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THE GAUSS MAP AND SPACELIKE SURFACES WITH PRESCRIBED MEAN CURVATURE IN MINKOWSKI 3-SPACE

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For an oriented spacelike surface M in Minkowski 3-space L^3, the Gauss map G is defined to be a mapping of M into the unit pseudosphere $\boldsymbol{H}$ in L^3, which assigns to each point p of M the point in $\boldsymbol{H}$ obtained by translating the timelike unit normal vector at p to the origin. Our primary object of this paper is to prove a representation formula for spacelike surfaces with prescribed mean curvature in terms of their Gauss maps.

It is well-known that the classical Weierstrass-Enneper representation formula describes minimal surfaces in Euclidean 3-space $\boldsymbol{R}^3$ in terms of their Gauss maps and auxiliary holomorphic functions ([8]). More generally, a remarkable representation formula has been discovered by Kenmotsu [3] for arbitrary surfaces in $\boldsymbol{R}^3$ with nonvanishing mean curvature, which describes these surfaces in terms of their Gauss maps and mean curvature functions. On the other hand, Kobayashi [4, 5] proved the Lorentzian version of the classical Weierstrass-Enneper representation formula for maximal surfaces in Minkowski 3-space L^3 (see also McNertney [10]) and applied it to the study of maximal surfaces with conelike singularities.

Motivated by these results, we shall prove, in §4 of this paper, that arbitrary oriented spacelike surfaces in L^3 satisfy a system of first order partial differential equations involving the mean curvature function H and the Gauss map G of the surface (Theorem 4.1). An interesting feature therein is that the complete integrability condition for the formula then yields a system of nonlinear second order partial differential equations which identifies the gradient of H and the tension field of G (Proposition 5.3). In particular, the condition simply means that the Gauss map G should be a harmonic mapping provided the mean curvature H is constant.

The converse of these observations will be discussed in §6. Our main result is that given a nowhere holomorphic smooth mapping G of a simply connected Riemman surface M into the pseudosphere $\boldsymbol{H}$ satisfying the complete integrability condition for some nonvanishing smooth function H on M, we can construct explicitly a spacelike immersion of M into L^3 such that the mean curvature of M is H and the Gauss map of M is given by G (Theorem 6.1). This allows us, in particular, to produce a wealth of spacelike surfaces of constant mean curvature in L^3, and more importantly, to relate

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the geometry of these surfaces to the theory of harmonic mappings through their Gauss maps.

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1. Preliminaries. We begin with fixing our terminology and notation. Let $L^3=(\boldsymbol{R}^3, \bar{g})$ denote Minkowski 3-space with flat Lorentzian metric $\bar{g}$ of signature $(+,+,-)$. In terms of the canonical coordinates (x^1, x^2, x^3) of $\boldsymbol{R}^3$, the metric $\bar{g}$, denoted also by $\langle\ ,\ \rangle$, can be expressed as $\bar{g}=(dx^1)^2+(dx^2)^2-(dx^3)^2$. Let M^2 be a connected smooth 2-manifold, and $X\colon M^2\to L^3$ be a smooth immersion of M^2 into L^3. Throughout this paper, we assume that X is a *spacelike immersion* or M^2 is a *spacelike surface* in L^3, that is, the pull back $X^*\bar{g}$ of the Lorentzian metric $\bar{g}$ via X is a positive definite metric on M^2 (cf. [1, 7]). Also, we always assume that M is orientable. It should be remarked that there exists no closed spacelike surface in L^3. Indeed, otherwise the Euclidean normal directions of the surface would all make an angle of more than $\pi/4$ with the horizontal plane, contradicting the fact that a closed surface in $\boldsymbol{R}^3$ has Euclidean normals in all directions.

Let $M=(M^2, g)$ denote the Riemannian 2-manifold M^2 with induced metric $g=X^*\bar{g}$ so that $X\colon M^2\to L^3$ is an isometric immersion. By $\xi=(\xi^1, \xi^2)$ we always denote an isothermal coordinates compatible with the orientation on M, by which g is expressed locally as

$$g=\lambda^2((d\xi^1)^2+(d\xi^2)^2)\,, \qquad \lambda>0\,. \tag{1.1}$$

It is well-known that (ξ^1, ξ^2) is defined around each point of M, and we may regard M as a Riemann surface by introducing complex coordinates by $z=\xi^1+\sqrt{-1}\xi^2$.

