK 2.13. In gathering these local data, we have to take into consideration the condition from the product formula $\prod_{p\mid D(K)}\varepsilon(p)=1$. Thus we see that $T_2=0$ except in the case $\varepsilon(p)=1$ for all p, i.e., the case of principal genus. Indeed, if $\varepsilon(p)=-1$ for some $p\neq 2$, then

$$(-1/p)=1 \Leftrightarrow (-1, K/\boldsymbol{Q})_p=1 \Leftrightarrow B_p \quad \text{is a division algebra}\,.$$

Therefore, we have $B_p(f_2)=\varnothing$ in this case. On the other hand, if $(-1/p)=-1$, then Proposition 2.3, (ii) implies that

$$R_p\simeq\begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix},$$

and it follows from Proposition 2.11 that $R_p(f_2)=\varnothing$. Also we note that if $2\mid d(K)$, then

$T_3=0$ for the genera of normal unimodular lattices.

3. $n=3$: Contribution from $f_2(X)$.

3.1. In this and the subsequent section, we calculate the contributions of non-central elements of the unitary group to the class number formula for the principal genus in the ternary case ($n=3$). To be more precise, let (V, H) be the ρ-Hermitian space defined in (0.5):

$$V=K^3\,, \qquad H(x, y):=x\cdot{}^t y^{\rho} \quad (x, y\in V)\,.$$

We denote by $\mathbb{G}$ the unitary group $\mathbb{U}(V, H)$, and by $\mathscr{L}$ the principal genus represented by the standard lattice $L:=\mathcal{O}^3$. We keep these notation in §§ 3, 4 and 5.

Throughout this section, we write $f(X)$ for $f_2(X)=(X-1)(X+1)^2$, and denote by T_2 the contribution from $G(f)$ to the formula (0.1). First we investigate, for each finite place p, the set $G_p(f)\cap U_p$ and the U_p-conjugacy classes in it. We follow the method of Asai [2], which makes an essential use of the results of Jacobowitz [19] that we described in [32, § 6].

For any $g\in G_p(f)$, we put

$$V_{1,p}=V_{1,p}^{(g)}:=V_p\cdot(g+1)\,, \qquad V_{2,p}=V_{2,p}^{(g)}:=V_p\cdot(g-1)\,.$$

Thus we have $V_p=V_{1,p}\oplus V_{2,p}$. Further, we put

$$L_{1,p}=L_{1,p}^{(g)}:=L_p\cap V_{1,p}\,, \qquad L_{2,p}=L_{2,p}^{(g)}:=L_p\cap V_{2,p}\,.$$

By an $\mathcal{O}_p$-*sublattice* of L_p, we understand a free $\mathcal{O}_p$-submodule of L_p (whose rank may be less than three). For any $\mathcal{O}_p$-sublattice Λ of L_p, we say that Λ is optimally embedded in L_p if it satisfies $\Lambda=(\Lambda\otimes_{\mathcal{O}_p}K_p)\cap L_p$. Clearly the above $L_{1,p}$, $L_{2,p}$ are optimally embedded in L_p. Let π be a prime element of K_p. When $K_p=\boldsymbol{Q}_p\oplus\boldsymbol{Q}_p$, we put $\pi=(p, p)$.

LEMMA 3.1 (cf. Asai [2]). (i) *Suppose a, b are distinct elements of $\mathcal{O}_p^\times$ such that $N_{K_p/\boldsymbol{Q}_p}(a)=N_{K_p/\boldsymbol{Q}_p}(b)=1$. Put $\psi(X)=(X-a)(X-b)^2$ and $l=\mathrm{ord}_\pi(a-b)$. Let g be an element of $G_p(\psi)\cap U_p$, and put $\Lambda_1=V_p\cdot(g-b)\cap L_p$, $\Lambda_2=V_p\cdot(g-a)\cap L_p$. Then the pair (Λ_1, Λ_2) satisfies the condition*:

(3.1) (i) *Λ_i is an optimally embedded $\mathcal{O}_p$-sublattice of L_p of rank i $(i=1, 2)$.*

(ii) *$H(\Lambda_1, \Lambda_2)=(0)$ (i.e., Λ_1, Λ_2 are orthogonal).*

(iii) *$\Lambda_1\oplus\Lambda_2\supseteqq\pi^l L_p$.*

Conversely, suppose that a pair (Λ_1, Λ_2) satisfies the condition (3.1). *Define an element g of $GL_{K_p}(V_p)$ by $g\,|\,\Lambda_1=a$, $g\,|\,\Lambda_2=b$. Then g belongs to $G_p(\psi)\cap U_p$. Moreover, this correspondence $g\longmapsto(\Lambda_1, \Lambda_2)$ induces the following bijection*:

$$G_p(\psi)\cap U_p//U_p\simeq\{U_p\text{-equivalence classes of }(\Lambda_1, \Lambda_2)\text{ satisfying (3.1)}\}\,.$$

Here (Λ_1, Λ_2) *and* (Λ_1', Λ_2') *are said to be* U_p*-equivalent, if there exists* $h \in U_p$ *such that* $\Lambda_i \cdot h = \Lambda_i'$ $(i=1, 2)$.

(ii) *In particular, when* $\psi = f$, *the value of the above* l *is equal to*

$$l = \begin{cases} 0 \cdots p \neq 2 \\ 1 \cdots p=2,\ (K/p) \neq 0 \\ 2 \cdots p=2,\ (K/p)=0 . \end{cases}$$

PROOF. Here we prove only the surjectivity of the map in (i), since the other part is proved quite easily. Suppose that (Λ_1, Λ_2) satisfies (3.1), and let g be defined as above. Then since Λ_1 and Λ_2 are orthogonal, g belongs to G_p. We show that $g \in U_p = \mathbb{U}(L_p, H)$. Let x be an arbitrary vector in L_p. By (3.1), (iii), we can write $x = \pi^{-l}(x_1 + x_2)$ with $x_1 \in \Lambda_1$, $x_2 \in \Lambda_2$. Then we have

$$x \cdot g = \pi^{-l}(x_1 \cdot g + x_2 \cdot g) = \pi^{-l}(ax_1 + bx_2) = \pi^{-l}(a-b)x_1 + bx .$$

The definition of l shows that $x \cdot g \in L_p$. Thus $L_p \cdot g \subset L_p$, hence $g \in U_p$. q.e.d.

3.2. From this lemma we see that there are at most two types of $(L_{1,p}, L_{2,p})$ corresponding to an element of $G_p(f) \cap U_p$:

$$L_p = L_{1,p} \oplus L_{2,p} \qquad \cdots \text{(say) Type I} \tag{3.2}$$

$$L_p \supsetneqq L_{1,p} \oplus L_{2,p} \supsetneqq pL_p \quad \cdots \text{(say) Type II} . \tag{3.3}$$

Type I occurs in all cases, while type II appears only if $p=2$. We say that a U_p-conjugacy class in $G_p(f) \cap U_p$ is of type I or II, according as the corresponding pair $(L_{1,p}, L_{2,p})$ is of type I or II. First we treat the case of type I.

LEMMA 3.2. (i) *If* $(K/p) \neq 0$, *then* $G_p(f) \cap U_p$ *contains a unique* U_p*-conjugacy class of type I. It is characterized by*

$$(d(L_{2,p}), K/\boldsymbol{Q})_p = +1 , \qquad L_{2,p} \text{ is normal unimodular} .$$

(ii) *If* $p \neq 2$ *and* $(K/p)=0$, *then* $G_p(f) \cap U_p$ *contains two* U_p*-conjugacy classes. They are classified by the corresponding lattices* $L_{2,p}$ *with the conditions*:

$$(d(L_{2,p}), K/\boldsymbol{Q})_p = \pm 1 , \qquad L_{2,p} \text{ is normal unimodular} .$$

(iii) *If* $p=2$ *and* $4 \| d(K)$, *then* $G_p(f) \cap U_p$ *contains three* U_p*-conjugacy classes of type* I. *They are classified by the corresponding lattices* $L_{2,p}$ *with the conditions*:

$$(d(L_{2,p}), K/\boldsymbol{Q})_p = \pm 1 , \qquad L_{2,p} \text{ is normal unimodular} , \quad \text{or}$$

$$(d(L_{2,p}), K/\boldsymbol{Q})_p = -1 , \qquad L_{2,p} \text{ is subnormal unimodular} .$$

(iv) *If* $p=2$ *and* $8 \| d(K)$, *then* $G_p(f) \cap U_p$ *contains four* U_p*-conjugacy classes of type* I. *They are classified by the corresponding lattices* $L_{2,p}$ *with the conditions*:

$(d(L_{2,p}), K/\boldsymbol{Q})_p = \pm 1$, $\quad L_{2,p}$ *is normal unimodular* , *or*

$(d(L_{2,p}), K/\boldsymbol{Q})_p = \pm 1$, $\quad L_{2,p}$ *is subnormal unimodular* .

PROOF. Let p be any finite place of $\boldsymbol{Q}$, and suppose that L_p is written as $L_p = L_{1,p} \oplus L_{2,p} = L'_{1,p} \oplus L'_{2,p}$. Then we have $d(L_{i,p})$, $d(L'_{i,p}) \in \boldsymbol{Z}_p^\times$, hence $L_{i,p}$, $L'_{i,p}$ are unimodular ($i=1, 2$) (see [32, Lemma 6.1]). Clearly, $(L_{1,p}, L_{2,p})$ and $(L'_{1,p}, L'_{2,p})$ are U_p-equivalent if any only if $L_{i,p}$ and $L'_{i,p}$ are isometric for $i=1, 2$. From (6.3) and (6.4) of [32] we see that $(L_{1,p}, L_{2,p})$ and $(L'_{1,p}, L'_{2,p})$ are U_p-equivalent if $(K/p) = = +1$. Suppose $(K/p) \neq +1$. From [32, Lemmas 6.2, 6.3], we see that $L_{2,p} \simeq L'_{2,p}$ implies $L_{1,p} \simeq L'_{1,p}$. Then [32, Lemma 6.3, (ii)] shows the assertions in this case. q.e.d.

3.3 Next we suppose $p=2$, and consider the case of type II. Suppose first that $(K/p) = +1$. Using the identification $V_p = \boldsymbol{Q}_p^3 \oplus \boldsymbol{Q}_p^3$, we put

$$L_{i,p}^0 := L_{i,p} \cap (\boldsymbol{Q}_p^3 \oplus \{0\}) \quad (i=1, 2) ,$$

and regard $L_{i,p}^0$ as a $\boldsymbol{Z}_p$-module in $\boldsymbol{Q}_p^3$.

LEMMA 3.3. *Suppose that $p=2$ and $(K/p) = +1$. Then $U_p(f)$ contains a unique U_p-conjugacy class of type* II. *It is characterized by the following lattices*:

$$(3.4) \qquad L_{1,p}^0 = \boldsymbol{Z}_p x_1 , \qquad L_{2,p}^0 = \boldsymbol{Z}_p x_2 \oplus \boldsymbol{Z}_p x_3 ;$$

$$x_1 = (1, 0, 0) , \qquad x_2 = (0, 1, 0) , \qquad x_3 = (1, 0, p) .$$

PROOF. Using (6.3) and (6.4) of [32], we see that it is enough to show that any pair $(L_{1,p}^0, L_{2,p}^0)$ of type II is transformed to the one given in (3.4) by an element h of $GL_3(\boldsymbol{Z}_p)$ $(\simeq U_p)$. An easy argument on elementary divisors show the existence of such h.

q.e.d.

Suppose next that $(K/p) \neq +1$. The following lemma, which is proved in Asai [2], is essential in our calculation.

LEMMA 3.4. *Suppose $p=2$ and $(K/p) \neq +1$. Let $(L_{1,p}, L_{2,p})$ be a pair satisfying* (3.3). *Then the Jordan splitting of each $L_{i,p}$ is written as*

$$L_{2,p} = L_{2,p}^{(1)} \oplus L_{2,p}^{(2)} ; \qquad \text{rank } L_{2,p}^{(j)} = 1 ,$$

$$s(L_{1,p}) = s(L_{2,p}^{(2)}) = p\,\mathcal{O}_p , \qquad s(L_{2,p}^{(1)}) = \mathcal{O}_p .$$

Using this, we can show:

LEMMA 3.5. *Suppose that $p=2$.*

(i) *If $(K/p) = -1$, then $U_p(f)$ contains a unique U_p-conjugacy class of type* II, *which is characterized by*

$$(3.5)\qquad L_{1,p}=\mathcal{O}_p x_1\,,\qquad L_{2,p}^{(1)}=\mathcal{O}_p x_2\,,\qquad L_{2,p}^{(2)}=\mathcal{O}_p x_3\,;$$
$$x_1=(1,0,1)\,,\qquad x_2=(0,1,0)\,,\quad x_3=(1,0,-1)\,.$$

Here we have $(d(L_{2,p}), K/\boldsymbol{Q})_p=-1$.

(ii) *If* $(K/p)=0$, $U_p(f)$ *contains two* U_p*-conjugacy classes of type* II. *Each class is characterized by*

$$(3.6)\qquad L_{1,p}=\mathcal{O}_p x_1\,,\qquad L_{2,p}^{(1)}=\mathcal{O}_p x_2\,,\qquad L_{2,p}^{(2)}=\mathcal{O}_p x_3\,;$$
$$x_1=(1,0,\varepsilon)\,,\qquad x_2=(0,1,0)\,,\qquad x_3=(\varepsilon^\rho,0,-1)\,.$$
$$1+N(\varepsilon)\in p\boldsymbol{Z}_p^\times\cap N(K_p^\times)\qquad\cdots\ (d(L_{2,p}), K/\boldsymbol{Q})_p=+1\,.$$
$$(3.7)\qquad L_{1,p}=\mathcal{O}_p x_1\,,\qquad L_{2,p}^{(1)}=\mathcal{O}_p x_2\,,\qquad L_{2,p}^{(2)}=\mathcal{O}_p x_3\,;$$
$$x_1=(1,0,\xi)\,,\qquad x_2=(0,1,0)\,,\qquad x_3=(\xi^\rho,0,-1)\,.$$
$$1+N(\xi)\in p\boldsymbol{Z}_p^\times-N(K_p^\times)\qquad\cdots\ (d(L_{2,p}), K/\boldsymbol{Q})_p=-1\,.$$

Here we write $N(*)$ *for* $N_{K_p/\boldsymbol{Q}_p}(*)$.

