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A THEOREM ON THE LIMIT SETS OF QUASICONFORMAL DEFORMATIONS OF INFINITELY GENERATED FUCHSIAN GROUPS OF THE FIRST KIND

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1. Introduction. Let Γ be a Fuchsian group. Denote by $\Lambda(\Gamma)$ and $\Omega(\Gamma)$ its limit set and region of discontinuity, respectively. Then Γ is said to be of the first kind if $\Omega(\Gamma)$ is not connected. If all elements of $\Gamma\setminus\{1\}$ are hyperbolic transformations, Γ is said to be purely hyperbolic. Let w be a quasiconformal automorphism of the Riemann sphere $\hat{C}$ which is compatible with Γ, that is, $w\circ\gamma\circ w^{-1}$ is a Möbius transformation for each $\gamma\in\Gamma$. Then $w\Gamma w^{-1}$ is a Kleinian group and is called a quasiconformal deformation of Γ. The limit set $\Lambda(w\Gamma w^{-1})$ coincides with $w(\Lambda(\Gamma))$, which is a quasicircle when Γ is of the first kind. For two Jordan curves J_1 and J_2 in the finite complex plane C we define the Fréchet distance $[J_1, J_2]$ as $\inf\max\{|z_1(t)-z_2(t)|;\ 0\leqq t\leqq 1\}$, where the infimum is taken over all possible parametrizations $z_k(t)$ of J_k $(k=1, 2)$.

In Chu [1] the following theorem is used as a key lemma to prove a theorem on the outradii of the Teichmüller spaces of finitely generated purely hyperbolic Fuchsian groups of the first kind.

THEOREM A. *Let J be a rectifiable Jordan curve in C and let $\delta>0$. Then there exists a quasiconformal deformation G of a finitely generated purely hyperbolic Fuchsian group of the first kind so that $[\Lambda(G), J]<\delta$.*

Theorem A is proved by means of a theorem of Maskit on finitely generated Kleinian groups (Maskit [4, Theorem 2]). The assumption of the rectifiability of J can be removed (see Lemma 4.1). In this note we prove the following theorem, which is an analogue of Theorem A.

THEOREM B. *Let J be a Jordan curve in C and let $\delta>0$. Then there exists a quasiconformal deformation G of an infinitely generated Fuchsian group of the first kind so that $[\Lambda(G), J]<\delta$.*

We prove Theorem B by constructing a group G explicitly. In §2 we prove two lemmas which are used in §4. In §3 we construct a quasiconformal mapping used in §5. In §4 we construct an infinitely generated Kleinian group G whose limit set $\Lambda(G)$ is

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a Jordan curve with $[\Lambda(G), J]<\delta$ and an infinitely generated Fuchsian group $\tilde{G}$ of the first kind. In § 5 we prove a lemma which we use in § 6 to show that G is a quasiconformal deformation of $\tilde{G}$.

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2. Preliminary lemmas. The purpose of this section is to prove two lemmas (Lemmas 2.1 and 2.2) which are used in § 4. For $\alpha \in \boldsymbol{C}$ and $r>0$ set $B(\alpha, r)=\{z \in \boldsymbol{C}; |z-\alpha| \leqq r\}$. Let $\alpha_n \in \boldsymbol{C}$ and $r_n>0$ $(n \in \boldsymbol{Z})$. Set $B_n=B(\alpha_n, r_n)$. Assume that the following conditions (1)–(3) are satisfied.

(1) $B_{n-1} \cap B_n$ consists of one point p_n. $B_m \cap B_n=\varnothing$ for $m \neq n, n \pm 1$.

(2) $r_{2n-1}=r_{2n}$ and p_{2n+1} is the mirror image of p_{2n-1} with respect to the perpendicular bisector of the segment $\alpha_{2n-1}\alpha_{2n}$.

(3) The radius r_n converges to 0 as n tends to $\pm\infty$. The center α_n converges to a point $p_\infty \in \boldsymbol{C} \setminus (\bigcup_{n \in \boldsymbol{Z}} B_n)$ as n tends to $\pm\infty$.

Let g_n be the parabolic transformation with the fixed point p_{2n} which sends p_{2n-1} to p_{2n+1}. Then by the condition (2)

$$g_n(z)=\frac{\alpha_{2n}z-p_{2n}^2}{z-\alpha_{2n-1}}, \tag{4}$$

where $p_{2n}^2-\alpha_{2n-1}\alpha_{2n}=(\alpha_{2n-1}-\alpha_{2n})^2/4$. For a Möbius transformation g with $g(\infty) \neq \infty$ let $I(g)$ be the isometric circle of g (see Lehner [2, II, 10]). Then by (4)

$$I(g_n)=\partial B_{2n-1} \quad \text{and} \quad I(g_n^{-1})=\partial B_{2n}. \tag{5}$$

Let G_n be the cyclic group generated by g_n. Let G be the group generated by $\{g_n; n \in \boldsymbol{Z}\}$.

LEMMA 2.1. (i) *G is the free product of G_n $(n \in \boldsymbol{Z})$. In particular, G is infinitely generated.*

(ii) *G is a Kleinian group and the Ford region $F(G)$ coincides with $\hat{\boldsymbol{C}} \setminus [(\bigcup_{n \in \boldsymbol{Z}} B_n) \cup \{p_\infty\}]$.*

PROOF. The lemma follows from (1), (2), (5) and two theorems of Lehner ([2, p. 118]). q.e.d.