We shall define a local Lorentzian frame field (e_1, e_2, e_3) adapted to M in L^3 in the following manner. Let $X(\xi)=(X^1(\xi^1, \xi^2), X^2(\xi^1, \xi^2), X^3(\xi^1, \xi^2))$ be a local expression of the immersion X with respect to an isothermal coordinates (ξ^1, ξ^2) on M. For $i=1$, 2, let

$$e_i=\frac{1}{\lambda}\frac{\partial X}{\partial \xi^i}=\frac{1}{\lambda}\left(\frac{\partial X^1}{\partial \xi^i}, \frac{\partial X^2}{\partial \xi^i}, \frac{\partial X^3}{\partial \xi^i}\right).$$

Then (e_1, e_2) defines an orthonormal tangent frame field on M compatible with the orientation. We then define $e_3=e_1\times e_2$. Here the exterior product $v\times w$ of two vectors v, w in L^3 is defined by $v\times w=-(\iota_w\iota_v dx^1\wedge dx^2\wedge dx^3)^\#$, ι_v and $\#$ denoting the interior product with respect to v and the operation of raising indices by the metric $\bar{g}$, respectively. Note that e_3 is timelike and defines a (Lorentzian) unit normal vector field on M, that is, $\langle e_3, e_3\rangle=-1$ and $\langle e_3, e_i\rangle=0$ for $i=1$, 2. In terms of local coordinates, e_3 is given explicitly by

$$(1.2)\qquad e_3=\frac{1}{\lambda^2}\left(\frac{\partial X^3}{\partial\xi^1}\frac{\partial X^2}{\partial\xi^2}-\frac{\partial X^2}{\partial\xi^1}\frac{\partial X^3}{\partial\xi^2},\ \frac{\partial X^1}{\partial\xi^1}\frac{\partial X^3}{\partial\xi^2}-\frac{\partial X^3}{\partial\xi^1}\frac{\partial X^1}{\partial\xi^2},\ \frac{\partial X^1}{\partial\xi^1}\frac{\partial X^2}{\partial\xi^2}-\frac{\partial X^2}{\partial\xi^1}\frac{\partial X^1}{\partial\xi^2}\right).$$

It should be noted that $\partial/\partial x^1\times\partial/\partial x^2=\partial/\partial x^3$, $\partial/\partial x^2\times\partial/\partial x^3=-\partial/\partial x^1$ and $\partial/\partial x^3\times\partial/\partial x^1=-\partial/\partial x^2$ due to our sign convention for the exterior product.

Let h denote the second fundamental form of M in L^3 (cf. [1, 7]). With respect to a Lorentzian frame field (e_1, e_2, e_3), h is represented by the matrix $(h_{ij})_{1\leq i,j\leq 2}$, where

$$h_{ij}=-\langle D_{e_i}e_j, e_3\rangle\,,$$

D denoting covariant differentiation in L^3. Then, by an elementary calculation, we see that the fundamental formulas of Gauss and Weingarten for M in L^3 are given as follows:

$$(1.3)\qquad \begin{aligned}\frac{\partial^2X}{\partial\xi^1\partial\xi^1}&=\frac{1}{\lambda}\frac{\partial\lambda}{\partial\xi^1}\frac{\partial X}{\partial\xi^1}-\frac{1}{\lambda}\frac{\partial\lambda}{\partial\xi^2}\frac{\partial X}{\partial\xi^2}+\lambda^2h_{11}e_3\,,\\ \frac{\partial^2X}{\partial\xi^1\partial\xi^2}&=\frac{1}{\lambda}\frac{\partial\lambda}{\partial\xi^2}\frac{\partial X}{\partial\xi^1}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial\xi^1}\frac{\partial X}{\partial\xi^2}+\lambda^2h_{12}e_3\,,\\ \frac{\partial^2X}{\partial\xi^2\partial\xi^2}&=-\frac{1}{\lambda}\frac{\partial\lambda}{\partial\xi^1}\frac{\partial X}{\partial\xi^1}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial\xi^2}\frac{\partial X}{\partial\xi^2}+\lambda^2h_{22}e_3\,.\end{aligned}$$

$$(1.4)\qquad \begin{aligned}\frac{\partial e_3}{\partial\xi^1}&=h_{11}\frac{\partial X}{\partial\xi^1}+h_{12}\frac{\partial X}{\partial\xi^2}\,,\\ \frac{\partial e_3}{\partial\xi^2}&=h_{21}\frac{\partial X}{\partial\xi^1}+h_{22}\frac{\partial X}{\partial\xi^2}\,.\end{aligned}$$

The mean curvature H of M is defined to be $H=(h_{11}+h_{22})/2$. If H vanishes identically on M, then M is said to be *maximal*. It is easy to see from (1.3) that M is maximal if and only if each component function of the immersion X is harmonic on M.