PROOF. We prove (i). Suppose $(K/p)=-1$, and let $(L_{1,p}, L_{2,p})$ be an arbitrary pair satisfying (3.3). It suffices to show that this is U_p-equivalent to $(\mathcal{O}_p x_1, \mathcal{O}_p x_2\oplus\mathcal{O}_p x_3)$ given by (3.5). Using Lemma 3.4, we put $L_{2,p}=L_{2,p}^{(1)}\oplus L_{2,p}^{(2)}$. Then [32, Lemma 6.2, (i)] shows that U_p acts transitively on the set of unimodular lines in V_p, if $(K/p)=-1$. Thus we can transform $L_{2,p}^{(1)}$ to $\mathcal{O}_p x_2$ by the U_p-action, so we may assume that $L_{2,p}^{(1)}=\mathcal{O}_p x_2$. Then we can write $L_{1,p}=\mathcal{O}_p(a,0,b)$, and $L_{2,p}^{(2)}=\mathcal{O}_p(-b^\rho,0,a)$ where a, b are elements of $\mathcal{O}_p^\times$ satisfying $N(a)+N(b)=2$. Then we can find $c\in\mathcal{O}_p^\times$ such that

$$a-cb^\rho\equiv ca^\rho+b\equiv 0\ (\mathrm{mod}\ 2)\,,\quad N(c)=1\,.$$

Now put

$$h:=\frac{1}{2}\begin{pmatrix} a-cb^\rho & 0 & ca^\rho+b\\ 0 & 2 & 0\\ a+cb^\rho & 0 & -ca^\rho+b\end{pmatrix}.$$

A straightforward calculation shows that $h\in U_p$ and $L_{1,p}h^{-1}=\mathcal{O}_p x_1$, $L_{2,p}^{(1)}h^{-1}=\mathcal{O}_p x_2$, $L_{2,p}^{(2)}h^{-1}=\mathcal{O}_p x_3$. This completes the proof of (i). The assertions (ii) is proved by a similar argument, so we omit the details.

3.4. Now we collect the above local results to obtain the G-conjugacy classes in $G(f)$ which are *locally integral.* For $g\in G(f)$, we put

$$V_1=V_1^{(g)}:=V\cdot(g+1)\,,\qquad V_2=V_2^{(g)}:=V\cdot(g-1)\,.$$

We abbreviate $(V_i, H|V_i)$ as V_i. Then we have

$$\mathbb{G}(g)=\mathbb{U}(V_1)\times\mathbb{U}(V_2)\,. \tag{3.8}$$

By [32, Proposition 4.7], we see that the G-conjugacy class $[g]$ is determined by $\{(d(V_{2,p}), K/\boldsymbol{Q})_p\}_p$ which can take arbitrary values subject to the following condition:

$$\prod_{p<\infty}(d(V_{2,p}), K/\boldsymbol{Q})_p=+1\,. \tag{3.9}$$

From Lemmas 3.2, 3.5, we get the following:

PROPOSITION 3.6. *Let $f(X)=(X-1)(X+1)^2$. Then a G-conjugacy class $[g]$ in $G(f)$ is locally integral, if and only if $(d(V_2), K/\boldsymbol{Q})_p=+1$ at any p such that $p\neq 2$ and $(K/p)\neq 0$. Hence there exist exactly 2^t or 2^{t-1} such classes in $G(f)//G$, according as $(K/2)=-1$ or $(K/2)\neq -1$.*

In order to know the contribution of each G-conjugacy class to our formula (0.1), we choose a representative g of each class, and calculate the mass $\mathbb{M}(\mathbb{G}(g); V)$ for a fixed V, and the indices $\mathrm{Ind}(\delta_p; g)$. According to the results of [32, § 5] and (3.8), the calculation of $\mathbb{M}(\mathbb{G}(g); V)$ is reduced to that of the local density $\alpha_p(L_{2,p})$, where $L_{2,p}$ corresponds to g as above. When $L_{2,p}$ is isometric to $\mathcal{O}_p^2$, this was given in Lemmas 5.2, 5.3, 5.4, and 5.5 of [32].

LEMMA 3.7. *Suppose $p\neq 2$, and let $L_{2,p}$ be as in Lemma 3.2, (ii) with $(d(L_{2,p}), K/\boldsymbol{Q})_p=\varepsilon$. Then we have $\alpha_p(L_{2,p})=2\left(1-\varepsilon\left(\frac{-1}{p}\right)p^{-1}\right)$.*

LEMMA 3.8. *Suppose $p=2$. Then we have:*

(i) *If $(K/p)=-1$ and $L_{2,p}$ is as in Lemma 3.5, (i), then we have $\alpha_p(L_{2,p})=9/4$.*

(ii) *If $(K/p)=0$ and $L_{2,p}$ is a normal unimodular lattice, then we have $\alpha_p(L_{2,p})=2$.*

(iii) *If $(K/p)=0$ and $L_{2,p}$ is a subnormal unimodular lattice, then we have $\alpha_p(L_{2,p})=3$ if $4\|d(K)$; and if $8\|d(K)$ we have $\alpha_p(L_{2,p})=4$ or 12, according as*

$$L_{2,p}\simeq\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad or\quad\begin{pmatrix}2&1\\1&2\end{pmatrix}.$$

We omit the proofs of these results, since they are proved by the same method as in [32, § 5] (see also Otremba [24]). We remark that they are also proved by the method of § 2.

LEMMA 3.9. *Let $p=2$, and let $g\in U_p(f)$ be such that the corresponding lattice $L_{2,p}$ is normal unimodular. Suppose further that $\sharp([g]\cap U_p//U_p)=2$, and let g, $\delta_p^{-1}g\delta_p$ be representatives of the two U_p-conjugacy classes. Then*:

(i) *If $(K/p)=+1$, then for the second class such that $(d(L_{2,p}), K/\boldsymbol{Q})_p=+1$, we have* $\mathrm{Ind}(\delta_p; g)=3$.

(ii) *If* $(K/p)=0$, *then we always have* $\mathrm{Ind}(\delta_p; g)=4$.

PROOF. We first prove (i). As in Lemma 3.3, we can replace G_p, U_p by $GL_3(\boldsymbol{Q}_p)$, $GL_3(\boldsymbol{Z}_p)$, respectively, and assume that g is the diagonal matrix diag$(1, -1, -1)$. And we may put

$$\delta_p=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & p \end{pmatrix}.$$

Then a direct computation shows that

$$G(g)_p \cap \delta_p U_p \delta_p^{-1}=\left\{\begin{pmatrix} a & 0 & 0 \\ 0 & b & a \\ 0 & d & e \end{pmatrix} \in GL_3(\boldsymbol{Z}_p);\ \begin{matrix} a\equiv e\equiv 1 \\ d\equiv 0 \pmod p \end{matrix}\right\}.$$

It follows that

$$\mathrm{Ind}(\delta_p; g)=\left[GL_2(\boldsymbol{Z}_p):\begin{pmatrix} \boldsymbol{Z}_p & \boldsymbol{Z}_p \\ p\boldsymbol{Z}_p & \boldsymbol{Z}_p \end{pmatrix} \cap GL_2(\boldsymbol{Z}_p)\right]=p+1\,.$$

Next we prove (ii). We may put

$$L_{2,p}=\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \qquad \varepsilon \in \boldsymbol{Z}_p^{\times}\,.$$

Using Lemma 3.5, we see as in the case (i), that

$$\mathrm{Ind}(\delta_p; g)=\left[\mathbb{U}(L_{2,p}):\begin{pmatrix} \mathcal{O}_p & \mathcal{O}_p \\ p\mathcal{O}_p & \mathcal{O}_p \end{pmatrix} \cap \mathbb{U}(L_{2,p})\right],$$

where $\mathbb{U}(L_{2,p})$ denotes the group of isometries of $L_{2,p}$. Denote by U_0 the subgroup of $\mathbb{U}(L_{2,p})$ given by the right hand side above. It is easy to show that $\sharp(U_0 \bmod p)\leq 2$, and the equality holds if and only if $\mathcal{O}_p^{(1)}$, the norm one group of $\mathcal{O}_p^{\times}$, contains an element η such that $\eta\equiv 1+\pi \pmod p$, which is seen to be exactly the case $8\| d(K)$. Now put

$$U_1:=\mathbb{U}(L_{2,p})\cap(1+\pi M_2(\mathcal{O}_p))\,, \qquad U_2:=\mathbb{U}(L_{2,p})\cap(1+pM_2(\mathcal{O}_p))\,.$$

Then we have $[\mathbb{U}(L_{2,p}): U_1]=2$. Also it is not difficult to show, by direct computation, that $[U_1: U_2]=4$ or 2 according as $4\| d(K)$ or $8\| d(K)$. q.e.d.

3.5. Now we can calculate the contribution T_2 of the locally integral G-conjugacy classes in $G(f_2)$ to the formula (0.1). Let $\{p_1, \cdots, p_t\}$ be the set of distinct prime divisors of $d(K)$. For a t-ple $\boldsymbol{\varepsilon}=(\varepsilon_1, \cdots, \varepsilon_t)$ such that $\varepsilon_i=\pm 1$, we denote by $G(f_2; \boldsymbol{\varepsilon})$ the locally integral G-conjugacy class such that $\varepsilon_i=d(d(V_2), K/\boldsymbol{Q})_{p_i}$ for $i=1, 2, \cdots, t$.

PROPOSITION 3.10. *Suppose* $(K/2)=+1$. *Then we have*

$$T_2=\frac{(B_{1,\chi})^2}{2^{t+2}3}\cdot(|d(K)|-1).$$

PROOF. By Proposition 3.6, we have $\prod_{i=1}^{t}\varepsilon_i=+1$. Fix a representative g in $G(f_2;\varepsilon)$. We can assume that at each place p, the lattice $L_{2,p}$ corresponding to g ($\in U_p$) is normal unimodular, and $(d(L_{2,p}),K/\boldsymbol{Q})_p=+1$ at $p=2$. Then by Lemma 3.7 and [32, Theorem 5.6], we have

$$\mathbb{M}(\mathbb{G}(g);V)=\mathbb{M}_1(L_1)\times\mathbb{M}_2(L_2)=(B_{1,\chi}/2^t)\times(B_{1,\chi}B_2/2^{t+2})\times\prod_{i=1}^{t}\left(p_i+\varepsilon_i\left(\frac{-1}{p_i}\right)\right).$$

On the other hand, from Lemma 3.9, we have

$$\prod_p\sum_{\delta_p}\mathrm{Ind}(\delta_p;g)=1+3=4.$$

Applying the following lemma, we get the assertion. q.e.d.

LEMMA 3.11. *Let X_i, Y_i ($1\le i\le t$) be indeterminates. Then we have an equality*

$$\sum_{\varepsilon_1,\cdots,\varepsilon_t}\prod_{i=1}^{t}(X_i+\varepsilon_iY_i)=2^{t-1}\left(\prod_{i=1}^{t}X_i+\varepsilon\prod_{i=1}^{t}Y_i\right),$$

where the sum is extended over the t-ples $(\varepsilon_1,\cdots,\varepsilon_t)$ *such that*

$$\varepsilon_i=\pm1,\qquad\prod_i\varepsilon_i=\varepsilon.$$

Proof is omitted.

Now suppose that $(K/2)=-1$. In this case (ε_i) can take arbitrary values. Suppose first that $\prod_i\varepsilon_i=+1$. Then for any $g\in G(f;(\varepsilon_i))$, we have $(d(V_2),K/\boldsymbol{Q})_p=+1$ at $p=2$, hence the U_p-conjugacy class of g is of type I at $p=2$. Thus in this case we have, in the same way as above

(3.10) the sum of contributions of $G(f;(\varepsilon_i))$ for all (ε_i) such that $\prod_i\varepsilon_i=+1$ is equal to

$$(B_{1,\chi})^2(|d(K)|-1)/3\cdot2^{t+4}.$$

Next suppose $\prod_i\varepsilon_i=-1$. Then for any $g\in G(f;(\varepsilon_i))$, we have $(d(V_2),K/\boldsymbol{Q})_p=-1$ at $p=2$, hence it is of type II. By using Lemma 3.8, we have

(3.11) the sum of contributions of $G(f;(\varepsilon_i))$ for all (ε_i) such that $\prod_i\varepsilon_i=-1$ is equal to

$$(B_{1,\chi})^2(|d(K)|+1)/2^{t+4}.$$

From (3.10) and (3.11), we get:

PROPOSITION 3.12. *Suppose* $(K/2)=-1$. *Then we have*:

$$T_2=\frac{(B_{1,\chi})^2}{2^{t+3}3}\cdot(2|d(K)|+1).$$

Finally we suppose that $(K/2)=0$, and $p_1=2$. In this case we always have $\prod_i \varepsilon_i=+1$. Fix $g\in G(f;(\varepsilon_i))$. As in the case $(K/2)=+1$, we may assume that the lattice $L_{2,p}$ is normal unimodular at each p. Then from Lemma 3.7, we can show

$$\begin{aligned}&\mathbb{M}(L_1)=|B_{1,\chi}|/2^t\\ (3.12)\qquad &\mathbb{M}(L_2)=(|B_{1,\chi}|B_2/2^{t+2})\prod_{i=2}^{t}\left(p_i+\varepsilon_i\left(\frac{-1}{p_i}\right)\right)\times\begin{cases}3 & \cdots\quad 4\|d(K)\\ 6 & \cdots\quad 8\|d(K).\end{cases}\end{aligned}$$

On the other hand, from Lemma 3.9, we have

$$(3.13)\qquad \prod_p\sum_{\delta_p}\mathrm{Ind}(\delta_p;g)=\begin{cases}5 & \cdots\quad 4\|d(K),\ \varepsilon_1=+1\\ 17/3 & \cdots\quad 4\|d(K),\ \varepsilon_1=-1\\ 11/2 & \cdots\quad 8\|d(K),\ \varepsilon_1=(-1,K/\boldsymbol{Q})_2\\ 31/6 & \cdots\quad 8\|d(K),\ \varepsilon_1\neq(-1,K/\boldsymbol{Q})_2.\end{cases}$$

From (3.12), (3.13) and Lemma 3.11, we now get:

PROPOSITION 3.13. *Suppose* $(K/2)=0$, $p_1=2$. *Then we have*

$$T_2=\frac{(B_{1,\chi})^2}{2^{t+4}3}\cdot(4|d(K)|-1).$$

4. $n=3$: **Contributions from** $f_3(X)$, $f_{41}(X)$, **and** $f_{42}(X)$. In this section we study the contributions to the formula (0.1) of the conjugacy classes which belong to $f_3(X)=(X-1)(X^2+1)$, $f_{41}(X)=(X-1)(X^2+X+1)$, and $f_{42}(X)=(X-1)(X^2-X+1)$. We sometimes denote these polynomials simply by $f(X)$. Throughout this section we assume that our base field K is *not* equal to $\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\sqrt{-3})$.