LEMMA 2.2. *The limit set $\Lambda(G)$ of G is a Jordan curve.*

Our proof of Lemma 2.2 is elementary but somewhat tedious. We give Lemmas 2.3 and 2.4, from which Lemma 2.2 follows. By Lemma 2.1 (i) each element $g \in G \setminus \{1\}$ has a unique expression as a reduced word in g_n $(n \in \boldsymbol{Z})$, that is, $g=g_{k_m}^{\varepsilon_m} \circ \cdots \circ g_{k_1}^{\varepsilon_1}$, where $m \geqq 1$, $\varepsilon_j=\pm 1$ and $k_j \in \boldsymbol{Z}$ $(j=1, \cdots, m)$ and $g_{k_j}^{\varepsilon_j} \neq g_{k_{j+1}}^{-\varepsilon_{j+1}}$ $(j=1, \cdots, m-1)$. The number m is called the length of g and is denoted by $l(g)$. For $g=1$ we define $l(g)=0$. Set $B(g_n)=B_{2n-1}$ and $B(g_n^{-1})=B_{2n}$. For $m \geqq 0$ set $\mathscr{B}_m=\{g(B(g_k^\varepsilon)); g \in G, l(g)=m, k \in \boldsymbol{Z}, \varepsilon=\pm 1, l(g \circ g_k^{-\varepsilon})=m+1\}$ and set $\mathscr{B}=\bigcup_{m=0}^{\infty} \mathscr{B}_m$. Then $\mathscr{B}_0=\{B_n\}_{n \in \boldsymbol{Z}}$. Set $\Lambda_1(G)=\{g(p_\infty);$

$g \in G\}$, $\Lambda_2(G)=\{g(p_n);\ g\in G, n\in \boldsymbol{Z}\}$ and $\Lambda_3(G)=\Lambda(G)\setminus(\Lambda_1(G)\cup\Lambda_2(G))$. A sequence of Jordan curves $\{J_n\}$ in $\boldsymbol{C}$ is said to nest about a point z if, for every n, J_{n+1} separates J_n and z, and if, for any choice of z_n on J_n, $\lim_{n\to\infty} z_n = z$ (see, for example, Maskit [4, p. 622]).

LEMMA 2.3. *For each $z\in\Lambda_3(G)$ there exists a sequence $\{B^{(j)}\}$ in $\mathscr{B}$ so that $\{\partial B^{(j)}\}$ nests about z.*

PROOF. The existence of a sequence $\{B^{(j)}\}$ in $\mathscr{B}$ such that $\partial B^{(j+1)}$ separates $\partial B^{(j)}$ and z follows from (1), (2), (5) and Lemma 2.1. Let $B^{(j)}=g^{(j)}(B_{n_j})$, where $g^{(j)}\in G$ and $B_{n_j}\in\mathscr{B}_0$. We may assume $g^{(j)}\neq 1$. By Lemma 2.1 (ii), $I(g)\subset\bigcup_{n\in\boldsymbol{Z}}B_n$ for all $g\in G\setminus\{1\}$. Hence $\partial B^{(j)}=g^{(j)}(\partial B_{n_j})\subset \mathrm{Cl}\,D((g^{(j)})^{-1})$, where $D((g^{(j)})^{-1})$ denotes the bounded component of $\boldsymbol{C}\setminus I((g^{(j)})^{-1})$. Since the radii of the isometric circles of distinct elements of $G\setminus\{1\}$ form a null sequence (Lehner [2, III, 1H]), the diameter of $\partial B^{(j)}$ tends to 0. q.e.d.

Let $\{\tilde{B}_n;\ n\in\boldsymbol{Z}\}$ be another family of closed disks which satisfies the conditions (1)–(3) with B_n, p_n, r_n, α_n and p_∞ replaced by the ones crowned with tildes. Define $\tilde{g}_n$, $\tilde{G}_n$, $\tilde{G}$, $B(\tilde{g}_n)$, $B(\tilde{g}_n^{-1})$, $\tilde{\mathscr{B}}_n$, $\tilde{\mathscr{B}}$ and $\Lambda_j(\tilde{G})$ $(1\leqq j\leqq 3)$ in the same way as the ones with the tildes removed. For $\tilde{g}=1$ define $\chi(\tilde{g})=1$. For $\tilde{g}=\tilde{g}_{k_m}^{\varepsilon_m}\circ\cdots\circ\tilde{g}_{k_1}^{\varepsilon_1}\in\tilde{G}\setminus\{1\}$ (a reduced word), where $m\geqq 1$, $\varepsilon_j=\pm 1$ and $k_j\in\boldsymbol{Z}$ $(j=1,\cdots,m)$, define $\chi(\tilde{g})=g_{k_m}^{\varepsilon_m}\circ\cdots\circ g_{k_1}^{\varepsilon_1}\in G$. Then by Lemma 2.1 (i), χ is a well-defined isomorphism of $\tilde{G}$ onto G. For $\tilde{B}=\tilde{g}(B(\tilde{g}_k^{\varepsilon}))\in\tilde{\mathscr{B}}_m$ $(m\geqq 0)$, where $\tilde{g}\in\tilde{G}$, $l(\tilde{g})=m$, $k\in\boldsymbol{Z}$, $\varepsilon=\pm 1$ and $l(\tilde{g}\circ\tilde{g}_k^{-\varepsilon})=m+1$, define $X(\tilde{B})=\chi(\tilde{g})(B(g_k^{\varepsilon}))\in\mathscr{B}_m$. Then X gives a one-to-one correspondence between the disks of $\tilde{\mathscr{B}}_m$ and those of $\mathscr{B}_m$ for each $m\geqq 0$.