Let $\phi=(1/2)(h_{11}-h_{22})-\sqrt{-1}h_{12}$, which represents, up to a factor, the (2,0)-part of the complexification of the second fundamental form h of M. Then from (1.4) we have

$$(1.5)\qquad \frac{\partial e_3}{\partial z}=H\frac{\partial X}{\partial z}+\phi\frac{\partial X}{\partial\bar z}\,,$$

where we set $\partial/\partial z=(1/2)(\partial/\partial\xi^1-\sqrt{-1}\,\partial/\partial\xi^2)$ and $\partial/\partial\bar z=(1/2)(\partial/\partial\xi^1+\sqrt{-1}\,\partial/\partial\xi^2)$. Note that $\phi(p)=0$ at a point $p\in M$ if and only if p is an umbilical point of M. It is also not difficult to see that the Gaussian curvature K of M is given by

$$(1.6)\qquad K=-H^2+|\phi|^2\,,$$

for $K=-(h_{11}h_{22}-h_{12}^2)$ by the equation of Gauss.

2. The Gauss map. For a spacelike surface M in L^3, the *Gauss map* G of M is by definition a mapping of M into L^3, which assigns to each point $p \in M$ the point in L^3 obtained by translating parallelly the unit normal vector $e_3(p)$ of M at p to the origin of L^3 (cf. [1, 7]). Note that, since $e_3(p)$ is a timelike unit vector at $p \in L^3$, the Gauss map G is in fact a mapping of M into the unit pseudosphere $\boldsymbol{H}$ in L^3. That is, the image of G is contained in a spacelike surface $\boldsymbol{H}$ in L^3 defined by

$$\boldsymbol{H} = \{(x^1, x^2, x^3) \in L^3 \mid (x^1)^2 + (x^2)^2 - (x^3)^2 = -1\},$$

which is a two-sheeted hyperboloid in L^3, and has constant Gaussian curvature $K \equiv -1$ with respect to the induced metric.

On $\boldsymbol{H}$ we may define a natural complex structure in the following manner. Let $U_1 = \boldsymbol{H} - \{(0, 0, 1)\}$ and $U_2 = \boldsymbol{H} - \{(0, 0, -1)\}$, and introduce complex coordinates by means of stereographic mappings $\psi_1 \colon U_1 \to \boldsymbol{C}$ and $\psi_2 \colon U_2 \to \boldsymbol{C}$, which are defined respectively by

$$(2.1)\qquad \begin{aligned} \psi_1(x) &= \frac{x^1 + \sqrt{-1}\,x^2}{1 - x^3}, \qquad x = (x^1, x^2, x^3) \in U_1, \\ \psi_2(x) &= \frac{x^1 - \sqrt{-1}\,x^2}{1 + x^3}, \qquad x = (x^1, x^2, x^3) \in U_2. \end{aligned}$$

In fact, $\psi_1(x)$ is the intersection of the line joining $x \in U_1$ and the north pole $(0, 0, 1) \in \boldsymbol{H}$, and the (x^1, x^2)-plane identified with $\boldsymbol{C}$ by setting $\zeta = x^1 + \sqrt{-1}\,x^2$. Similarly, ψ_2 represents the stereographic mapping from the south pole $(0, 0, -1) \in \boldsymbol{H}$. It should be noted that the images of ψ_1 and ψ_2 are contained in the set $\boldsymbol{C} - \{|\zeta| = 1\}$, and the inverse mappings ψ_1^{-1} and ψ_2^{-1} of ψ_1 and ψ_2 are given respectively by

$$(2.2)\qquad \begin{aligned} \psi_1^{-1}(\zeta) &= \left(\frac{2\,\mathrm{Re}\,\zeta}{1 - |\zeta|^2}, \frac{2\,\mathrm{Im}\,\zeta}{1 - |\zeta|^2}, -\frac{1 + |\zeta|^2}{1 - |\zeta|^2}\right), \quad \zeta \in \boldsymbol{C} - \{|\zeta| = 1\}, \\ \psi_2^{-1}(\zeta) &= \left(\frac{2\,\mathrm{Re}\,\zeta}{1 - |\zeta|^2}, -\frac{2\,\mathrm{Im}\,\zeta}{1 - |\zeta|^2}, \frac{1 + |\zeta|^2}{1 - |\zeta|^2}\right), \quad \zeta \in \boldsymbol{C} - \{|\zeta| = 1\}. \end{aligned}$$