4.1. We define the algebraic number θ by

$$\theta=\begin{cases}\sqrt{-1} & \cdots\quad \text{if}\ \ f(X)=f_3(X)\\ \omega:=(-1+\sqrt{-3})/2 & \cdots\quad \text{if}\ \ f(X)=f_{41}(X)\quad\text{or}\quad f_{42}(X).\end{cases}$$

Put $\boldsymbol{M}=K(\theta)$. Since we have assumed that $d(K)\neq-3,\ -4$, $\boldsymbol{M}$ is a biquadratic field containing K. Let $\boldsymbol{N}$ be the real quadratic subfield of $\boldsymbol{M}$. We denote by R, S the rings of integers of $\boldsymbol{M}$, $\boldsymbol{N}$, respectively, and by σ the notrivial automorphism of $\boldsymbol{M}/\boldsymbol{N}$. Also we denote by $d(\boldsymbol{M}/\boldsymbol{N})$, $d(\boldsymbol{N})$ the relative discriminant of $\boldsymbol{M}/\boldsymbol{N}$, and the discriminant of $\boldsymbol{N}$, respectively.

We first note that some of the fundamental arithmetic properties of biquadratic fields (containing an imaginary quadratic field), which we need in our study, are described, for example, in Hasse [14] or Fujisaki [8].

LEMMA 4.1. *Let the notation be as above.*

(i) *If* $f(X)=f_3(X)$ *we have*

$$N_{N/\boldsymbol{Q}}(d(\boldsymbol{M}/\boldsymbol{N}))=d(K)d(\boldsymbol{Q}(\sqrt{-1}))/d(\boldsymbol{N})=-4d(K)/d(\boldsymbol{N}),$$

and if $f(X)=f_{41}(X)$ *or* $f_{42}(X)$, *we have*

$$N_{N/\boldsymbol{Q}}(d(\boldsymbol{M}/\boldsymbol{N}))=d(K)d(\boldsymbol{Q}(\omega))/d(\boldsymbol{N})=-3d(K)/d(\boldsymbol{N}).$$

(ii) *Suppose that* $f(X)=f_3(X)$. *Then* $\boldsymbol{M}/\boldsymbol{N}$ *is unramified outside the places lying above* 2 *and the infinite places. More precisely*:

(a) *If* $(K/2)\neq 0$, $\boldsymbol{M}/\boldsymbol{N}$ *is unramified outside the infinite places.*

(b) *If* $d(K)\equiv 12 \pmod{32}$, *then* 2 *remains prime in* $\boldsymbol{N}$, *and it ramifies at* $\boldsymbol{M}/\boldsymbol{N}$.

(c) *If* $d(K)\equiv -4 \pmod{32}$, *then* 2 *decomposes in* $\boldsymbol{N}$: $2=P_0P_0^\sigma$, *and* P_0, P_0^σ *ramify at* $\boldsymbol{M}/\boldsymbol{N}$.

(d) *If* $8\|d(K)$, *then* 2 *ramifies in* $\boldsymbol{N}$: $2=P_0^2$, *and* P_0 *ramifies at* $\boldsymbol{M}/\boldsymbol{N}$.

(iii) *Suppose that* $f(X)=f_{41}(X)$ *or* $f_4(X)$. *Then* $\boldsymbol{M}/\boldsymbol{N}$ *is unramified outside the places lying above* 3 *and the infinite places. More precisely*:

(a) *If* $(K/3)\neq 0$, $\boldsymbol{M}/\boldsymbol{N}$ *is unramified outside the infinite places.*

(b) *If* $d(K)\equiv 3 \pmod 9$, *then* 3 *remains prime in* $\boldsymbol{N}$, *and it ramifies at* $\boldsymbol{M}/\boldsymbol{N}$.

(c) *If* $d(K)\equiv -3 \pmod 9$, *then* 3 *decomposes in* $\boldsymbol{N}$: $3=P_0P_0^\sigma$, *and* P_0, P_0^σ *ramify at* $\boldsymbol{M}/\boldsymbol{N}$.

PROOF. The assertion (i) is shown in Fujisaki [8]. Assertions (ii) and (iii) are proved easily by (i). q.e.d.

Now let g be an element of $G(f)$. We define the subspaces V_1, V_2 of $V=K^3$ by

$$V_1:=\begin{cases} V\cdot(g^2+1) & \cdots \quad f(X)=f_3(X) \\ V\cdot(g^2+g+1) & \cdots \quad f(X)=f_{41}(X) \\ V\cdot(g^2-g+1) & \cdots \quad f(X)=f_{42}(X) \end{cases}$$

$$V_2:=V\cdot(g-1).$$

We often abbreviate the Hermitian space $(V_i, H|V_i)$ as V_i. By the Hasse principle for conjugacy classes in $\mathbb{G}$ (cf. [32, Proposition 4.8]), we see that the G-conjugacy class $[g]_{\boldsymbol{Q}}$ is determined by the system $\{[g]_p\}_{p<\infty}$ of local conjugacy classes. Here each $[g]_p$ is parametrized as follows:

(4.1) *The case* $f(X)=f_3(X)$.

(4.1.A) Suppose $(K/p)\neq +1$, and $\sqrt{-1}\notin K_p$. This is easily seen to be equivalent to either $p\equiv -1 \pmod 4$, $(K/p)=0$, or $p=2$, $K_p\neq \boldsymbol{Q}_p(\sqrt{-1})$. Then $G_p(f)//G_p$ contains exactly two conjugacy classes. Namely, it is determined by the invariant $(d(V_{2,p}), K/\boldsymbol{Q})_p$, which can take both values ± 1.

(4.1.B) Suppose $(K/p)\neq +1$, and $K_p=\boldsymbol{Q}_p(\sqrt{-1})$, which amounts to either $p\equiv -1$ (mod 4), $(K/p)=-1$, or $p=2$, $K_p=\boldsymbol{Q}_p(\sqrt{-1})$. Then $G_p(f)//G_p$ contains exactly four conjugacy classes described as follows: Put

$$V_{2,p,1}:=V_{2,p}\cdot(g+\sqrt{-1}), \qquad V_{2,p,2}:=V_{2,p}\cdot(g-\sqrt{-1}).$$

Then $[g]_p$ is determined by the pair of invariants

$$((d(V_{2,p,1}), K/\boldsymbol{Q})_p, \quad (d(V_{2,p,2}), K/\boldsymbol{Q})_p)$$

which can take any values among $(\pm 1, \pm 1)$.

(4.1.C) Suppose $(K/p)\neq +1$ and $\sqrt{-1}\in \boldsymbol{Q}_p$; namely, $p\equiv 1$ (mod 4) and $(K/p)\neq +1$. Then $G_p(f)//G_p$ contains a unique conjugacy class, for which we have $(d(V_{2,p}), K/\boldsymbol{Q})_p=(-1, K/\boldsymbol{Q})_p=+1$.

(4.1.D) Suppose $(K/p)=+1$. Then $G_p(f)//G_p$ contains a unique conjugacy class, for which we have $(d(V_{2,p}), K/\boldsymbol{Q})_p=+1$.

(4.2) *The case* $f(X)=f_{41}(x)$ *or* $f_{42}(X)$.

(4.2.A) Suppose $(K/p)\neq +1$, and $\omega\notin K_p$. This is easily seen to be equivalent to either $p\equiv -1$ (mod 3), $(K/p)=0$, or $p=3$, $K_p\neq \boldsymbol{Q}_p(\omega)$. Then $G_p(f)//G_p$ contains exactly two conjugacy classes. Namely, it is determined by the invariant $(d(V_{2,p}), K/\boldsymbol{Q})_p$, which can take both values ± 1.

(4.2.B) Suppose $(K/p)\neq +1$, and $K_p=\boldsymbol{Q}_p(\omega)$, which amounts to either $p\equiv -1$ (mod 3), $(K/p)=-1$, or $p=3$, $K_p=\boldsymbol{Q}_p(\omega)$. Then $G_p(f)//G_p$ contains exactly four conjugacy classes described as follows: Put

$$V_{2,p,1}=\begin{cases} V_{2,p}\cdot(g-\omega^2) \\ V_{2,p}\cdot(g+\omega^2)\end{cases}; \qquad V_{2,p,2}=\begin{cases} V_{2,p}\cdot(g-\omega) & \cdots\ f(X)=f_{41}(X) \\ V_{2,p}\cdot(g+\omega) & \cdots\ f(X)=f_{42}(X).\end{cases}$$

Then $[g]_p$ is determined by the pair of invariants

$$((d(V_{2,p,1}), K/\boldsymbol{Q})_p, \quad (d(V_{2,p,2}), K/\boldsymbol{Q})_p)$$

which can take any values among $(\pm 1, \pm 1)$.

(4.2.C) Suppose $(K/p)\neq +1$ and $\omega\in \boldsymbol{Q}_p$; namely, $p\equiv 1$ (mod 3) and $(K/p)\neq +1$. Then $G_p(f)//G_p$ contains a unique conjugacy class, for which we have $(d(V_{2,p}), K/\boldsymbol{Q})_p=(-1, K/\boldsymbol{Q})_p=(3, K/\boldsymbol{Q})_p$.

(4.2.D) Suppose $(K/p)=+1$. Then $G_p(f)//G_p$ contains a unique conjugacy class, for which we have $(d(V_{2,p}), K/\boldsymbol{Q})_p=+1$.

These assertions are obtained directly from [32, Proposition 4.6]. Also, from [32, Proposition 4.7], we see that the range of the system $\{[g]_p\}_{p<\infty}$ is determined by the following conditions:

$$\begin{array}{lll} & (d(V_{2,p}), K/\boldsymbol{Q})_p=+1 & \text{for almost all } p, \\ (4.3) & (d(V_{2,p,j}), K/\boldsymbol{Q})_p=+1 & \text{for almost all } p\ (j=1, 2) \text{ in the cases (4.1.B), (4.2.B),} \end{array}$$

and

$$\prod_{p<\infty}(d(V_{2,p}), K/\boldsymbol{Q})_p = +1 .$$

4.2. Now we consider the parametrization of the U_p-conjugacy classes. Let L_p be the standard lattice $\mathcal{O}_p^3$, and put

$$L_{1,p} := V_{1,p} \cap L_p , \qquad L_{2,p} := V_{2,p} \cap L_p .$$

Therefore, $L_{2,p}$ (together with the restriction of H) is a ρ-Hermitian $\mathcal{O}_p$-lattice of rank two. Moreover, it is an $\mathcal{O}_p[\theta]$-module contained in the $\boldsymbol{M}_p$-module $V_{2,p}$, where the $\boldsymbol{M}_p$-module structure is defined via $x\cdot\theta = x\cdot g$ $(x\in V_{2,p})$. Then there exists a unique $\mathcal{O}_p$-order R'_p of $\boldsymbol{M}_p$ such that $R'_p \supseteqq \mathcal{O}_p[\theta]$, and $L_{2,p}$ becomes a *proper* R'_p-module. Recall that $L_{2,p}$ is called a proper R'_p-module, if one has

$$R'_p = \{a\in \boldsymbol{M}_p;\ L_{2,p}\cdot a \subseteqq L_{2,p}\} .$$

It is easy to see that if we replace g by $u^{-1}gu$ $(u\in U_p)$, then we get $L_{2,p}\cdot u$ instead of $L_{2,p}$, and an isomorphic proper R'_p-module structure on it. Thus, in order to know the U_p-conjugacy classes in $U_p(f)$ $(:= G_p(f)\cap U_p)$, we may assume that $L_{2,p}(\subseteqq V_p)$ is a fixed representative in a U_p-orbit. Then the U_p-conjugacy class of g is completely determined by the U_p-equivalence class of the proper R'_p-module structure on $L_{2,p}$.

However, in many cases it turns out to be enough to know the $\mathbb{U}(L_{2,p})$-equivalence class of it. The latter is described as follows. Suppose that a proper R'_p-module structure is given on the (fixed) $L_{2,p}$. We always assume that the action of θ belongs to $\mathbb{U}(L_{2,p})$.

LEMMA 4.2. *Let R'_p be an $\mathcal{O}_p$-order in $\boldsymbol{M}_p$. Then any proper R'_p-ideal in $\boldsymbol{M}_p$ is a free R'_p-module of rank one.*

When $\boldsymbol{M}_p$ is a field, this result is well-known (cf. Ihara [17]). When $\boldsymbol{M}_p$ is not a field, we can prove it by componentwise argument. The detail will be omitted. By this lemma, we may write $L_{2,p} = x\cdot R'_p$, where $x\in L_{2,p}$ is uniquely determined modulo $(R'_p)^\times$. Then we see (cf. [32, §4]) that there exists a unique $z\in \boldsymbol{N}_p^\times$ such that the equality

$$H(x\cdot a, x\cdot b) = \mathrm{Tr}_{\boldsymbol{M}_p/K_p}(zab^\sigma)$$

holds for any $a, b\in R'_p$. The class of z modulo $N_{\boldsymbol{M}_p/K_p}((R'_p)^\times)$ is uniquely determined by the proper R'_p-module structure on $L_{2,p}$.

LEMMA 4.3. *The above correspondence induces the following canonical bijection*:

$$\left\{\begin{array}{l}\mathbb{U}(L_{2,p})\text{-}\textit{equivalence classes of}\\ \textit{proper } R'_p\textit{-module structures}\\ \textit{on } L_{2,p}\end{array}\right\} \simeq \{R'_p/L_{2,p}\}/N_{\boldsymbol{M}_p/\boldsymbol{N}_p}((R'_p)^\times) .$$

Here $\{R'_p/L_{2,p}\}$ denotes the set of all $z\in \boldsymbol{N}_p^\times$ such that the $\mathcal{O}_p$-lattice R'_p, equipped with the ρ-Hermitian form $(a, b)\longmapsto \mathrm{Tr}_{\boldsymbol{M}_p/K_p}(zab^\sigma)$), is isometric to $(L_{2,p}, H|L_{2,p})$.

Note that the set $\{R'_p/L_{2,p}\}$ is stable under the multiplication by any element of $N_{\boldsymbol{M}_p/N_p}((R'_p)^\times)$. This lemma is verified easily and we omit the proof. The next lemma is an integral version of [32, Lemma 4.3].

LEMMA 4.4. *Let R'_p be an $\mathcal{O}_p$-order of $\boldsymbol{M}_p$ containing $\mathcal{O}_p[\theta]$. For each $\mathcal{O}_p$-basis $\{a_1, a_2\}$ of R'_p, put*

$$\Delta_{R'_p}[a_1, a_2] := \det(\mathrm{Tr}_{\boldsymbol{M}_p/K_p}(a_j a_k^\sigma)_{j,k}) .$$

Then $\Delta_{R'_p}[a_1, a_2]$ belongs to $\boldsymbol{Q}_p^\times \cap N_{\boldsymbol{M}_p/N_p}(\boldsymbol{M}_p^\times)$; and its class modulo $N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)$ does not depend on the choice of a_1, a_2. Moreover, we have the following relation:

$$d(R'_p, \mathrm{Tr}_{\boldsymbol{M}_p/K_p}(zab^\sigma)) \equiv \Delta_{R'_p}[a_1, a_2] \cdot N_{N_p/\boldsymbol{Q}_p}(z) \ (\mathrm{mod}\ N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)) .$$

We shall abbreviate $\Delta_{R'_p}[a_1, a_2]$ as $\Delta_{R'_p}$, if there is no fear of confusion.