LEMMA 2.4. *$\Lambda(\tilde{G})$ is homeomorphic to $\Lambda(G)$.*

PROOF. Define a mapping F of $\Lambda(\tilde{G})$ to $\Lambda(G)$ in the following way. For $z=\tilde{g}(\tilde{p}_\infty)\in\Lambda_1(\tilde{G})$ and $=\tilde{g}(\tilde{p}_n)\in\Lambda_2(\tilde{G})$ set $F(z)=\chi(\tilde{g})(p_\infty)\in\Lambda_1(G)$ and $=\chi(\tilde{g})(p_n)\in\Lambda_2(G)$, respectively. For $z\in\Lambda_3(\tilde{G})$ let $\{\tilde{B}^{(j)}\}$ be a sequence in $\tilde{\mathscr{B}}$ such that $\{\partial\tilde{B}^{(j)}\}$ nests about z. It follows from the proof of Lemma 2.3 that the sequence $\{\partial X(\tilde{B}^{(j)})\}$ also nests about the point $F(z)\in\Lambda_3(G)$, which is independent of the choice of $\{\tilde{B}^{(j)}\}$. Then F is a bijection of $\Lambda(\tilde{G})$ onto $\Lambda(G)$. Let $\tilde{g}\in\tilde{G}$ and $j, n\in\boldsymbol{Z}$ with $j\geqq 0$. Set

$$\tilde{N}_{1,j}(\tilde{g}(\tilde{p}_\infty)) = \Lambda(\tilde{G})\cap\ \tilde{g}([\textstyle\bigcup_{|m|>j}\tilde{B}_m]\cup\{\tilde{p}_\infty\}),\ \tilde{N}_{2,j}(\tilde{g}(\tilde{p}_{2n}))=\Lambda(\tilde{G})\cap\tilde{g}(\tilde{g}_n^{-j}(\tilde{B}_{2n-1})\cup\tilde{g}_n^{j}(\tilde{B}_{2n}))$$

and

$$\tilde{N}_{2,j}(\tilde{g}(\tilde{p}_{2n-1})) = \Lambda(\tilde{G})\cap\tilde{g}((\tilde{g}_{n-1}\circ\cdots\circ\tilde{g}_{n-j})(\tilde{B}_{2n-2j-2})\cup(\tilde{g}_n^{-1}\circ\cdots\circ\tilde{g}_{n+j-1}^{-1})(\tilde{B}_{2n+2j-1})).$$

Define $N_{1,j}(g(p_\infty))$, $N_{2,j}(g(p_{2n}))$ and $N_{2,j}(g(p_{2n-1}))$ similarly. For $z\in\Lambda_3(\tilde{G})$ set $\tilde{N}_{3,j}(z)=\Lambda(\tilde{G})\cap\tilde{B}^{(j)}$ and $N_{3,j}(F(z))=\Lambda(G)\cap X(\tilde{B}^{(j)})$. Then $\tilde{N}_{k,j}(z)$ $(1\leqq k\leqq 3)$ are closed neighborhoods of $z \in \Lambda_k(\tilde{G})$ in $\Lambda(\tilde{G})$, which become arbitrarily small as j tends to ∞. The same holds for the ones with the tildes removed. On the other hand, $F(\Lambda(\tilde{G}) \cap \tilde{B}) = \Lambda(G) \cap X(\tilde{B})$ for each $\tilde{B} \in \tilde{\mathscr{B}}$. Hence $F(\tilde{N}_{k,j}(z)) = N_{k,j}(F(z))$ for $z \in \Lambda_k(\tilde{G})$ $(1 \leqq k \leqq 3)$. Therefore F is a homeomorphism. q.e.d.

PROOF OF LEMMA 2.2. Choose an infinitely generated Fuchsian group of the first kind as $\tilde{G}$ in Lemma 2.4 (see, for example, $\tilde{G}$ in §4). Then $\Lambda(\tilde{G})$ is a circle. Therefore by Lemma 2.4 $\Lambda(G)$ is a Jordan curve. q.e.d.

3. Construction of a quasiconformal mapping. Let a, b and c be positive numbers. Let $3s = \min(a, b/c)$. Let $h[a, b, c]$ be monotone increasing diffeomorphisms of class C^1 of $[0, a]$ onto $[0, b]$ which satisfy the following.

$$h[a, b, c](a-\omega) = b - h[a, b, c](\omega) \quad \text{for} \quad \omega \in [0, a]. \tag{6}$$

$$h[a, b, c](\omega) = c\omega \quad \text{for} \quad \omega \in [0, s]. \tag{7}$$

$$h[a, a, 1](\omega) = \omega \quad \text{for} \quad \omega \in [0, a]. \tag{8}$$

(For example, let $h[a, b, c](\omega) = c\omega$ for $\omega \in [0, s]$, $= b - c(a-\omega)$ for $\omega \in [a-s, a]$ and $= \beta(\omega - a/2)\exp(\gamma(\omega - a/2)^2) + b/2$ for $\omega \in (s, a-s)$, where $1 + 2\gamma(a/2 - s)^2 = c(a-2s)/(b-2cs)$ and $b - 2cs = \beta(a-2s)\exp(\gamma(a/2-s)^2)$.) Let Y be the positive imaginary axis $\{iy; y>0\}$. For $r>0$ and $\theta \in (0, 2\pi)$, set $A(r, \theta) = \{r(\exp(i\omega)-1); \omega \in [0, \theta]\}$, $A'(r, \theta) = A(r, \theta) \setminus \{r(\exp(i\theta)-1)\}$, $L(r, \theta) = A(r, \theta) \cup Y$ and $W(r, \theta) = \{q \cdot \exp(i\omega) - r; q > r, 0 < \omega < \theta\} \cap \{z; \operatorname{Re} z < 0\}$. Let $r, \tilde{r} > 0$ and $\theta, \tilde{\theta} \in (0, 2\pi)$. Let h be a monotone increasing homeomorphism of $[0, \theta]$ onto $[0, \tilde{\theta}]$. Then the mapping defined by $r(\exp(i\omega)-1) \mapsto \tilde{r}(\exp(ih(\omega))-1)$ for $\omega \in [0, \theta]$ and $iy \mapsto iy$ for $y > 0$ is a homeomorphism of $L(r, \theta)$ onto $L(\tilde{r}, \tilde{\theta})$. Denote this mapping by $f[r, \theta; \tilde{r}, \tilde{\theta}; h]$.