It is then immediate to see that $\psi_1(x)\psi_2(x) = -1$ for $x \in U_1 \cap U_2$, and $\{\psi_1, \psi_2\}$ defines a complex structure on $\boldsymbol{H}$, since $\psi_2 \circ \psi_1^{-1}(\zeta) = -1/\zeta$ and $\psi_1 \circ \psi_2^{-1}(\zeta) = -1/\zeta$. It is also not difficult to see that ψ_1 and ψ_2 are conformal with respect to the induced metric on $\boldsymbol{H}$ and the flat metric on $\boldsymbol{C}$. (Indeed, the induced metric on $\boldsymbol{H}$ can be written as $4|d\zeta|^2/(1 - |\zeta|^2)^2$, ζ being complex coordinates defined by stereographic mappings.)

In consequence, we obtain the following sequence of mappings:

$$M \xrightarrow{\ G\ } \boldsymbol{H} \subset L^3 \xrightarrow{\ \psi_i\ } \boldsymbol{C} - \{|\zeta| = 1\}, \qquad i = 1, 2.$$

We often refer to the composite mapping $\Psi_i = \psi_i \circ G$ for $i = 1, 2$ also as the Gauss map of M (into $\boldsymbol{C}$). Moreover, we omit the subscript i in Ψ_i, and write simply as Ψ, if there is no confusion or if the statement under consideration holds for both Ψ_i.

3. Beltrami equation. Let M be a spacelike surface immersed in L^3 by a mapping $X: M \to L^3$, and Ψ denote the Gauss map of M into $\boldsymbol{C}$ as in §2. The goal of this section is to prove that Ψ satisfies a Beltrami equation. To start with, we prove the following lemma.

LEMMA 3.1. *If $X=(X^1, X^2, X^3): M \to L^3$ is a spacelike immersion, then*

$$\frac{\partial(X^1+\sqrt{-1}X^2)}{\partial z} = -\Psi_1 \frac{\partial X^3}{\partial z}, \tag{3.1}$$

$$\frac{\partial X^3}{\partial z} = -\Psi_1 \frac{\partial(X^1-\sqrt{-1}X^2)}{\partial z}, \tag{3.2}$$

$$\frac{\partial X^3}{\partial z}\frac{\partial(X^1+\sqrt{-1}X^2)}{\partial \bar{z}} = -\frac{\lambda^2 \Psi_1}{(1-|\Psi_1|^2)^2}. \tag{3.3}$$

PROOF. Since $z=\xi^1+\sqrt{-1}\xi^2$ for which (ξ^1, ξ^2) is an isothermal coordinates on M, it follows from (1.1) that

$$\left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial \bar{z}} \right\rangle = \frac{\lambda^2}{2}, \qquad \left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial z} \right\rangle = \left\langle \frac{\partial X}{\partial \bar{z}}, \frac{\partial X}{\partial \bar{z}} \right\rangle = 0. \tag{3.4}$$

On the other hand, if (e_1, e_2, e_3) is a Lorentzian frame field adapted to M in L^3, then we see from (1.2)

$$e_3 = -\frac{2\sqrt{-1}}{\lambda^2}\left(\frac{\partial X^3}{\partial z}\frac{\partial X^2}{\partial \bar{z}} - \frac{\partial X^2}{\partial z}\frac{\partial X^3}{\partial \bar{z}}, \frac{\partial X^1}{\partial z}\frac{\partial X^3}{\partial \bar{z}} - \frac{\partial X^3}{\partial z}\frac{\partial X^1}{\partial \bar{z}}, \frac{\partial X^1}{\partial z}\frac{\partial X^2}{\partial \bar{z}} - \frac{\partial X^2}{\partial z}\frac{\partial X^1}{\partial \bar{z}}\right), \tag{3.5}$$

and also from (2.1)

$$\Psi_1 = \frac{e_3^1+\sqrt{-1}e_3^2}{1-e_3^3}, \tag{3.6}$$

$$(1-|\Psi_1|^2)(1-e_3^3)=2, \tag{3.7}$$

where we put $e_3=(e_3^1, e_3^2, e_3^3)$. On substituting (3.5) into (3.6), and making use of (3.4) and (3.7), we can then check (3.1), (3.2) and (3.3) without difficulty by a straightforward calculation.