LEMMA 4.5. *Suppose that either* (i) $p \neq 2$, (ii) $p=2$, $(K/p) \neq 0$, *or* (iii) $p=2$, $4 \| d(K)$, $(d(L_{2,p}), K/\boldsymbol{Q})_p = +1$. *Moreover, suppose that $L_{2,p}$ is unimodular. Then the set $\{R'_p/L_{2,p}\}$ coincides with the set of all $z \in N_p^\times$ satisfying*

$$(4.4) \qquad \begin{aligned} &\mathrm{Tr}_{\boldsymbol{M}_p/K_p}(zR'_p) \subseteq \mathcal{O}_p , \\ &N_{N_p/\boldsymbol{Q}_p}(z) \equiv (\Delta_{R'_p})^{-1} d(L_{2,p}) \bmod N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times) . \end{aligned}$$

PROOF. In the above three cases, the isometry class of a unimodular plane is determined by its discriminant (see [32, § 6]). The assertion follows from this and Lemma 4.4. q.e.d.

From the above results we see that the $\mathcal{O}_p$-order R'_p which is attached to an element of $U_p(f)$ plays an essential role in the description of $U_p(f) \,//\, U_p$. Thus we need the parametrization of all such orders. Calculating the discriminant of $\mathcal{O}_p[\theta]$, one can easily prove the following two lemmas.

LEMMA 4.6. *Suppose $f(X)=f_3(X)$ and let p be a finite place of $\boldsymbol{Q}$.*

(i) *If either $p \neq 2$, or $p=2$, $(K/p) \neq 0$, we have $\mathcal{O}_p[\theta] = R_p$ ($=$the maximal order of $\boldsymbol{M}_p$). Moreover, we have*

$$\Delta_{R_p} \equiv 4 \ (\mathrm{mod}\ N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)) .$$

(ii) *Suppose that $p=2$, $4 \| d(K)$, and let π be a prime element of K_p. Then the $\mathcal{O}_p$-orders of $\boldsymbol{M}_p$ containing $\mathcal{O}_p[\theta]$ are ordered as*

$$R_p = R_{0,p} \supsetneqq R_{1,p} \supsetneqq R_{2,p} = \mathcal{O}_p[\theta] ,$$

where $R_{j,p} := \mathcal{O}_p + \pi^j R_p$. Moreover, we have

$$\Delta_{R_{0,p}} \equiv 1 , \qquad \Delta_{R_{1,p}} \equiv 4/N_{K_p/\boldsymbol{Q}_p}(\pi) , \qquad \Delta_{R_{2,p}} \equiv 4 \ (\mathrm{mod}\ N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)) .$$

(iii) *Suppose that* $p=2$, $8\|d(K)$, *and let* π *be as above. Then the* $\mathcal{O}_p$*-orders of* $\boldsymbol{M}_p$ *containing* $\mathcal{O}_p[\theta]$ *are ordered as*

$$R_p = R_{0,p} \supsetneqq R_{1,p} = \mathcal{O}_p[\theta]\,,$$

with $R_{j,p} := \mathcal{O}_p + \pi^j R_p$. *Moreover, we have*

$$\Delta_{R_{0,p}} \equiv 4/N_{K_p/\boldsymbol{Q}_p}(\pi)\,, \qquad \Delta_{R_{1,p}} \equiv 4 \pmod{N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)}\,.$$

LEMMA 4.7. *Suppose that* $f(X)=f_{41}(X)$ *or* $f_{42}(X)$, *and let* p *be a finite place of* $\boldsymbol{Q}$.

(i) *If either* $p\neq 3$, *or* $p=3$, $(K/p)\neq 0$, *we have* $\mathcal{O}_p[\theta]=R_p$ (*= the maximal order of* $\boldsymbol{M}_p$). *Moreover, we have*

$$\Delta_{R_p} \equiv 3 \pmod{N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)}\,.$$

(ii) *Suppose that* $p=3$, $(K/p)=0$, *and let* π *be a prime element of* K_p. *Then the* $\mathcal{O}_p$*-orders of* $\boldsymbol{M}_p$ *containing* $\mathcal{O}_p[\theta]$ *are ordered as*

$$R_p = R_{0,p} \supsetneqq R_{1,p} = \mathcal{O}_p[\theta]\,,$$

with $R_{j,p} := \mathcal{O}_p + \pi^j R_p$. *Moreover, we have*

$$\Delta_{R_{0,p}} \equiv 3/N_{K_p/\boldsymbol{Q}_p}(\pi)\,, \qquad \Delta_{R_{1,p}} \equiv 3 \pmod{N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)}\,.$$

4.3. We shall now determine the U_p-conjugacy classes in $U_p(f)$. Firstly we see that there are cases (Lemma 4.8, 4.9), which are settled quite easily.

LEMMA 4.8. *Suppose* $f(X)=f_3(X)$ *and* $p\neq 2$.

(i) *Assume that* $[g]_p \cap U_p \neq \varnothing$, *and define* $L_{1,p}$, $L_{2,p}$ *as above. Then* L_p *splits as* $L_p = L_{1,p} \oplus L_{2,p}$, *hence each* $L_{i,p}$ *is a unimodular* $\mathcal{O}_p$*-lattice. Moreover, the unimodular plane* $L_{2,p}$ *becomes a proper* R_p*-module in the above manner.*

(ii) *The condition* $[g]_p \cap U_p \neq \varnothing$ *is satisfied by a unique* G_p*-conjugacy class in* $G_p(f)$. *In the case* (4.1.A), *it is characterized by* $(d(V_{2,p}), K/\boldsymbol{Q})_p = +1$. *In the case* (4.1.B), *it is characterized by* $(d(V_{2,p,j}), K/\boldsymbol{Q})_p = +1$ $(j=1,2)$.

(iii) *For the class* $[g]_p$ *as in* (ii), *the set* $[g]_p \cap U_p$ *consists of a single* U_p*-conjugacy class.*

PROOF. First we prove (i). Suppose $[g]_p \cap U_p \neq \varnothing$. We may assume that $g \in U_p$. Note that g^2 belongs to $U_p(f_2)$. By applying Lemma 3.1 to $(L_{1,p}, L_{2,p})$, we have $L_p = L_{1,p} \oplus L_{2,p}$. Also, Lemma 4.6, (i) shows that $\mathcal{O}_p[\theta]=R_p$ so that $L_{2,p}$ is a proper R_p-module. Next we prove (ii). By Lemma 3.2, the U_p-equivalence class of $L_{2,p}$ is determined by $(d(L_{2,p}), K/\boldsymbol{Q})_p$. Since $L_{2,p}$ admits a proper R_p-module structure, we see from Lemma 4.3, 4.5 together with Lemma 4.6, (i), that there exists $z \in (1/2)S_p^\times$ such that $N_{\boldsymbol{N}_p/\boldsymbol{Q}_p}(2z) \equiv d(L_{2,p}) \bmod N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)$. Now suppose that the situation is as in (4.1.A). Then $\boldsymbol{M}_p/\boldsymbol{N}_p$ is unramified by Lemma 4.1, (ii), hence $2z \in N_{\boldsymbol{M}_p/\boldsymbol{N}_p}(\boldsymbol{M}_p^\times)$. Therefore we have $(d(L_{2,p}), K/\boldsymbol{Q})_p = +1$. Next suppose we are in (4.1.B). Then we can write $\boldsymbol{M}_p =$

$K_p \oplus K_p$, $N_p = \boldsymbol{Q}_p \oplus \boldsymbol{Q}_p$. Here $K_p/\boldsymbol{Q}_p$ is unramified, since $p \neq 2$. So again we have $2z \in N_{M_p/N_p}(M_p^{\times})$. Writing $2z = (z_1, z_2)$, we have $(z_j, K/\boldsymbol{Q})_p = (d(V_{2,p,j}), K/\boldsymbol{Q})_p = +1$ $(j = 1, 2)$. Finally we prove (iii). It suffices, by Lemmas 4.3, 4.5, to show that if $N_{N_p/\boldsymbol{Q}_p}(2z) \in N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^{\times})$ $(2z \in S_p^{\times})$, then $2z$ should be an element of $N_{M_p/N_p}(R_p^{\times})$. Since R_p is the maximal order, this in fact follows from the translation theorem in local class field theory. q.e.d.

By a similar argument, we have the following:

LEMMA 4.9. *Suppose* $f(X) = f_{41}(X)$ *or* $f_{42}(X)$, *and* $p \neq 3$.

(i) *Assume that* $[g]_p \cap U_p \neq \varnothing$, *and define* $L_{1,p}$, $L_{2,p}$ *as above. Then* L_p *splits as* $L_p = L_{1,p} \oplus L_{2,p}$, *hence each* $L_{i,p}$ *is a unimodular* $\mathcal{O}_p$*-lattice. Moreover, the unimodular plane* $L_{2,p}$ *becomes a proper* R_p*-module.*

(ii) *The condition* $[g]_p \cap U_p \neq \varnothing$ *is satisfied by a unique* G_p*-conjugacy class in* $G_p(f)$. *In the case* (4.2.A), *it is characterized by* $(d(V_{2,p}), K/\boldsymbol{Q})_p = (3, K/\boldsymbol{Q})_p$. *In the case* (4.2.B), *it is characterized by* $(d(V_{2,p,j}), K/\boldsymbol{Q})_p = +1$ $(j = 1, 2)$.

(iii) *For the class* $[g]_p$ *as in* (ii), *the set* $[g]_p \cap U_p$ *consists of a single* U_p*-conjugacy class.*

Now we treat the remaining complicated cases. To avoid unnecessary work, we first make the following observation, which follows from the condition (4.3), together with the results in Lemmas 4.8, 4.9.

LEMMA 4.10. *Let* $[g]_{\boldsymbol{Q}}$ *be a locally integral G-conjugacy classes in* $G(f)$. *Then we have*:

(i) *If* $f(X) = f_3(X)$, *then the corresponding subspace* V_2 *satisfies* $(d(V_{2,p}), K/\boldsymbol{Q})_p = +1$ *at* $p = 2$.

(ii) *If* $f(X) = f_{41}(X)$ *or* $f_{42}(X)$, *then* V_2 *satisfies* $(d(V_{2,p}), K/\boldsymbol{Q})_p = (3, K/\boldsymbol{Q})_p$ *at* $p = 3$.

From now on, we treat only such G_p-conjugacy classes, at $p = 2$ or 3, that appear in the above lemma. It is easily seen from (4.1) and (4.2), that such a G_p-conjugacy class is *unique*, except in the cases:

(A) $f(X) = f_3(X)$, $p = 2$, and $d(K) \equiv -4 \pmod{32}$,

(B) $f(X) = f_{41}(X), f_{42}(X)$, $p = 3$, and $d(K) \equiv -3 \pmod 9$.

Before we study these cases in detail, we fix notation. We denote by π a prime element of K_p, and by $K^{(1)}$, $M^{(1)}$ the unitary group of one-dimensional Hermitian spaces over $K/\boldsymbol{Q}$, M/N:

$$K^{(1)} := \mathrm{Ker}(N_{K/\boldsymbol{Q}}), \qquad M^{(1)} := \mathrm{Ker}(N_{M/N}).$$

For any $g \in U_p(f)$, we attach $(L_{1,p}, L_{2,p})$ as before, and define the types of them as follows (cf. § 3):

Type I : $L_p = L_{1,p} \oplus L_{2,p}$

Type II: $L_p \supsetneqq L_{1,p} \oplus L_{2,p} \supsetneqq pL_p$, with

$$(d(L_{2,p}), K/\boldsymbol{Q})_p=\begin{cases} +1 \cdots f(X)=f_3(X)\,, & p=2 \\ (3, K/\boldsymbol{Q})_p \cdots f(X)=f_{41}(X)\,, \quad f_{42}(X)\,, & p=3\,. \end{cases}$$

If $(K/p)=+1$, we substitute $(L^0_{1,p}, L^0_{2,p})$ for $(L_{1,p}, L_{2,p})$ as in §3; namely, writing $V_p=V^0_p\oplus V^0_p$, $V^0_p=\boldsymbol{Q}^3_p$, $L^0_p=\boldsymbol{Z}^3_p$, we have $L^0_{i,p}:=L_{i,p}\cap(V^0_p\oplus\{0\})$. We regard $L^0_{i,p}$ as a $\boldsymbol{Z}_p$-module contained in V^0_p. Then the type of $(L^0_{1,p}, L^0_{2,p})$ is defined in the same way as above.

LEMMA 4.11. *For any $g\in U_p(f)$ which belongs to a G_p-conjugacy class satisfying the conditions of Lemma* 4.10, *the corresponding pair $(L_{1,p}, L_{2,p})$ (or $(L^0_{1,p}, L^0_{2,p})$ if $(K/p)=+1$) belongs to one of the two types defined above.*

PROOF. Consider the action of g^2 (resp. g^2+g) when $f=f_3$ (resp. f_{41} or f_{42}). Then we get easily the assertion, as in the proof of Lemma 4.8, (i). q.e.d.

Let us assume that $f(X)=f_3(X)$ and $p=2$. There are five cases (1)–(5) to be considered separately.

Case (1): $(K/p)=+1$ ($f=f_3$; $p=2$). In this case X^2+1 is irreducible over $\boldsymbol{Q}_p$ and $G_p(f)$ consists of a single conjugacy class. It satisfies $(d(V_2), K/\boldsymbol{Q})_p=+1$.