LEMMA 3.1. *Let $r, \tilde{r} > 0$ and $\theta, \tilde{\theta} \in (0, 2\pi)$. Let $A' = A'(r, \theta)$, $\tilde{A}' = A'(\tilde{r}, \tilde{\theta})$, $L = L(r, \theta)$, $W = W(r, \theta)$ and $\tilde{W} = W(\tilde{r}, \tilde{\theta})$. Let $h = h[\theta, \tilde{\theta}, r/\tilde{r}]$* and *$f = f[r, \theta; \tilde{r}, \tilde{\theta}; h]$. Then there exist open neighborhoods U and $\tilde{U}$ of A' and $\tilde{A}'$ in* Cl W *and* Cl $\tilde{W}$, *respectively, and a homeomorphism $\hat{f}$ of U onto $\tilde{U}$ so that $\hat{f}$ is quasiconformal in $W \cap U$ and that $\hat{f} = f$ on $L \cap U$.*

PROOF. Let $v(z) = -2r/z$ and $\tilde{v}(z) = -2\tilde{r}/z$. Then $v(r(\exp(i\omega)-1)) = \tilde{v}(\tilde{r}(\exp(i\omega)-1)) = 1 + i \cdot t(\omega)$, where $t(\omega) = \cot(\omega/2)$. Define a mapping ψ_0 of $(0, \infty)$ onto itself by $\psi_0(y) = (\tilde{r}/r)y$. Define a mapping ψ_1 of $(t(\theta), \infty)$ onto $(t(\tilde{\theta}), \infty)$ by $\psi_1(t(\omega)) = t(h(\omega))$ for $\omega \in (0, \theta)$. Then we have

$$\tilde{v} \circ f \circ v^{-1}(iy) = i\psi_0(y) \quad \text{for} \quad y \in (0, \infty), \tag{9}$$

and

$$\tilde{v} \circ f \circ v^{-1}(1+iy) = 1 + i\psi_1(y) \quad \text{for} \quad y \in [t(\theta), \infty), \tag{10}$$

where

$$\psi_1(y) = t \circ h \circ t^{-1}(y) = \cot(h(2\,\mathrm{Arccot}\, y)/2)\,. \tag{11}$$

Both ψ_0 and ψ_1 are monotone increasing and satisfy

$$\psi'_0(y) = \frac{\tilde{r}}{r} \quad \text{and} \quad \psi'_1(y) = \frac{\sin^2(\omega(y)/2)}{\sin^2(h(\omega(y))/2)} \cdot h'(\omega(y))\,, \tag{12}$$

where $\omega(y) = 2\,\mathrm{Arccot}\, y$. Since $h(\omega) = (r/\tilde{r})\omega$ sufficiently near 0 by (7) and since $\cot(\beta\,\mathrm{Arccot}\, x) = (1/\beta)x + O(x^{-1})$ $(x \to \infty)$ for $\beta \neq 0$, (11) shows $\psi_1(y) = (\tilde{r}/r)y + O(y^{-1})$ $(y \to \infty)$. Hence

$$\lim_{y \to \infty} \{\psi_1(y) - \psi_0(y)\} = 0\,. \tag{13}$$

Let $N = \{x + iy;\ 0 \leqq x \leqq 1,\ y > t(\theta)\} \cup \{\infty\}$ and $\tilde{N} = \{x + iy;\ 0 \leqq x \leqq 1;\ y > x\psi_1(t(\theta)) + (1-x)\psi_0(t(\theta))\} \cup \{\infty\}$. Define a homeomorphism w of N onto $\tilde{N}$ by $w(x+iy) = x + i\{x\psi_1(y) + (1-x)\psi_0(y)\}$ and $w(\infty) = \infty$. Then

$$w(iy) = i\psi_0(y),\ w(1+iy) = 1 + i\psi_1(y)\,, \tag{14}$$

and $\mu[w] = (\partial w/\partial\bar{z})/(\partial w/\partial z)$ is given by

$$\mu[w](x+iy) = \frac{1 - x\psi'_1(y) - (1-x)\psi'_0(y) + i\{\psi_1(y) - \psi_0(y)\}}{1 + x\psi'_1(y) + (1-x)\psi'_0(y) + i\{\psi_1(y) - \psi_0(y)\}} \tag{15}$$

in $\mathrm{Int}\, N$. It follows from (12), (13) and (15) that

$$\lim_{y \to \infty} \mu[w](x+iy) = \frac{1 - x\psi'_1(\infty) - (1-x)\psi'_0(\infty)}{1 + x\psi'_1(\infty) + (1-x)\psi'_0(\infty)}\,, \tag{16}$$

uniformly in $x \in (0, 1)$, where $\psi'_0(\infty) = \psi'_1(\infty) = \tilde{r}/r$. By (12), (15) and (16), it holds that

$$\text{ess.}\sup\{\mu[w](z);\ z \in \mathrm{Int}\, N\} < 1\,. \tag{17}$$

Set $U = v^{-1}(N)$, $\tilde{U} = \tilde{v}^{-1}(\tilde{N})$ and $\hat{f} = \tilde{v}^{-1} \circ w \circ v$. Then U and $\tilde{U}$ are open neighborhoods of A' and $\tilde{A}'$ in $\mathrm{Cl}\, W$ and $\mathrm{Cl}\, \tilde{W}$, respectively, and $\hat{f}$ is a homeomorphism of U onto $\tilde{U}$. By (17), $\hat{f}$ is quasiconformal in $W \cap U$. By (9), (10) and (14), $\hat{f} = f$ on $L \cap U$. q.e.d.

Lemma 3.1 together with a theorem on quasiconformal mappings (Lehto-Virtanen [3, p. 45, Theorem 8.3]) yields the following lemma.

LEMMA 3.2. *Let $j = 1, 2$. Let $r_j, \tilde{r}_j > 0$ and $\theta_j, \tilde{\theta}_j \in (0, 2\pi)$. Let $A'_j = A'(r_j, \theta_j)$, $\tilde{A}'_j = A'(\tilde{r}_j, \tilde{\theta}_j)$, $W_j = W(r_j, \theta_j)$ and $\tilde{W}_j = W(\tilde{r}_j, \tilde{\theta}_j)$. Let $h_j = h[\theta_j, \tilde{\theta}_j, r_j/\tilde{r}_j]$ and $f_j = f[r_j, \theta_j; \tilde{r}_j, \tilde{\theta}_j; h_j]$. Let ρ be the reflection in the imaginary axis. Then there exist open neighborhoods U and $\tilde{U}$ of $A'_1 \cup \rho(A'_2)$ and $\tilde{A}'_1 \cup \rho(\tilde{A}'_2)$ in $\mathrm{Cl}[W_1 \cup \rho(W_2)]$ and $\mathrm{Cl}[\tilde{W}_1 \cup \rho(\tilde{W}_2)]$, respectively, and a homeomorphism $\hat{f}$ of U onto $\tilde{U}$ so that $\hat{f}$ is quasiconformal in $[W_1 \cup Y \cup \rho(W_2)] \cap U$* and that *$\hat{f} = f_1$ on A'_1 and $= \rho \circ f_2 \circ \rho$* on $\rho(A'_2)$.

4. Construction of groups. Let J be a Jordan curve in $\boldsymbol{C}$. The following lemma is well known (see, for example, Moise [5, Ch. 6, Theorem 2]).

LEMMA 4.1. *For each $\delta_1>0$ there exists a piecewise linear Jordan curve $K(\delta_1)$ with $[K(\delta_1), J]<\delta_1$.*

Let $\delta>0$. Let $K=K(\delta/3)$ be a piecewise linear Jordan curve in Lemma 4.1. Suppose that we obtain K by joining the points $v_0, v_1, \cdots, v_m=v_0$ by segments in this order. Choose interior points u_1 and u_2 of the segment v_1v_2 so close to each other that a circle Σ passing through u_1 and u_2 lies in the $(\delta/3)$-neighborhood of the segment u_1u_2. Let K' be a Jordan curve obtained by replacing the open segment u_1u_2 by a component Σ_1 of $\Sigma\setminus\{u_1, u_2\}$. Let p_∞ be a point of Σ_1. Let $v(K')=\{u_1, u_2, v_0, \cdots, v_{m-1}\}$. It is not difficult to construct a covering of $K'\setminus\{p_\infty\}$ by closed disks V_n $(n\in\boldsymbol{Z})$ which satisfy the following conditions.

(18) $V_{n-1}\cap V_n$ consists of one point p_{2n-1}, where $p_{2n-1}\in K'\setminus v(K')$. $V_m\cap V_n=\varnothing$ for $m\neq n, n\pm 1$.

(19) $d(V_n)<\delta/3$ and $d(V_n)$ (resp. the center of V_n) converges to 0 (resp. p_∞) as n tends to $\pm\infty$, where $d(V_n)$ denotes the diameter of V_n.

(20) ∂V_n intersects K' at exactly two points p_{2n-1} and p_{2n+1}, where ∂V_n and K' make right angles.

By the conditions (18) and (19) there exists an integer $N>0$ so that $p_{2n+1}\in\Sigma_1$ for all $|n|\geqq N$. Let $\tilde{p}_\infty=p_\infty$. Let $\tilde{V}_n=V_n$ for $|n|\geqq N+1$. Cover $\Sigma\setminus([\bigcup_{|n|>N}\tilde{V}_n]\cup\{p_\infty\})$ with $2N+1$ closed disks $\tilde{V}_n$ $(|n|\leqq N)$ so that the family $\{\tilde{V}_n\}_{n\in\boldsymbol{Z}}$ satisfies (18) and (20) with V_n, p_{2n-1} and $K'\setminus v(K')$ replaced by $\tilde{V}_n, \tilde{p}_{2n-1}$ and $\Sigma\setminus\{u_1, u_2\}$, respectively. In V_n there exist two closed disks $B_{2n-1}=B(\alpha_{2n-1}, r_{2n-1})$ and $B_{2n}=B(\alpha_{2n}, r_{2n})$ with $r_{2n-1}=r_{2n}$ so that $B_{2n-1}\cap B_{2n}$ consists of one point p_{2n} and that $B_{2n-1}\cap V_{n-1}=\{p_{2n-1}\}$ and $B_{2n}\cap V_{n+1}=\{p_{2n+1}\}$. Similarly there exist $\tilde{B}_n, \tilde{\alpha}_n, \tilde{r}_n$ and $\tilde{p}_{2n}$. Then the family $\mathscr{B}_0=\{B_n\}_{n\in\boldsymbol{Z}}$ (resp. $\tilde{\mathscr{B}}_0=\{\tilde{B}_n\}_{n\in\boldsymbol{Z}}$) satisfies the conditions (1)–(3) (resp. (1)–(3) with B_n, p_n, r_n, α_n and p_∞ replaced by the ones crowned with tildes). Also the following conditions are satisfied.

(21). $\partial\tilde{B}_n$ intersects Σ perpendicularly.

(22) $p_n=\tilde{p}_n$ for $|n|\geqq 2N+1$, and $\alpha_n=\tilde{\alpha}_n$ and $r_n=\tilde{r}_n$ for $|n+1/2|\geqq 2N+3/2$.

Define g_n, G_n and G (resp. $\tilde{g}_n$, $\tilde{G}_n$ and $\tilde{G}$) as in § 2 by using the family $\mathscr{B}_0$ (resp. $\tilde{\mathscr{B}}_0$). Then by Lemma 2.1 both G and $\tilde{G}$ are infinitely generated Kleinian groups.The condition (21) shows that each $\tilde{g}_n\in\tilde{G}$ keeps the bounded and unbounded components of $\hat{\boldsymbol{C}}\setminus\Sigma$ invariant. Hence $\tilde{G}$ is Fuchsian. Since the Ford region $F(\tilde{G})$ has no free sides, $\tilde{G}$ is of the first kind and $\Lambda(\tilde{G})=\Sigma$ (Lehner [2, p. 144]). Thus $\tilde{G}$ is an infinitely generated Fuchsian group of the first kind. On the other hand, $\Lambda(G)$ is contained in $(\bigcup_{n\in\boldsymbol{Z}}B_n)\cup\{p_\infty\}$ by Lemma 2.1 (ii) and is a Jordan curve by Lemma 2.2. Hence both K' and $\Lambda(G)$ are Jordan curves contained in $(\bigcup_{n\in\boldsymbol{Z}}V_n)\cup\{p_\infty\}$. This together with the condition (19) implies $[\Lambda(G), K']\leqq\delta/3$. Therefore $[\Lambda(G), J]\leqq[\Lambda(G), K']+[K', K]+$

$[K, J] < \delta$.