LEMMA 4.12. *Suppose $f(X)=f_3(X)$, $p=2$, and $(K/p)=+1$. Then $L_{2,p}$ becomes a proper R_p-module and there is a unique U_p-conjugacy class of each type. Let $[g]_{U_p}$ (resp. $[\delta^{-1}g\delta]_{U_p}$, $\delta\in G_p$) be the U_p-conjugacy class such that the corresponding $(L^0_{1,p}, L^0_{2,p})$ is of type* I (*resp. type* II). *Then we have*

$$[G(g)_p\cap U_p: G(g)_p\cap\delta U_p\delta^{-1}]=1\,.$$

PROOF. Let $(L^0_{1,p}, L^0_{2,p})$ be a fixed representative (cf. Lemmas 3.2, 3.3). By Lemma 4.6, (i), we see that $\boldsymbol{Z}_p[\theta]=\boldsymbol{Z}_p[\sqrt{-1}]$ is the maximal order of $\boldsymbol{Q}_p(\sqrt{-1})$, hence $L_{2,p}$ is a proper R_p-module. Therefore our U_p-conjugacy class of type I is unique. Suppose that $(L^0_{1,p}, L^0_{2,p})$ is of type II. We may assume it to be as in (3.4). It is then easy to see that L^0_p is a free module with basis (0, 0, 1) over the ring $\boldsymbol{Z}_p[(1, \sqrt{-1})]$. It follows from Lemma 4.3 that our U_p-conjugacy class of type II is also unique. Now these arguments show that, if we identify $G(g)_p$ with $\boldsymbol{Q}^\times_p\times\boldsymbol{Q}_p(\sqrt{-1})^\times$ we have

$$G(g)_p\cap U_p=\boldsymbol{Z}^\times_p\times\boldsymbol{Z}_p[\sqrt{-1}]^\times\,,$$

$$G(g)_p\cap\delta U_p\delta^{-1}=\boldsymbol{Z}_p[(1, \sqrt{-1})]^\times\,.$$

Since the right hand sides are the same, we get the last assertion. q.e.d.

Case (2): $(K/p)=-1$ ($f=f_3$; $p=2$). In this case X^2+1 is irreducible over K_p and $G_p(f)$ consists of a unique G_p-conjugacy class such that $(d(V_p), K/\boldsymbol{Q})_p=+1$. Suppose that $g\in U_p(f)$. Lemma 3.5, (i) shows that for any $g\in U_p(f)$, the corresponding

$(L_{1,p}, L_{2,p})$ is of type I, so that $L_p = L_{1,p} \oplus L_{2,p}$. Let $L_{2,p}$ be fixed. Then an argument similar to that in the proof of Lemma 4.8, (iii) gives the following result.

LEMMA 4.13. *Suppose* $f(X) = f_3(X)$, $p=2$, $(K/p) = -1$, *and* $(d(V_{2,p}), K/\boldsymbol{Q})_p = +1$. *Then* $(L_{1,p}, L_{2,p})$ *is of type* I, *and* $L_{2,p}$ *becomes a proper* R_p*-module. Moreover,* $U_p(f)$ *contains a unique* U_p*-conjugacy class of this type.*

Case (3): $d(K) \equiv 12 \pmod{32}$ $(f = f_3; p=2)$. In this case we may put $K_p = \boldsymbol{Q}_p(\sqrt{-5})$, $\mathcal{O}_p = \boldsymbol{Z}_p[\sqrt{-5}]$, and $\pi = 1 + \sqrt{-5}$. Here X^2+1 is irreducible over K_p and $G_p(f)$ contains a unique conjugacy class such that $(d(V_p), K/\boldsymbol{Q})_p = +1$.

LEMMA 4.14. *Suppose* $f(X) = f_3(X)$, $p=2$, *and* $d(K) \equiv 12 \pmod{32}$. *Then there is no element of type* II *in* $U_p(f)$.

PROOF. Suppose there exists an elment $g \in U_p(f)$ of type II. By Lemma 3.5, (ii), we may put

$$L_{1,p} = \mathcal{O}_p x_1\,, \qquad L_{2,p} = \mathcal{O}_p x_2 \oplus \mathcal{O}_p x_3\,;$$
$$x_1 = (1, 0, \sqrt{-5})\,, \quad x_2 = (0, 1, 0)\,, \quad x_3 = (-\sqrt{-5}, 0, -1)\,.$$

Put ${}^t h = ({}^t x_1, {}^t x_2, {}^t x_3)$ and $g_0 = hgh^{-1}$. Then we see that g_0 is written in the following form

$$g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -6b^\rho & a^\rho \end{pmatrix}; \qquad a, b \in \mathcal{O}_p\,, \quad N_{K_p/\boldsymbol{Q}_p}(a) + 6N_{K_p/\boldsymbol{Q}_p}(b) = 1\,, \ \mathrm{Tr}_{K_p/\boldsymbol{Q}_p}(a) = 0\,.$$

Calculating the (1, 1)-entry of $g = h^{-1}g_0 h \in GL_3(\mathcal{O}_p)$, we have $1 + 5a^\rho \equiv 0 \pmod 2$. This is a contradiction, because $\mathrm{Tr}_{K_p/\boldsymbol{Q}_p}(a) = 0$. q.e.d.

Next we suppose that $(L_{1,p}, L_{2,p})$ is of type I. By Lemma 3.2, (ii), $L_{2,p}$ is normal unimodular, which we assume being fixed.

LEMMA 4.15. *Suppose* $f(X) = f_3(X)$, $p=2$, $d(K) \equiv 12 \pmod{32}$, *and* $(d(V_{2,p}), K/\boldsymbol{Q})_p = +1$. *Then* $(L_{1,p}, L_{2,p})$ *is of type* I, *and we have the following assertions.*

(i) $L_{2,p}$ *is either a proper* $R_{0,p}$*-module, or a proper* $R_{2,p}$*-module.*

(ii) *There is a unique* U_p*-conjugacy class* $[g]_{U_p}$ *of this type such that the corresponding* $L_{2,p}$ *is a proper* $R_{0,p}$*-module. Let* δ *run through a complete set of representatives in* $G(g)_p \backslash G_p / U_p$ *such that* $\delta^{-1} g \delta \in U_p$ *and* $L_{2,p}$ *corresponding to* $\delta^{-1} g \delta$ *is a proper* $R_{2,p}$*-module. Then we have*

$$\sum_{\delta} [G(g)_p \cap U_p : G(g)_p \cap \delta U_p \delta^{-1}] = 6\,.$$

PROOF. First we prove (i). By Lemma 4.3, it suffices to show that $\{R_{j,p}/L_{2,p}\} \neq \emptyset$ if and only if $j=0$ or 2. From Lemmas 4.5 and 4.6, (i), we have

$$\{R_{0,p}/L_{2,p}\}=(N_{N_p/Q_p})^{-1}(N_{K_p/Q_p}(\mathcal{O}_p^\times))\neq\varnothing\,,$$

$$\{R_{2,p}/L_{2,p}\}=\frac{1}{2}\{R_{0,p}/L_{2,p}\}\neq\varnothing\,,$$

and

$$z\in\{R_{1,p}/L_{2,p}\}\Rightarrow N_{N_p/Q_p}(z)\in(1/2)\boldsymbol{Z}_p^\times\,.$$

Since $N/\boldsymbol{Q}$ is unramified at $p=2$ (cf. Lemma 4.1), we see that $\{R_{1,p}/L_{2,p}\}=\varnothing$. Next we prove (ii). The above expression and the translation theorem in local class field theory show that $\{R_{0,p}/L_{2,p}\}=N_{\boldsymbol{M}_p/N_p}(R_{0,p}^\times)$. This, together with Lemma 4.3, shows the uniqueness of $[g]_{U_p}$. Similarly, we see that the number of δ's is equal to $[N_{\boldsymbol{M}_p/N_p}(R_{0,p}^\times):N_{\boldsymbol{M}_p/N_p}(R_{2,p}^\times)]$.

Now put $R_{j,p}^{(1)}:=\boldsymbol{M}_p^{(1)}\cap R_{j,p}^\times$. Then it is easy to see that, for any δ as above, one has $[G(g)_p\cap U_p:G(g)_p\cap\delta U_p\delta^{-1}]=[R_{0,p}^{(1)}:R_{2,p}^{(1)}]$. We thus have

$$\begin{aligned}\sum_\delta[G(g)_p\cap U_p:G(g)_p\cap\delta U_p\delta^{-1}]&=[N_{\boldsymbol{M}_p/N_p}(R_{0,p}^\times):N_{\boldsymbol{M}_p/N_p}(R_{2,p}^\times)]\cdot[R_{0,p}^{(1)}:R_{2,p}^{(1)}]\\&=[R_{0,p}^\times:R_{2,p}^\times]=p(p+1)=6\,.\end{aligned}$$

q.e.d.

Case (4): $d(K)\equiv-4\pmod{32}$ $(f=f_3;p=2)$. *In this case we may put* $K_p=\boldsymbol{Q}_p(\sqrt{-1})$, $\mathcal{O}_p=\boldsymbol{Z}_p[\sqrt{-1}]$, and $\pi=1+\sqrt{-1}$. Here X^2+1 is decomposed over K_p as $(X+\sqrt{-1})(X-\sqrt{-1})$, hence $G_p(f)\,//\,G_p$ contains exactly two conjugacy classes such that $(d(V_{2,p}),K/\boldsymbol{Q})_p=+1$. They are distinguished by the invariant

$$(d(V_{2,p,1}),K/\boldsymbol{Q})_p=(d(V_{2,p,2}),K/\boldsymbol{Q})_p\ (=\pm1)\,,$$

where $V_{2,p,j}$ is as in (4.1.B).

By an argument similar to that in Case (3), we see that there is no such element of type II in $U_p(f)$.

Let $(L_{1,p},L_{2,p})$ be a pair of type I, where $L_{2,p}$ is assumed to be fixed.

LEMMA 4.16. *Suppose* $f(X)=f_3(X)$, $p=2$, $d(K)\equiv-4\pmod{32}$, *and* $(d(V_{2,p}),K/\boldsymbol{Q})_p=+1$. *Then* $(L_{1,p},L_{2,p})$ *is of type* I, *and we have the following assertions*:

(i) $L_{2,p}$ *is either a proper* $R_{0,p}$*-module, or a proper* $R_{2,p}$*-module.*

(ii) *There are exactly two* U_p*-conjugacy classes* $[g_1]_{U_p}$, $[g_2]_{U_p}$ *of this type such that the corresponding* $L_{2,p}$ *are proper* $R_{0,p}$*-modules. They belong to distinct* G_p*-conjugacy classes*: $[g_1]_p\neq[g_2]_p$. *For* $i=1,2$, *let* $\delta=\delta(i)$ *run through a complete set of representatives in* $G(g_i)_p\backslash G_p/U_p$ *such that* $\delta^{-1}g_i\delta\in U_p$ *and* $L_{2,p}$, *which corresponds to* $\delta^{-1}g_i\delta$, *is a proper* $R_{2,p}$*-module.*

Then we have

$$\sum_\delta[G(g_i)_p\cap U_p:G(g_i)_p\cap\delta U_p\delta^{-1}]=2\,.$$

PROOF. We have $M_p = K_p \oplus K_p$, $N_p = \boldsymbol{Q}_p \oplus \boldsymbol{Q}_p$, $R_p = R_{0,p} = \mathcal{O}_p \oplus \mathcal{O}_p$, and $S_p = \boldsymbol{Z}_p \oplus \boldsymbol{Z}_p$. These expressions, together with Lemmas 4.5, 4.6, (ii), show that

$$\{R_{0,p}/L_{2,p}\} = S_p^\times \cap (N_{N_p/\boldsymbol{Q}_p})^{-1}(N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^\times)) \neq \varnothing ,$$

$$\{R_{2,p}/L_{2,p}\} = \frac{1}{2}\{R_{0,p}/L_{2,p}\} \neq \varnothing , \qquad \{R_{1,p}/L_{2,p}\} = \varnothing .$$

(i) follows from this. We also see that $\{R_{0,p}/L_{2,p}\}/N_{\boldsymbol{M}_p/N_p}(R_p^\times)$ is represented by two elements $(1, 1)$ and $(\varepsilon, \varepsilon)$, where $\varepsilon \in \boldsymbol{Z}_p^\times$, $(\varepsilon, K/\boldsymbol{Q})_p = -1$. They correspond to $[g_1]_{U_p}$, $[g_2]_{U_p}$. Finally using the argument in the proof of Lemma 4.15, (ii), we have

$$\sum_{\delta} [G(g_i)_p \cap U_p : G(g_i)_p \cap \delta U_p \delta^{-1}] = [R_{0,p}^\times : R_{2,p}^\times] .$$

Now it is easy to see that $[R_{0,2}^\times : R_{2,p}^\times] = 2$. q.e.d.

Case (5): $8 \| d(X)$ $(f = f_3; p = 2)$. In this case $X^2 + 1$ is irreducible over K_p and $G_p(f) // G_p$ contains a unique conjugacy class such that $(d(V_{2,p}), K/\boldsymbol{Q})_p = +1$. As in Cases (3), (4), we see that there is no element of type II in $U_p(f)$. But in the present case, the U_p-conjugacy classes of type I are divided into two subtypes, which we have to treat separately:

type I-1: $L_p = L_{1,p} \oplus L_{2,p}$, $\quad (d(L_{2,p}), K/\boldsymbol{Q}_p) = +1$,
$L_{2,p}$ is *normal* unimodular,

type I-2: $L_p = L_{1,p} \oplus L_{2,p}$, $\quad (d(L_{2,p}), K/\boldsymbol{Q}_p) = +1$,
$L_{2,p}$ is *subnormal* unimodular.

Let $(L_{1,p}, L_{2,p})$ be of type I-k $(k = 1, 2)$, with $L_{2,p}$ being fixed.

LEMMA 4.17. *Suppose $f(X) = f_3(X)$, $p = 2$, $8 \| d(K)$, and $(d(V_{2,p}), K/\boldsymbol{Q})_p = +1$. Then $(L_{1,p}, L_{2,p})$ is of type* I-1 *or* I-2. *Moreover, we have the following assertions*:

(i) *$L_{2,p}$ is a proper $R_{1,p}$ (resp. $R_{0,p}$)-module, if it is of type* I-1 (resp. I-2).