5. A quasiconformal mapping between the fundamental regions. Let G and $\tilde{G}$ be the groups in §4. Let Ω_1 and Ω_2 (resp. $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$) be the bounded and unbounded components of $\Omega(G)$ (resp. $\Omega(\tilde{G})$), respectively. Let $F = F(G)$ and $\tilde{F} = F(\tilde{G})$ be the Ford regions. Let $F_j = F \cap \Omega_j$ and $\tilde{F}_j = \tilde{F} \cap \tilde{\Omega}_j$ $(j = 1, 2)$. Then by Lemma 2.1 (ii), $\partial F_j = [\bigcup_{n \in \boldsymbol{Z}} (\partial B_n \cap \mathrm{Cl}\, \Omega_j)] \cup \{p_\infty\}$ and $\partial \tilde{F}_j = [\bigcup_{n \in \boldsymbol{Z}} (\partial \tilde{B}_n \cap \mathrm{Cl}\, \tilde{\Omega}_j)] \cup \{\tilde{p}_\infty\}$ $(j = 1, 2)$. In particular, ∂F_j and $\partial \tilde{F}_j$ are Jordan curves. The purpose of this section is to prove the following lemma.

LEMMA 5.1. *Let $j = 1$ or 2. Then there exists a homeomorphism $\hat{\varphi}_j$ of $\mathrm{Cl}\, F_j$ onto $\mathrm{Cl}\, \tilde{F}_j$ which is quasiconformal in F_j and which satisfies the following for all $n \in \boldsymbol{Z}$.*

$$\hat{\varphi}_j(p_\infty) = \tilde{p}_\infty \quad \textit{and} \quad \hat{\varphi}_j(p_n) = \tilde{p}_n \,. \tag{23}$$

$$\tilde{g}_n \circ \hat{\varphi}_j = \hat{\varphi}_j \circ g_n \qquad \textit{on} \quad \partial F_j \cap \partial B_{2n-1} \,. \tag{24}$$

First we prove the following lemma.

LEMMA 5.2. *Let D and $\tilde{D}$ be Jordan domains in $\hat{\boldsymbol{C}}$. Let ∂D and $\partial \tilde{D}$ be positively oriented with respect to D and $\tilde{D}$, respectively. Let φ be an orientation-preserving homeomorphism of ∂D onto $\partial \tilde{D}$. Suppose that for each point $\zeta \in \partial D$ there exist open neighborhoods U_ζ and $\tilde{U}_{\varphi(\zeta)}$ of ζ and $\varphi(\zeta)$, respectively, and a homeomorphism $\hat{\varphi}_\zeta$ of $(\mathrm{Cl}\, D) \cap U_\zeta$ onto $(\mathrm{Cl}\, \tilde{D}) \cap \tilde{U}_{\varphi(\zeta)}$ so that $\hat{\varphi}_\zeta$ is quasiconformal in $D \cap U_\zeta$ and that $\hat{\varphi}_\zeta = \varphi$ on $(\partial D) \cap U_\zeta$. Then there exists a homeomorphism $\hat{\varphi}$ of $\mathrm{Cl}\, D$ onto $\mathrm{Cl}\, \tilde{D}$ so that $\hat{\varphi}$ is quasiconformal in D and $\hat{\varphi} = \varphi$ on ∂D.*

PROOF. Let ξ and $\tilde{\xi}$ be conformal mappings of the open unit disk Δ onto D and $\tilde{D}$, respectively. Let $w = \tilde{\xi}^{-1} \circ \varphi \circ \xi$. Let $\partial \Delta$ be positively oriented with respect to Δ. Then w is an orientation-preserving homeomorphism of $\partial \Delta$ onto itself. By the assumption for each $z \in \partial \Delta$ there exist open neighborhoods U_z and $U_{w(z)}$ of z and $w(z)$, respectively, and a homeomorphism $\hat{w}_z$ of $(\mathrm{Cl}\, \Delta) \cap U_z$ onto $(\mathrm{Cl}\, \Delta) \cap \tilde{U}_{w(z)}$ so that $\hat{w}_z$ is quasiconformal in $\Delta \cap U_z$ and that $\hat{w}_z = w$ on $(\partial \Delta) \cap U_z$. By the reflection principle $\hat{w}_z$ can be extended to a quasiconformal mapping of $((\mathrm{Cl}\, \Delta) \cap \tilde{U}_z) \cup \{x;\ 1/\bar{x} \in \Delta \cap U_z\}$ (Lehto-Virtanen [3, p. 47]). Hence it follows from a theorem of Lehto-Virtanen ([3, Theorem II. 8.1]) and a theorem of Rickman ([6, Theorem 4]) that w has a quasiconformal extension $\hat{w}$ to $\hat{\boldsymbol{C}}$ with $\hat{w} = w$ on $\partial \Delta$. Since $\hat{w}$ is orientation-preserving, $\hat{w}$ maps Δ onto itself. Therefore $\hat{\varphi} = \tilde{\xi} \circ \hat{w} \circ \xi^{-1}$ is a required extension. q.e.d.

PROOF OF LEMMA 5.1. We assume $j = 1$. The proof for $j = 2$ is similar. First we construct a homeomorphism φ_1 of ∂F_1 onto $\partial \tilde{F}_1$ satisfying both (23) and (24) with $\hat{\varphi}_1$ replaced by φ_1. Next we show that φ_1 is extended to $\hat{\varphi}_1$.

We may assume, if necessary by replacing the suffices n of B_n (resp. $\tilde{B}_n$) by $-n$ for all $n \in \boldsymbol{Z}$, that F_1 (resp. $\tilde{F}_1$) lies on the left of the directed circular arc $p_n p_{n+1}$ of ∂F_1

(resp. $\tilde{p}_n\tilde{p}_{n+1}$ of $\partial\tilde{F}_1$). Let $p_n-\alpha_n=(p_{n+1}-\alpha_n)\exp(i\theta_n)$ and $\tilde{p}_n-\tilde{\alpha}_n=(\tilde{p}_{n+1}-\tilde{\alpha}_n)\exp(i\tilde{\theta}_n)$ ($\theta_n, \tilde{\theta}_n\in(0, 2\pi)$). Set $h_n=h[\theta_n, \tilde{\theta}_n, r_n/\tilde{r}_n]$ and $f_n=f[r_n, \theta_n; \tilde{r}_n, \tilde{\theta}_n; h_n]$ for $n\in\boldsymbol{Z}$ (see §3). Then f_n is a homeomorphism of $A_n=A(r_n, \theta_n)$ onto $\tilde{A}_n=A(\tilde{r}_n, \tilde{\theta}_n)$ with $f_n(0)=0$ and $f_n(r_n(\exp(i\theta_n)-1))=\tilde{r}_n(\exp(i\tilde{\theta}_n)-1)$. Let $\sigma_n(z)=-r_{n-1}(z-p_n)/(\alpha_{n-1}-p_n)$ and $\tilde{\sigma}_n(z)=-\tilde{r}_{n-1}(z-\tilde{p}_n)/(\tilde{\alpha}_{n-1}-\tilde{p}_n)$. Then $\sigma_n(p_n)=0, \sigma_n(\alpha_{n-1})=-r_{n-1}$ and $\sigma_n(\partial F_1\cap\partial B_{n-1})=A_{n-1}$. The same holds for the ones crowned with tildes. Set

$$f_n^*=\tilde{\sigma}_{n+1}^{-1}\circ f_n\circ\sigma_{n+1} \qquad \text{on} \quad \partial F_1\cap\partial B_n\,. \tag{25}$$

Then f_n^* is a homeomorphism of $\partial F_1\cap\partial B_n$ onto $\partial\tilde{F}_1\cap\partial\tilde{B}_n$ with $f_n^*(p_n)=\tilde{p}_n$ and $f_n^*(p_{n+1})=\tilde{p}_{n+1}$. Now define

$$\varphi_1(z)=\begin{cases} f_n^*(z) & \text{for} \quad z\in\partial F_1\cap\partial B_n \quad (n\in\boldsymbol{Z}) \\ \tilde{p}_\infty & \text{for} \quad z=p_\infty\,. \end{cases} \tag{26}$$

Then φ_1 is a homeomorphism of ∂F_1 onto $\partial\tilde{F}_1$ satisfying (23) with $\hat{\varphi}_1$ replaced by φ_1. Let $\tau_n(z)=-(z-r_n)\exp(i\theta_n)-r_n$ and $\tilde{\tau}_n(z)=-(z-\tilde{r}_n)\exp(i\tilde{\theta}_n)-\tilde{r}_n$. Since $\sigma_n(p_{n+1})=r_n(1-\exp(-i\theta_n))$ and $\sigma_n(\alpha_n)=r_n$, we have $\sigma_{n+1}=\tau_n\circ\sigma_n$. Similarly $\tilde{\sigma}_{n+1}=\tilde{\tau}_n\circ\tilde{\sigma}_n$. Then it follows from (6) and (25) that for $\omega\in[0, \theta_n]$

$$\begin{aligned} &\rho\circ\tilde{\sigma}_n\circ f_n^*\circ\sigma_n^{-1}\circ\rho(r_n(\exp(i\omega)-1))=\rho\circ\tilde{\tau}_n^{-1}\circ f_n\circ\tau_n\circ\rho(r_n(\exp(i\omega)-1)) \\ &=\rho\circ\tilde{\tau}_n^{-1}\circ f_n(r_n(\exp(i(\theta_n-\omega))-1))=\rho\circ\tilde{\tau}_n^{-1}(\tilde{r}_n(\exp(ih_n(\theta_n-\omega))-1)) \\ &=\rho\circ\tilde{\tau}_n^{-1}(\tilde{r}_n(\exp(i(\tilde{\theta}_n-h_n(\omega)))-1))=\tilde{r}_n(\exp(ih_n(\omega))-1)=f_n(r_n(\exp(i\omega)-1))\,, \end{aligned}$$

where ρ is the reflection in the imaginary axis. Hence

$$\rho\circ\tilde{\sigma}_n\circ f_n^*\circ\sigma_n^{-1}\circ\rho=f_n \qquad \text{on} \quad A_n\,. \tag{27}$$

By (2), $r_{2n-1}=r_{2n}, \theta_{2n-1}=\theta_{2n}, \tilde{r}_{2n-1}=\tilde{r}_{2n}$ and $\tilde{\theta}_{2n-1}=\tilde{\theta}_{2n}$. Hence $A_{2n-1}=A_{2n}$ and $f_{2n-1}=f_{2n}$. Therefore (27) shows that $\rho\circ\tilde{\sigma}_{2n}\circ f_{2n}^*\circ\sigma_{2n}^{-1}\circ\rho=f_{2n-1}$ on A_{2n-1}. On the other hand, by (5), $\sigma_{2n}\circ g_n=\rho\circ\sigma_{2n}$ on ∂B_{2n-1} and $\tilde{\sigma}_{2n}\circ\tilde{g}=\rho\circ\tilde{\sigma}_{2n}$ on $\partial\tilde{B}_{2n-1}$. Therefore we have $\tilde{g}_n\circ f_{2n-1}^*=f_{2n}^*\circ g_n$ on $\partial F_1\cap\partial B_{2n-1}$. This together with (26) shows that φ_1 satisfies (24) with $\hat{\varphi}_1$ replaced by φ_1.