(ii) *There is a unique U_p-conjugacy class $[g]_{U_p}$ such that the corresponding $L_{2,p}$ is a proper $R_{0,p}$-module. Let δ run through all representatives of $G(g)_p \backslash G_p / U_p$ such that $\delta^{-1} g \delta$ is an element of U_p and of type* I-1. *Then we have*

$$\sum_{\delta} [G(g)_p \cap U_p : G(g)_p \cap \delta U_p \delta^{-1}] = 2 .$$

PROOF. Let $L_{2,p}$ be a unimodular plane such that $(d(L_{2,p}), K/\boldsymbol{Q})_p = +1$. Then, by Lemma 4.4, $\{R_{j,p}/L_{2,p}\}$ is contained in the set of all $z \in \boldsymbol{N}_p^\times$ satisfying (4.4), with $R_{j,p}$ instead of R'_p. From this fact and Lemma 4.6, (iii), we have

$$(*)_0 \quad \{R_{0,p}/L_{2,p}\} \subseteqq (N_{\boldsymbol{N}_p/\boldsymbol{Q}_p})^{-1}\left(N_{\boldsymbol{K}_p/\boldsymbol{Q}_p}\left(\frac{1}{\pi}\mathcal{O}_p^\times\right)\right) \neq \varnothing ,$$

$$(*)_1 \quad \{R_{1,p}/L_{2,p}\} \subseteqq \frac{1}{2}(N_{\boldsymbol{N}_p/\boldsymbol{Q}_p})^{-1}(N_{K_p/\boldsymbol{Q}_p}(\mathcal{O}_p^{\times})) \neq \varnothing .$$

Now, let ω be a prime element of $\boldsymbol{N}_p$. Since $8 \| d(K)$, we see that the different of $\boldsymbol{N}_p/\boldsymbol{Q}_p$ is $2\omega \cdot S_p$, hence we have

$$\mathrm{Tr}_{\boldsymbol{M}_p/K_p}(zaa^{\sigma}) \in \mathrm{Tr}_{\boldsymbol{N}_p/\boldsymbol{Q}_p}(z \cdot S_p) = 2\boldsymbol{Z}_p \quad (a \in R_{0,p})$$

for any $z \in \{R_{0,p}/L_{2,p}\}$. Therefore $L_{2,p}$ is a subnormal lattice if it is a proper $R_{0,p}$-module. Thus we have shown (i). It follows that the equality holds in the inclusion $(*)_j$ above, if either

$$j=0\,, \quad L_{2,p}=\text{subnormal}\,, \quad \text{or} \quad j=1\,, \quad L_{2,p}=\text{normal}\,.$$

Then the assertion (ii) can be proved by an argument similar to those in the proof of the preceding lemmas, with the fact that $[R_{0,p}^{\times} : R_{1,p}^{\times}]=2$. q.e.d.

Now let $f(X)=f_{41}(X)$ or $f_{42}(X)$, and $p=3$. There are four cases (6)–(9) to be considered separately. However, in some cases the arguments are quite parallel to those in the above. So we shall omit the proof in such cases. We first note that to study $U_p(f_{42}) // U_p$ is simpler than $U_p(f_{41}) // U_p$, because of the following:

LEMMA 4.18. *Suppose $f(X)=f_{42}(X)$ and $p=3$. Then, for any element of $U_p(f)$ which satisfies the condition of Lemma* 4.10, (ii), *the corresponding $(L_{1,p}, L_{2,p})$ is of type* I.

This is proved easily by Lemma 3.1 (cf. Proof of Lemma 4.8, (i) or Lemma 4.11).

Case (6): $(K/p)=+1$. $(f=f_{41}, f_{42}; p=3)$. In this case $X^2 \pm X+1$ are irreducible over $\boldsymbol{Q}_p$ and $G_p(f)$ consists of a single G_p-conjugacy class. It satisfies $(d(V_{2,p}), K/\boldsymbol{Q})_p = (3, K/\boldsymbol{Q})_p$.

LEMMA 4.19. *Suppose that $f(X)=f_{41}(X)$ or $f_{42}(X)$, $p=3$, and $(K/p)=+1$.*

(i) *If $f(X)=f_{42}(X)$, then $(L_{1,p}, L_{2,p})$ is of type* I, *$L_{2,p}$ is a proper R_p-module, and $U_p(f)$ consists of a unique U_p-conjugacy class.*

(ii) *If $f(X)=f_{41}(X)$, then for each type, $L_{2,p}$ is a proper R_p-module and there is a unique U_p-conjugacy class of that type. Let $[g]_{U_p}$ (resp. $[\delta^{-1}g\delta]_{U_p}$, $\delta \in G_p$) be the U_p-conjugacy class of type* I *(resp. type* II*). Then we have*

$$[G(g)_p \cap U_p : G(g)_p \cap \delta U_p \delta^{-1}]=2\,.$$

Proof is omitted (see the proof of Lemma 4.12).

Case (7): $(K/p)=-1$ $(f=f_{41}, f_{42}; p=3)$. In this case, $X^2 \pm X+1$ are irreducible over K_p, and $G_p(f) // G_p$ contains a unique class such that $(d(V_{2,p}), K/\boldsymbol{Q})_p = (3, K/\boldsymbol{Q})_p$ $(=-1)$.

LEMMA 4.20. *Suppose that $f(X)=f_{41}(X)$ or $f_{42}(X)$, $p=3$, and $(K/p)=-1$.*

(i) *If $f(X)=f_{42}(X)$, then $[g]_p \cap U_p = \varnothing$.*

(ii) *If* $f(X)=f_{41}(X)$, *then* $(L_{1,p}, L_{2,p})$ *is of type* II, *and* $L_{2,p}$ *is a proper* R_p*-module. There is a unique* U_p*-conjugacy class in* $[g]_p \cap U_p$. *Moreover, we have*

$$[G(g)_p : G(g)_p \cap U_p] = 4 .$$

PROOF. We first prove (i). Suppose that $[g]_p \cap U_p \neq \emptyset$, and let $(L_{1,p}, L_{2,p})$ be the pair corresponding to an element of this set. By Lemma 4.18, this is of type I, hence $L_{2,p}$ is unimodular. Then we have $(d(L_{2,p}), K/\boldsymbol{Q})_p = +1$ (note that $(K/p) \neq 0$), which is a contradiction. Next we prove (ii). From the above we see that $(L_{1,p}, L_{2,p})$ is of type II, and by Lemma 4.7, (i), $L_{2,p}$ is a proper R_p-module. Here an argument similar to that in the proof of Lemma 3.5, (i) shows that any $(L_{1,p}, L_{2,p})$ of type II can be transformed by the U_p-action to the following:

$$L_{1,p} = \mathcal{O}_p x_1 , \qquad L_{2,p} = \mathcal{O}_p x_2 \oplus \mathcal{O}_p x_3 ;$$
$$x_1 = (1, 1, 1) , \qquad x_2 = (0, 1, -1) , \qquad x_3 = (-2, 1, 1) .$$

So we assume that $(L_{1,p}, L_{2,p})$ is this standard one. Then it is easy to see that $\{R_p/L_{2,p}\}$ coincides with the set of all $z \in N_p^{\times}$ satisfying (4.4), with R_p instead of R'_p. It follows from this and Lemma 4.3, that there is a unique $\mathbb{U}(L_{2,p})$-equivalence class of proper R_p-module structures on $L_{2,p}$. This means that, for any $g_1, g_2 \in U_p$ corresponding to the above $(L_{1,p}, L_{2,p})$, there exists $h \in G_p$ such that $h^{-1} g_1 h = g_2$, $L_{1,p} \cdot h = L_{1,p}$, and $L_{2,p} \cdot h = L_{2,p}$. Now we put ${}^t x := ({}^t x_1, {}^t x_2, {}^t x_3) (\in GL_3(K_p))$, $g_1 = x^{-1} u_1 x$, $g_2 = x^{-1} u_2 x$ $(u_1, u_2 \in U_p)$, and $h = x^{-1} v x$. Then we have $v^{-1} u_1 v = u_2$, and we may write

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & S_1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & 0 \\ 0 & S_2 \end{pmatrix}, \quad v = \begin{pmatrix} c & 0 \\ 0 & T \end{pmatrix}.$$

Then T is seen to have the form

$$T = \begin{pmatrix} a & b \\ -3\varepsilon b^{\rho} & \varepsilon a^{\rho} \end{pmatrix}$$

with $a, b, \varepsilon \in \mathcal{O}_p$, $N(a) + 3N(b) = 1$, $N(\varepsilon) = 1$ $(N(*) = N_{K_p/\boldsymbol{Q}_p}(*))$. Then, a straightforward calculation shows that we have

$$h \in GL_3(\mathcal{O}_p) \Leftrightarrow c^{-1} \varepsilon a^{\rho} \equiv 1 \pmod{p} .$$

Therefore, replacing c by $-c$ if necessary, we can find h in U_p. It follows that $[g]_p \cap U_p$ consists of a single U_p-conjugacy class. By a similar argument, we have

$$[G(g)_p : G(g)_p \cap U_p] = 4 .$$

q.e.d.

REMARK. In the assertion (ii) above, $[g]_p \cap U_p \neq \emptyset$ is easily seen. Indeed the element

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

belongs to this set.

Case (8): $d(K)\equiv 3 \pmod 9$ ($f=f_{41}, f_{42}; p=3$). In this case we may put $K=\boldsymbol{Q}_p(\sqrt{3})$, $\mathcal{O}_p=\boldsymbol{Z}_p[\sqrt{3}]$, and $\pi=\sqrt{3}$. Here $X^2\pm X+1$ are irreducible over K_p and $G_p(f)//G_p$ contains a unique conjugacy class such that $(d(V_{2,p}), K/\boldsymbol{Q}_p)=(3, K/\boldsymbol{Q})_p(=-1)$.

LEMMA 4.21. *Suppose that $f(X)=f_{41}(X)$ or $f_{42}(X)$, $p=3$, and $d(K)\equiv 3 \pmod 9$, and $(d(V_{2,p}), K/\boldsymbol{Q})_p=-1$.*

(i) *If $f(X)=f_{42}(X)$, then $(L_{1,p}, L_{2,p})$ is of type* I, *$L_{2,p}$ is a proper R_p-module, and $U_p(f)$ contains a unique U_p-conjugacy class of this type.*

(ii) *If $f(X)=f_{41}(X)$, then $L_{2,p}$ is a proper $R_{0,p}$ (resp. $R_{1,p}$)-module if $(L_{1,p}, L_{2,p})$ is of type* I *(resp. type* II*). There is a unique U_p-conjugacy class $[g]_{U_p}$ of type* I. *Let δ run through all representatives of $G(g)_p\backslash G_p/U_p$ such that $\delta^{-1}g\delta$ belongs to U_p and is of type* II. *Then we have*

$$\sum_{\delta}[G(g)_p\cap U_p\colon G(g)_p\cap\delta U_p\delta^{-1}]=6\,.$$

Proof is omitted (see the proof of Lemma 4.20).

Case (9): $d(K)\equiv -3 \pmod 9$ ($f=f_{41}, f_{42}; p=3$). In this case we may put $K_p=\boldsymbol{Q}_p(\sqrt{-3})$, $\mathcal{O}_p=\boldsymbol{Z}_p[\sqrt{-3}]$, and $\pi=\sqrt{-3}$. Here X^2+X+1 (resp. X^2-X+1) is decomposed over K_p as $(X-\omega)(X-\omega^2)$ (resp. $(X+\omega)(X+\omega^2)$), hence $G_p(f)//G_p$ contains exactly two conjugacy classes such that $(d(V_{2,p}), K/\boldsymbol{Q})_p=(3, K/\boldsymbol{Q})_p$ $(=+1)$. They are determined by the invariant $(d(V_{2,p,1}), K/\boldsymbol{Q})_p=(d(V_{2,p,2}), K/\boldsymbol{Q})_p$ $(=\pm 1)$.

LEMMA 4.22. *Suppose that $f(X)=f_{41}(X)$ or $f_{42}(X)$, $p=3$, and $d(K)\equiv -3 \pmod 9$, and $(d(V_{2,p}), K/\boldsymbol{Q}_p)=+1$.*

(i) *If $f(X)=f_{42}(X)$, then $(L_{1,p}, L_{2,p})$ is of type* I, *$L_{2,p}$ is a proper R_p-module. Moreover, $U_p(f)$ contains two U_p-conjugacy classes $[g_1]_{U_p}$ and $[g_2]_{U_p}$ of this type. They belong to distinct G_p-conjugacy classes: $[g_1]_p\neq[g_2]_p$.*

(ii) *If $f(X)=f_{41}(X)$, then $L_{2,p}$ is a proper $R_{0,p}$ (resp. $R_{1,p}$)-module if $(L_{1,p}, L_{2,p})$ is of type* I *(resp. type* II*). There are exactly two U_p-conjugacy classes $[g_1]_{U_p}$, $[g_2]_{U_p}$ in $U_p(f)$ which are of type* I; *and they belong to distinct G_p-conjugacy classes. Let δ run through all representatives of $G(g)_p\backslash G_p/U_p$ such that $\delta^{-1}g\delta$ belongs to U_p and is of type* II. *Then we have*

$$\sum_{\delta}[G(g)_p\cap U_p\colon G(g)_p\cap\delta U_p\delta^{-1}]=\begin{cases} 12 \cdots (d(V_{2,p,j}), K/\boldsymbol{Q})_p=+1 \\ 0 \ \cdots (d(V_{2,p,j}), K/\boldsymbol{Q})_p=-1\,. \end{cases}$$

Proof is again omitted (see the proof of Lemma 4.20).

4.4. We now summarize the results obtained above, and calculate the contributions T_3, T_{41}, and T_{42} to the formula (0.1), from the conjugacy classes belonging to $f_3(X)$, $f_{41}(X)$, and $f_{42}(X)$ respectively.

We first describe the locally integral conjugacy classes in $G(f)$ for each $f(X)$.

PROPOSITION 4.23. *Suppose* $f(X)=f_3(X)$.

(i) *If* $d(K)\not\equiv -4 \pmod{32}$, *then there exists a unique locally integral conjugacy class in* $G(f)//G$. *It is characterized by*

$$(d(V_{2,p}), K/\boldsymbol{Q})_p=+1 \qquad \text{at all} \quad p<\infty\,,$$

$$(d(V_{2,p,j}), K/\boldsymbol{Q})_p=+1 \ (j=1,2) \qquad \text{at any} \quad p \quad \text{in the case (4.1}B\text{)}.$$

(ii) *If* $d(K)\equiv -4 \pmod{32}$, *then there are exactly two locally integral conjugacy classes in* $G(f)//G$. *They are characterized by*

$$(d(V_{2,p}), K/\boldsymbol{Q})_p=+1 \qquad \text{at all} \quad p<\infty\,,$$

$$(d(V_{2,p,j}), K/\boldsymbol{Q})_p=+1 \ (j=1,2) \qquad \text{at any} \quad p\neq 2 \quad \text{in the case (4.1.}B\text{)}\,.$$

$$(d(V_{2,p,1}), K/\boldsymbol{Q})_p=(d(V_{2,p,2}), K/\boldsymbol{Q})_p=\pm 1 \qquad \text{at} \quad p=2\,.$$

PROPOSITION 4.24. *Suppose* $f(X)=f_{41}(X)$.

(i) *If* $d(K)\not\equiv -3 \pmod 9$, *then there exists a unique locally integral conjugacy class in* $G(f)//G$. *It is characterized by*

$$(d(V_{2,p}), K/\boldsymbol{Q})_p=(3, K/\boldsymbol{Q})_p \qquad \text{at all} \quad p<\infty\,,$$

$$(d(V_{2,p,j}), K/\boldsymbol{Q})_p=+1 \ (j=1,2) \qquad \text{at any} \quad p \quad \text{in the case (4.2.B)}\,.$$

(ii) *If* $d(K)\equiv -3 \pmod 9$, *then there are exactly two locally integral conjugacy classes in* $G(f)//G$. *They are characterized by*

$$(d(V_{2,p}), K/\boldsymbol{Q})_p=(3, K/\boldsymbol{Q})_p \qquad \text{at all} \quad p<\infty\,,$$

$$(d(V_{2,p,j}), K/\boldsymbol{Q})_p=+1 \ (j=1,2) \qquad \text{at any} \quad p\neq 3 \quad \text{in the case (4.2.B)}\,.$$

$$(d(V_{2,p,1}), K/\boldsymbol{Q})_p=(d(V_{2,p,2}), K/\boldsymbol{Q})_p=\pm 1 \qquad \text{at} \quad p=3\,.$$

PROPOSITION 4.25. *Suppose* $f(X)=f_{42}(X)$.

(i) *If* $(K/p)=+1$ *or* $d(K)\equiv 3 \pmod 9$, *then there exists a unique locally integral conjugacy class in* $G(f)//G$. *It is characterized by the same condition as in Proposition* 4.24, (i).

(ii) *If* $(K/p)=-1$, *then there is no locally integral conjugacy class in* $G(f)//G$.

(iii) *If* $d(K)\equiv -3 \pmod 9$, *then there are exactly two locally integral conjugacy classes in* $G(f)//G$. *They are characterized by the same condition as in Propostion* 4.24, (ii).

Now, let $[g]=[g]_{\boldsymbol{Q}}$ be a locally integral conjugacy class in $G(f)\,//\,G$. Once an idélic arithmetic subgroup V of $\mathbb{G}(g)_{\mathbb{A}}$ is fixed, we may consider each factor of $\boldsymbol{h}([g];\,\mathscr{L})$ in (0.2). In order to apply the results in [32, § 5], we choose V as follows: Firstly we identify $\mathbb{G}(g)$ with $K^{(1)}\times \boldsymbol{M}^{(1)}$, and define $V=\prod_v V_v$ by

$$V_p := \mathcal{O}_p^{(1)}\times R_p^{(1)} \qquad (\text{for each } p<\infty)$$

$$V_\infty := K_\infty^{(1)}\times \boldsymbol{M}_\infty^{(1)}.$$

Therefore, the first factor $\mathbb{M}(V)$ of $\boldsymbol{h}([g];\,\mathscr{L})$ is nothing but $\mathbb{M}_1(K/\boldsymbol{Q})\times\mathbb{M}_1(\boldsymbol{M}/\boldsymbol{N})$, where each factor is the standard mass in the (one-dimensional) principal genus. By [32, Theorems 5.6, 5.7], we have

$$\mathbb{M}_1(K/\boldsymbol{Q})=2^{-t}|\,B_{1,\chi}\,|\,,$$

$$\mathbb{M}_1(\boldsymbol{M}/\boldsymbol{N})=2^{-1-T}B_{1,\chi}\cdot B_{1,\psi}\,,$$

with T being the number of distinct prime divisors of the relative discriminant $d(\boldsymbol{M}/\boldsymbol{N})$, and

$$\chi(p)=(d(K)/p)$$

$$\psi(p)=\begin{cases}(-1/p)\ \cdots\ f(X)=f_3(X)\\[4pt] (-3/p)\ \cdots\ f(X)=f_{41}(X)\quad\text{or}\quad f_{42}(X)\,.\end{cases}$$

We note that, by Dirichlet's formula for the class numbers of imaginary quadratic fields, we have $B_{1,\chi}=-h(K)$ (note that we are assuming $K\neq \boldsymbol{Q}(\sqrt{-1}),\,\boldsymbol{Q}(\sqrt{-3})$). Also we have $B_{1,\psi}=-1/2$ or $-1/3$, according as $f=f_3$ or $f=f_{41}$, f_{42}. Thus we have

$$\mathbb{M}(V)=\begin{cases}2^{-T-t-2}h(K)^2 \qquad \cdots\ f(X)=f_3(X)\\[4pt] 2^{-T-t-1}\cdot 3^{-1}h(K)^2\cdots\ f(X)=f_{41}(X)\,,\quad f_{42}(X)\,.\end{cases} \tag{4.5}$$

By Lemma 4.1, we have:

(4.6) If $f(X)=f_3(X)$,

$$T=\begin{cases}0\ \cdots\ (K/2)\neq 0\\ 1\ \cdots\ d(K)\equiv 12\ (\mathrm{mod}\ 32)\quad\text{or}\quad 8\,\|\,d(K)\\ 2\ \cdots\ d(D)\equiv -4\ (\mathrm{mod}\ 32)\,.\end{cases}$$

(4.7) If $f(X)=f_{41}(X), f_{42}(X)$

$$T=\begin{cases}0\ \cdots\ (K/3)\neq 0\\ 1\ \cdots\ d(K)\equiv 3\ (\mathrm{mod}\ 9)\\ 2\ \cdots\ d(D)\equiv -3\ (\mathrm{mod}\ 9)\,.\end{cases}$$

As for the second factor $\prod_p(\sum_\delta \mathrm{Ind}_p(\delta; g))$ of $\boldsymbol{h}([g]; \mathscr{L})$, it is easy to show that $\mathrm{Ind}_p(\delta; g)=1$ if and only if the pair $(L_{1,p}, L_{2,p})$ corresponding to $\delta^{-1}g\delta$ is of type I, and $L_{2,p}$ is a proper R_p-module. Therefore, Lemmas 4.8, 4.9, 4.12, 4.13, 4.15, 4.16, 4.17, 4.19, 4.20, 4.21 and 4.22 together imply the following results:

(4.8) If $f(X)=f_3(X)$, and $p\neq 2$, then

$$\sum_\delta \mathrm{Ind}_p(\delta: g)=1 .$$

(4.9) If $f(X)=f_3(X)$, and $p=2$, then

$$\sum_\delta \mathrm{Ind}_p(\delta: g)=\begin{cases} 2 \cdots (K/p)=+1 \\ 1 \cdots (K/p)=-1 \\ 7 \cdots d(K)\equiv 12 \pmod{32} \\ 3 \cdots d(K)\equiv -4 \pmod{32} \\ 3 \cdots 8\| d(K) . \end{cases}$$

(4.10) If $f(X)=f_{41}(X), f_{42}(X)$; and $p\neq 3$, then

$$\sum_\delta \mathrm{Ind}_p(\delta: g)=1 .$$

(4.11) If $f(X)=f_{42}(X)$, and $p=3$, then

$$\sum_\delta \mathrm{Ind}_p(\delta: g)=1 .$$

(4.12) If $f(X)=f_{41}(X)$, and $p=3$, then

$$\sum_\delta \mathrm{Ind}_p(\delta: g)=\begin{cases} 3 \cdots (K/p)=+1 \\ 4 \cdots (K/p)=-1 \\ 7 \cdots d(K)\equiv 3 \pmod 9 \\ 13 \cdots d(K)\equiv -3 \pmod 9),\ (d(V_{2,p,j}), K/\boldsymbol{Q})_3=+1 \\ 1 \cdots d(K)\equiv -3 \pmod 9),\ (d(V_{2,p,j}), K/\boldsymbol{Q})_3=-1 . \end{cases}$$

5. $n=3$: Explicit formulas (main results).

5.1. We collect all data that we obtained in §§3, 4, and putting them into the general formula (0.1), we get an explicit formula for the class number $\boldsymbol{h}(\mathscr{L})$ of the principal genus $\mathscr{L}$ of the ternary positive definite Hermitian space (V, H).

Let $K=\boldsymbol{Q}(\sqrt{-m})$ be an imaginary quadratic field with discriminant $d(K)$, where m is a positive square-free integer. For each polynomial $f_i(X)$ listed in Lemma 1.2, we put

$$T_i:=\sum_{[g]} \boldsymbol{h}([g]; \mathscr{L}), \tag{5.1}$$

where the sum is extended over the locally integral G-conjugacy classes $[g]$ in $G(f_i)$. Thus

T_i is the contribution to our class number formula from the conjugacy classes belonging to $G(f_i)$.

MAIN THOEREM 5.1. *The class number $\boldsymbol{h}_3$ of the principal genus in the ternary positive definite Hermitian space, with respect to the unitary group, is given as follows: $\boldsymbol{h}_3 = 1$ if $d(K) = -3$, or -4; and in the other cases,*

$$\boldsymbol{h}_3 = 2T_1 + 2T_2 + 2T_3 + 2T_{41} + 2T_{42} + 4T_5 + 4T_6\,, \tag{5.2}$$

where the contribution T_i from $G(f_i)$ is given by

$$T_1 = h(K)B_{3,\chi}/2^{t+4}3^2\,,$$

$$T_2 = (h(K)^2/2^{t+4}3) \times \begin{cases} 4|d(K)|-4 & \cdots \quad -m \equiv 1 \\ 4|d(K)|+2 & \cdots \quad -m \equiv 5 \\ 4|d(K)|-1 & \cdots \quad \textit{otherwise} \end{cases} \pmod 8$$

$$T_3 = (h(K)^2/2^{t+3}) \times \begin{cases} 4 & \cdots \quad -m \equiv 1 \\ 7 & \cdots \quad -m \equiv 3 \\ 2 & \cdots \quad -m \equiv 5 \\ 3 & \cdots \quad -m \equiv \pm 2, 7 \end{cases} \pmod 8$$

$$T_{41} = (h(K)^2/2^{t+2}3) \times \begin{cases} 6 & \cdots \quad -m \equiv 1 \\ 8 & \cdots \quad -m \equiv 2 \\ 7 & \cdots \quad -m \equiv 0 \end{cases} \pmod 3$$

$$T_{42} = (h(K)^2/2^{t+2}3) \times \begin{cases} 2 & \cdots \quad -m \equiv 1 \\ 0 & \cdots \quad -m \equiv 2 \\ 1 & \cdots \quad -m \equiv 0 \end{cases} \pmod 3$$

$$T_5 = \begin{cases} 1/16 & \cdots \quad \textit{if} \quad d(K) = -8 \\ 0 & \cdots \quad \textit{otherwise} \end{cases}$$

$$T_6 = \begin{cases} 1/14 & \cdots \quad \textit{if} \quad d(K) = -7 \\ 0 & \cdots \quad \textit{otherwise} \end{cases}$$

where t is the number of distinct prime divisors of $d(K)$, χ is the Dirichlet character attached to K, and $B_{3,\chi}$ is the third generalized Bernoulli number attached to χ.

PROOF. The fact that $\boldsymbol{h}_3 = 1$ for $d(K) = -3$, -4 was proved in [32, Proposition 5.13]. So we assume, in the following, that $d(K) \neq -3$, -4. Recall that T_1 was already given in [32, Theorem 5.6]. The evaluation for T_2 was carried out in Propositions 3.10, 3.12 and 3.13. In the same way as for T_2, we can get easily the above expressions for T_3, T_{41}, T_{42} from the results in §4. Thus it remains to evaluate T_5 and T_6. Let ζ_m be a primitive m-th root of unity. The point here is that the order $\mathcal{O}[\zeta_8]$ (resp. $\mathcal{O}[\zeta_7]$) is the maximal order of $K(\zeta_8)$ (resp. $K(\zeta_7)$), in the unique case $K=\boldsymbol{Q}(\sqrt{-2})$ (resp. $\boldsymbol{Q}(\sqrt{-7})$) where we have non-trivial contribution. Using this fact, one can easily show that the locally integral G-conjugacy class, or the U_p-conjugacy class corresponding to the characteristic polynomial in question, is unique. Thus we see that the contribution $T_i=\boldsymbol{h}([g];\mathscr{L})$ $(i=5,6)$ is nothing but the mass of the centralizer $\boldsymbol{Q}(\zeta_8)^{(1)}$ (resp. $\boldsymbol{Q}(\zeta_7)^{(1)}$), which we evaluated in [32, Theorem 5.7]. q.e.d.

5.2 Class numbers for $\mathbb{SU}(3)$. As in §1, we regard $\mathbb{G}^{(1)}:=\mathbb{SU}(V,H)$ as an algebraic group over $\boldsymbol{Q}$. Let $\mathscr{L}_0^{(1)}$ be any $\mathbb{G}^{(1)}$-genus contained in the principal genus $\mathscr{L}_0$ with respect to $\mathbb{G}$. Then we consider the class number $\boldsymbol{h}_3^{(1)}:=\boldsymbol{h}^{(1)}(\mathscr{L}_0^{(1)})$, i.e., the number of $G^{(1)}$-orbits in $\mathscr{L}_0^{(1)}$. Our second main result is concerned, as well as its evaluation, with the relation between $\boldsymbol{h}_3^{(1)}$ and the unitary class number $\boldsymbol{h}_3$.

MAIN THEOREM 5.2. (i) *The two class numbers $\boldsymbol{h}_3^{(1)}$ and $\boldsymbol{h}_3$ are related by*

$$\boldsymbol{h}_3=\boldsymbol{h}_1\boldsymbol{h}_3^{(1)}, \tag{5.3}$$

where $\boldsymbol{h}_1=h(K)/2^{t-1}$ is the one-dimensional unitary class number. In particular, $\boldsymbol{h}_3^{(1)}$ depends only on the $\mathbb{G}$-genus $\mathscr{L}_0$.

(ii) *More precisely, suppose that $d(K)\neq -3,-4,-7,-8$, and let $f_i(X)$ be as in the list of Lemma* 1.2. *Then the contribution $T_i^{(1)}$ of the $G^{(1)}$-conjugacy classes in $G^{(1)}(f_i)$ to the general formula, which is similar to* (0.1), *is related to T_i by*

$$T_i=(h(K)/2^t)T_i^{(1)} \qquad (\textit{for each } i)\,. \tag{5.4}$$

(iii) *An explicit formula for $\boldsymbol{h}_3^{(1)}$ is given as follows:*

$\boldsymbol{h}_3^{(1)}=1$ (resp. 2) if $K=\boldsymbol{Q}(\sqrt{-1})$, $\boldsymbol{Q}(\sqrt{-3})$ (resp. $\boldsymbol{Q}(\sqrt{-2})$, $\boldsymbol{Q}(\sqrt{-7})$), and otherwise,

$$\boldsymbol{h}_3^{(1)}=T_1^{(1)}+T_2^{(1)}+T_3^{(1)}+T_{41}^{(1)}+T_{42}^{(1)}\,,$$

with

$$T_1^{(1)}=B_{3,\chi}/144$$

$$T_2^{(1)}=(h(K)/48)[4\,|\,d(K)\,|-1-3\chi(2)]$$

$$T_3^{(1)}=(h(K)/8)[3+\chi(2)+\{1+(2,K/\boldsymbol{Q})_2\}\{1+(5,K/\boldsymbol{Q})_2\}]$$

$$T_{41}^{(1)}=(h(K)/12)[7-\chi(3)]$$

$$T_{42}^{(1)}=h(K)/12)[1+\chi(3)]\,.$$

We shall give a table for $\boldsymbol{h}_3^{(1)}$ below. Here we note that our result for the relation between $\boldsymbol{h}_3$ and $\boldsymbol{h}_3^{(1)}$, which holds *without any condition*, is rather remarkable since in general this kind of relation can be shown only under some conditions. See [32, §2].