Next we show that φ_1 is extended to $\mathrm{Cl}\,F_1$. Let ∂F_1 and $\partial\tilde{F}_1$ be positively oriented with respect to F_1 and $\tilde{F}_1$, respectively. Then φ_1 is orientation-preserving. Now by Lemma 5.2 it is sufficient to prove that the following (E_ζ) holds for each $\zeta\in\partial F_1$: (E_ζ) There exist neighborhoods U_ζ and $\tilde{U}_{\varphi_1(\zeta)}$ of ζ and $\varphi_1(\zeta)$ in $\mathrm{Cl}\,F_1$ and $\mathrm{Cl}\,\tilde{F}_1$, respectively, and a homeomorphism $\hat{\varphi}_{1,\zeta}$ of U_ζ onto $\tilde{U}_{\varphi(\zeta)}$ so that $\hat{\varphi}_{1,\zeta}$ is quasiconformal in $F_1\cap\mathrm{Int}\,U_\zeta$ and that $\hat{\varphi}_{1,\zeta}=\varphi_1$ on $(\partial F_1)\cap U_\zeta$. First let $\zeta\in(\partial F_1)\cap\Omega_1$. Then $\zeta\in(\partial F_1\cap\partial B_n)\setminus\{p_n, p_{n+1}\}$ for some $n\in\boldsymbol{Z}$. Hence $\sigma_{n+1}^{-1}(\zeta)$ is, in particular, a point of A'_n. By (25) and (26), $\sigma_{n+1}\circ\varphi_1\circ\sigma_{n+1}^{-1}=f_n$ on A_n. Therefore Lemma 3.1 shows that (E_ζ) holds.

Secondly let $\zeta=p_n$ for some $n\in\boldsymbol{Z}$. Since $\sigma_n(\partial F_1\cap\partial B_{n-1})=A_{n-1}$ and $\sigma_n(\partial F_1\cap\partial B_n)=\rho(A_n)$, (25), (26) and (27) show $\tilde{\sigma}_n\circ\varphi_1\circ\sigma_n^{-1}=f_{n-1}$ on A_{n-1} and $=\rho\circ f_n\circ\rho$ on $\rho(A_n)$. Hence Lemma 3.2 shows that (E_ζ) holds.

Finally let $\zeta = p_\infty$. By (22), $\sigma_{n+1} = \tilde{\sigma}_{n+1}$ for all n with $|n+1/2| \geqq 2N+3/2$. By (8) and (22), $f_n(z) = z$ for $z \in A_n$ with $|n+1/2| \geqq 2N+3/2$. Hence by (25) and (26) there exists a neighborhood U_ζ of ζ in $\mathrm{Cl}\, F_1$ so that $\varphi_1(z) = z$ for $z \in (\partial F_1) \cap U_\zeta$. Let $\tilde{U}_{\varphi(\zeta)} = U_\zeta$ and $\hat{\varphi}_{1,\zeta}$ be the identity mapping. Then (E_ζ) holds. q.e.d.

6. Proof of Theorem B. Let G and $\tilde{G}$ be the groups constructed in §4. Then G is an infinitely generated Kleinian group whose limit set $\Lambda(G)$ is a Jordan curve with $[\Lambda(G), J] < \delta$ and $\tilde{G}$ is an infinitely generated Fuchsian group of the first kind. Let χ be the isomorphism of $\tilde{G}$ onto G defined in §2. Let $j = 1$ or 2. Let Ω_j, $\tilde{\Omega}_j$, F_j and $\tilde{F}_j$ be as in §5. Let $\hat{\varphi}_j$ be the mapping in Lemma 5.1. Define a mapping Φ_j of $\bigcup_{\tilde{g} \in \tilde{G}} \tilde{g}(\mathrm{Cl}\, \tilde{F}_j)$ $(\supset \tilde{\Omega}_j)$ by

$$\Phi_j = \chi(\tilde{g})^{-1} \circ \hat{\varphi}_j^{-1} \circ \tilde{g} \qquad \text{on} \quad \tilde{g}^{-1}(\mathrm{Cl}\, \tilde{F}_j) \quad (\tilde{g} \in \tilde{G}) . \tag{28}$$

By Lemma 5.1, Φ_j is a well-defined homeomorphism of $\tilde{\Omega}_j$ onto Ω_j which is quasiconformal off the set $\bigcup_{\tilde{g} \in \tilde{G}} \tilde{g}(\partial \tilde{F}_j)$. Hence Φ_j is a quasiconformal mapping of $\tilde{\Omega}_j$ onto Ω_j by a theorem of Lehto-Virtanen ([3, p. 45, Theorem 8.3]). Since $\Lambda(\tilde{G})$ and $\Lambda(G)$ are Jordan curves, Φ_j can be extended to a homeomorphism of $\mathrm{Cl}\, \tilde{\Omega}_j$ onto $\mathrm{Cl}\, \Omega_j$. By (23) and (28), $\Phi_1 = \Phi_2$ on the set $\bigcup_{\tilde{g} \in \tilde{G}} \tilde{g}(\{\tilde{p}_\infty\} \cup \{\tilde{p}_n;\, n \in \boldsymbol{Z}\})$, which is dense in $\Lambda(\tilde{G})$ by a theorem of Lehner ([2, p. 102]). Hence $\Phi_1 = \Phi_2$ on $\Lambda(\tilde{G})$. Set $\Phi = \Phi_j$ on $\mathrm{Cl}\, \tilde{\Omega}_j$ $(j = 1, 2)$. Then Φ is a homeomorphism of $\hat{\boldsymbol{C}}$ onto itself which is quasiconformal off the circle $\Lambda(\tilde{G})$. Hence Φ is a quasiconformal automorphism of $\hat{\boldsymbol{C}}$. On the other hand, it follows from (28) that $\chi(\tilde{g}) \circ \Phi = \Phi \circ \tilde{g}$ $(\tilde{g} \in \tilde{G})$ on $\Omega(\tilde{G})$, hence, by continuity, on $\hat{\boldsymbol{C}}$. Therefore G is a quasiconformal deformation of $\tilde{G}$. q.e.d.

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