PROOF. If K is one of the exceptional fields, it is easy to check the assertion, since we know that $\boldsymbol{h}_3=1$ or 2. So we assume that $d(K)\neq -3, -4, -7, -8$, and prove (ii), from which (i) and (iii) follow immediately. First note that our general formula (0.1) remains valid, if we replace $\mathbb{G}$ by $\mathbb{G}^{(1)}$. However, we have to notice that, in general, the Hasse principle (see [32, Proposition 4.8]), fails to hold for the $\mathbb{G}^{(1)}$-conjugacy classes. But for certain types of conjugacy classes in $\mathbb{G}^{(1)}$, we still have the Hasse principle:

LEMMA 5.3. *Let g be an element of $G^{(1)}$, and suppose that the determinant* det: $G\to K^{(1)}$ *maps the centralizer $G(g)$ onto $K^{(1)}$. Then the $G^{(1)}$-conjugacy class $[g]^{(1)}$ is uniquely determined by its image in the set of $G^{(1)}_{\mathbb{A}}$-conjugacy classes.*

This lemma is easily proved. Now we note that, under the above assumption on $d(K)$, the condition of Lemma 5.3 is satisfied for any element g in $G^{(1)}(f_i)$, since $f_i(X)$ has a linear factor $(X-1)$. Also it is easy to see that the U_p-conjugacy class $[g]_{U_p}$ is decomposed into $[\det U_p\colon \det(U_p(g))]$ $U_p^{(1)}$-conjugacy classes $[\delta^{-1}g\delta]_{U_p^{(1)}}$. On the other hand, one has the equality

$$[\det U_p\colon \det(U_p(g))]\cdot \mathrm{ind}_p^{(1)}(\delta\,;\,g)=\mathrm{Ind}_p(1\,;\,g)\,.$$

where $\mathrm{Ind}_p^{(1)}(\delta\,;\,g)$ is defined similarly as $\mathrm{Ind}_p(\delta\,;\,g)$ with $G^{(1)}$ instead of G (cf. [32, § 3]). Now the contribution $T_i^{(1)}$ is calculated in exactly the same way as T_i. Noting the above equality and the relation between the two masses $\mathbb{M}(\mathbb{G}(g))$, $\mathbb{M}(\mathbb{G}^{(1)}(g))$, which is analogous to that given in [32, Proposition 5.8], we get the assertion of Theorem 5.2.

q.e.d.

TABLE OF CLASS NUMBERS OF THE PRINCIPAL GENERA ($n=1, 2, 3$). In the following table, we give, for each imaginary quadratic field K with discriminant $|d(K)|\leq 250$, the class numbers $\boldsymbol{h}_n^{(1)}$ of the principal genera, in the positive definite standard Hermitian spaces (K^n, H), for $n=2$, and 3, with respect to the special unitary group. Note that $\boldsymbol{h}_1^{(1)}=1$, and $\boldsymbol{h}_1=h(K)/2^{t-1}$, and that $\boldsymbol{h}_3=\boldsymbol{h}_1\cdot\boldsymbol{h}_3^{(1)}$.

(I)	$d(K)$	ramified primes	$\boldsymbol{h}_1$	$\boldsymbol{h}_2^{(1)}$	$\boldsymbol{h}_3^{(1)}$
(1)	−3	3	1	1	1
(2)	−4	2	1	1	1
(3)	−7	7	1	1	2
(4)	−8	2	1	1	2
(5)	−11	11	1	2	2
(6)	−15	3*5	1	2	5
(7)	−19	19	1	2	3
(8)	−20	2*5	1	3	7
(9)	−23	23	3	3	10
(10)	−24	2*3	1	2	7

(l)	$d(K)$	ramified primes	h_1	$h_2^{(1)}$	$h_3^{(1)}$
(11)	−31	31	3	3	13
(12)	−35	5*7	1	4	10
(13)	−39	3*13	2	4	21
(14)	−40	2*5	1	4	12
(15)	−43	43	1	4	8
(16)	−47	47	5	5	31
(17)	−51	3*17	1	4	16
(18)	−52	2*13	1	5	18
(19)	−55	5*11	2	6	31
(20)	−56	2*7	2	4	31
(21)	−59	59	3	6	26
(22)	−67	67	1	6	17
(23)	−68	2*17	2	6	40
(24)	−71	71	7	7	66
(25)	−79	79	5	7	59
(26)	−83	83	3	8	43
(27)	−84	2*3*7	1	6	56
(28)	−87	3*29	3	6	77
(29)	−88	2*11	1	6	40
(30)	−91	7*13	1	8	40
(31)	−95	5*19	4	10	107
(32)	−103	103	5	9	88
(33)	−104	2*13	3	8	97
(34)	−107	107	3	10	66
(35)	−111	3*37	4	8	133
(36)	−115	5*23	1	12	62
(37)	−116	2*29	3	9	118
(38)	−119	7*17	5	10	172
(39)	−120	2*3*5	1	8	97
(40)	−123	3*41	1	8	72
(41)	−127	127	5	11	125
(42)	−131	131	5	12	121
(43)	−132	2*3*11	1	8	115
(44)	−136	2*17	2	10	120
(45)	−139	139	3	12	104
(46)	−143	11*13	5	14	227
(47)	−148	2*37	1	11	113
(48)	−151	151	7	13	200
(49)	−152	2*19	3	10	181
(50)	−155	5*31	2	16	148

(l)	$d(K)$	ramified primes	h_1	$h_2^{(1)}$	$h_3^{(1)}$
(51)	−159	3∗53	5	10	267
(52)	−163	163	1	14	111
(53)	−164	2∗41	4	12	239
(54)	−167	167	11	15	308
(55)	−168	2∗3∗7	1	8	182
(56)	−179	179	5	16	213
(57)	−183	3∗61	4	12	303
(58)	−184	2∗23	2	12	215
(59)	−187	11∗17	1	16	170
(60)	−191	191	13	17	423
(61)	−195	3∗5∗13	1	16	228
(62)	−199	199	9	17	368
(63)	−203	7∗29	2	16	252
(64)	−211	211	3	18	243
(65)	−212	2∗53	3	15	341
(66)	−215	5∗43	7	22	537
(67)	−219	3∗73	2	14	290
(68)	−223	223	7	19	411
(69)	−227	227	5	20	339
(70)	−228	2∗3∗19	1	12	341
(71)	−231	3∗7∗11	3	12	558
(72)	−232	2∗29	1	16	303
(73)	−235	5∗47	1	24	284
(74)	−239	239	15	21	669
(75)	−244	2∗61	3	17	436
(76)	−247	13∗19	3	22	481
(77)	−248	2∗31	4	16	510

6. Dimension of automorphic forms. We give here an explicit formula for the dimension of automorphic forms of "*weight* ρ" on $G_{\mathbb{A}}^{(1)}$, where $(\rho, \mathscr{F})$ is any irreducible continuous representation of the compact group

$$G_\infty^{(1)} \simeq \mathbb{SU}(2) \quad \text{or} \quad \mathbb{SU}(3) \quad (n=2 \text{ or } 3).$$

6.1. Let $(\rho, \mathscr{F})$ be as above and extend it to a representation of $G_{\mathbb{A}}^{(1)}$ through the projection $G_{\mathbb{A}}^{(1)} \to G_\infty^{(1)}$. Denote by $M_\rho(\mathscr{L})$ the space over $\boldsymbol{C}$ consisting of the $\mathscr{F}$-valued functions f on $G_{\mathbb{A}}^{(1)}$ satisfying $f(uxa)=\rho(u)f(x)$ for any $u\in \boldsymbol{U}=\boldsymbol{U}(L)$, $x\in G_{\mathbb{A}}^{(1)}$ and $a\in G^{(1)}$, where we fix an $\mathcal{O}$-lattice L in a given genus $\mathscr{L}$.

PROPOSITION 6.1. *The dimension of $M_\rho(\mathscr{L})$ is given by*

$$\dim M_\rho(\mathscr{L}) = \sum_{f\in F} \operatorname{tr}(\rho(f)) \sum_g \boldsymbol{h}([g]; \mathscr{L})$$

where the sums are the same as in the formula (0.1), *and* $\mathrm{tr}(\rho(f))$ *is the character of* ρ *at any element of* $G_\infty(f)$.

PROOF. Let Γ_i $(1\le i\le \boldsymbol{h}=\boldsymbol{h}^{(1)})$ be the finite group given by [32, (3.2)], and put $\mathscr{F}_i:=\{m\in\mathscr{F};\ \rho(\gamma)m=m$ for any $\gamma\in\Gamma_i\}$. Then it is easy to see that the mapping $M_\rho(\mathscr{L})\to\bigoplus_i\mathscr{F}_i$, $f\longmapsto(\rho(\xi_i)^{-1}f(\xi_i))$ is an isomorphism. It follows that

$$\dim M_\rho(\mathscr{L})=\sum_i \dim\mathscr{F}_i=\sum_i\frac{1}{\#[\Gamma_i]}\sum_{\gamma\in\Gamma_i}\mathrm{tr}(\rho(\gamma))=\sum_i\frac{\mathrm{tr}(\rho(f))}{\#[\Gamma_i]}\sum_{f\in F}\#[\Gamma_i(f)]\,.$$

Now the assertion follows from [32, Proposition 3.1].

Thus, to know the dimension of $M_\rho(\mathscr{L})$, we have only to compute the character $\mathrm{tr}(\rho(f))$ for each polynomial $f\in F$ listed in Lemmas 1.1, 1.2.

6.2. First suppose that $n=2$. Then the irreducible representations of $\mathbb{SU}(2)$ is parametrized by a non-negative integer k, to which corresponds the k-th symmetric tensor representation ρ_k. As is well known, we have

$$\text{(6.1)}\qquad \mathrm{tr}(\rho_k(1))=\mathrm{tr}(\rho_k(f_1))=k+1\,,\qquad \mathrm{tr}(\rho_k(f))=(\zeta^{k+1}-\zeta^{-k-1})/(\zeta-\zeta^{-1})\,,$$

where in the second formula, $f\ne f_1$ and ζ, ζ^{-1} are the roots of $f(X)$. It follows that

$$\text{(6.2)}\qquad \mathrm{tr}(\rho_k(f_2))=[1,0,-1,0;4]\,,\ \ \mathrm{tr}(\rho_k(f_3))=[1,-1,0;3]\,,$$

where $t=[t_0,t_1,\cdots,t_{q-1};q]$ means that we have $t=t_i$ if $k\equiv i\pmod q$.

Next suppose that $n=3$. Then the irreducible representations are parametrized by the pairs (k_1,k_2) of non-negative integers such that $k_1\ge k_2\ge 0$. The corresponding representation $\rho=\rho_{(k_1,k_2)}$ has the degree

$$\text{(6.3)}\qquad \mathrm{tr}(\rho(f_1))=(k_1-k_2+1)(k_1+2)(k_2+1)/2\,,$$

and the character

$$\text{(6.4)}\qquad \mathrm{tr}(\rho(f))=\sum_{i=1}^{3}\frac{\varepsilon_i^a}{(\varepsilon_i-\varepsilon_j)(\varepsilon_i-\varepsilon_k)}\cdot\frac{\varepsilon_j^b-\varepsilon_k^b}{(\varepsilon_j-\varepsilon_k)}\qquad(\{i,j,k\}=\{1,2,3\})\,,$$

where ε_1, ε_2, ε_3 $(\varepsilon_1\varepsilon_2\varepsilon_3=1)$ are the distinct roots of $f(X)$, and we put $a:=k_1+2$, $b:=k_2+1$.

It follows that, explicitly we have

$$\text{(6.5)}\qquad \mathrm{tr}(\rho(f_2))=\begin{cases}(k_1+2)/2 & \cdots\ (k_1,k_2)\equiv(0,0)\\ 0 & \cdots\ (k_1,k_2)\equiv(0,1)\\ (-k_1+k_2-1)/2 & \cdots\ (k_1,k_2)\equiv(1,0)\\ -(k_2+1)/2 & \cdots\ (k_1,k_2)\equiv(1,1)\end{cases}\qquad \pmod 2$$

$$\mathrm{tr}(\rho(f_3)) = \begin{cases} 1 & \cdots \quad (k_1, k_2) \equiv (0, 0), (1, 0), (1, 1) \\ -1 & \cdots \quad (k_1, k_2) \equiv (0, 2), (3, 1), (3, 2) \qquad (\mathrm{mod}\ 3) \\ 0 & \cdots \quad \textit{otherwise} \end{cases}$$

$$\mathrm{tr}(\rho(f_{41})) = \begin{cases} 1 & \cdots \quad (k_1, k_2) \equiv (0, 0) \\ -1 & \cdots \quad (k_1, k_2) \equiv (2, 1) \qquad (\mathrm{mod}\ 3) \\ 0 & \cdots \quad \textit{otherwise} \end{cases}$$

$$\mathrm{tr}(\rho(f_{42})) = \begin{cases} 3 & \cdots \quad (k_1, k_2) \equiv (2, 1) \\ 2 & \cdots \quad (k_1, k_2) \equiv (1, 0), (1, 1), (2, 0), (2, 2), (3, 1), (3, 2) \\ 1 & \cdots \quad (k_1, k_2) \equiv (0, 0), (3, 0), (3, 3) \\ -1 & \cdots \quad (k_1, k_2) \equiv (2, 4), (5, 1), (5, 4) \qquad (\mathrm{mod}\ 6)\,. \\ -2 & \cdots \quad (k_1, k_2) \equiv (0, 2), (2, 4), (1, 3), (1, 4), (5, 2), (5, 3) \\ -3 & \cdots \quad (k_1, k_2) \equiv (0, 3) \\ 0 & \cdots \quad \textit{otherwise}\,. \end{cases}$$

CORRECTION TO [31], [32]:

[31]: Page 324, line 5 (in the table). Read "(mod 32)" for "(mod 22)".
[31]: Page 324, line 7 from bottom. Read "odd (resp. even)" for "1 (resp. 2)".
[31]: Page 325, line 13 from bottom. Read "1 (resp. 2)" for "odd (resp. even)".
[32]: Page 20, line 15 from bottom. Read "2^t" for "2^{t-1}".

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