We shall now compute the derivatives of the Gauss map Ψ. First we prove:

PROPOSITION 3.2. *The complex derivatives of the Gauss map Ψ_1 are given by*

$$\frac{\partial \Psi_1}{\partial \bar{z}} = \frac{H}{2}(1-|\Psi_1|^2)^2 \frac{\partial (X^1+\sqrt{-1}X^2)}{\partial \bar{z}}, \tag{3.8}$$

$$\frac{\partial \Psi_1}{\partial z} = \frac{\phi}{2}(1-|\Psi_1|^2)^2 \frac{\partial (X^1+\sqrt{-1}X^2)}{\partial \bar{z}}. \tag{3.9}$$

PROOF. Differentiating (3.6) with respect to $\bar{z}$ and applying (1.5), we get

$$\frac{\partial \Psi_1}{\partial \bar{z}} = \frac{1}{1-e_3^3}\left[H\frac{\partial X^1}{\partial \bar{z}} + \bar{\phi}\frac{\partial X^1}{\partial z} + \sqrt{-1}\left(H\frac{\partial X^2}{\partial \bar{z}} + \bar{\phi}\frac{\partial X^2}{\partial z}\right)\right] + \frac{1}{1-e_3^3}\Psi_1\left[H\frac{\partial X^3}{\partial \bar{z}} + \bar{\phi}\frac{\partial X^3}{\partial z}\right].$$

Then, by (3.1) and (3.2) together with (3.7), it is verified that

$$\begin{aligned}\frac{\partial \Psi_1}{\partial \bar{z}} &= \frac{1-|\Psi_1|^2}{2}\left[H\frac{\partial (X^1+\sqrt{-1}X^2)}{\partial \bar{z}} + \bar{\phi}\frac{\partial (X^1+\sqrt{-1}X^2)}{\partial z}\right] \\ &\quad - \frac{1-|\Psi_1|^2}{2}\left[|\Psi_1|^2 H\frac{\partial (X^1+\sqrt{-1}X^2)}{\partial \bar{z}} + \bar{\phi}\frac{\partial (X^1+\sqrt{-1}X^2)}{\partial z}\right] \\ &= \frac{H}{2}(1-|\Psi_1|^2)^2\frac{\partial (X^1+\sqrt{-1}X^2)}{\partial \bar{z}},\end{aligned}$$

thus proving (3.8). (3.9) can be proved in a similar fashion.

By the same argument we can also prove the following

PROPOSITION 3.3. *The complex derivatives of the Gauss map Ψ_2 are given by*

$$\frac{\partial \Psi_2}{\partial \bar{z}} = \frac{H}{2}(1-|\Psi_2|^2)^2 \frac{\partial (X^1-\sqrt{-1}X^2)}{\partial \bar{z}}, \tag{3.10}$$

$$\frac{\partial \Psi_2}{\partial z} = \frac{\phi}{2}(1-|\Psi_2|^2)^2 \frac{\partial (X^1-\sqrt{-1}X^2)}{\partial \bar{z}}. \tag{3.11}$$

From these propositions the following theorem is now immediate.

THEOREM 3.4. *The Gauss map Ψ of a spacelike surface M in L^3 satisfies a Beltrami equation*

$$H\frac{\partial \Psi}{\partial z} = \phi\frac{\partial \Psi}{\partial \bar{z}}. \tag{3.12}$$

It is well-known that the Gauss map of a minimal surface in Euclidean 3-space is a holomorphic mapping into the Riemann sphere (cf. [8]). In connection with this, we may point out the following

PROPOSITION 3.5. *Let M be a spacelike surface in L^3. Then at $p\in M$*

$$H(p)=0 \Longleftrightarrow \frac{\partial\Psi}{\partial\bar{z}}(p)=0\,, \tag{3.13}$$

$$\phi(p)=0 \Longleftrightarrow \frac{\partial\Psi}{\partial z}(p)=0\,. \tag{3.14}$$

PROOF. It is verified from Lemma 3.1 that

$$\left|\frac{\partial(X^1+\sqrt{-1}X^2)}{\partial\bar{z}}\right|=\frac{\lambda}{|1-|\Psi_1|^2|}\,. \tag{3.15}$$

Hence from Proposition 3.2 we get

$$\left|\frac{\partial\Psi_1}{\partial\bar{z}}\right|=\alpha|H|\,, \qquad \left|\frac{\partial\Psi_1}{\partial z}\right|=\alpha|\phi|\,, \tag{3.16}$$

where $\alpha=\lambda|1-|\Psi_1|^2|/2$. Since $\alpha\neq 0$, this proves the proposition when $p\in\Psi_1^{-1}(C)$. The proof for the case $p\in\Psi_2^{-1}(C)$ is similar.

4. Representation formula. Given a spacelike surface M in L^3, we shall now prove a representation formula for M in terms of the Gauss map Ψ and the mean curvature H of M.

THEOREM 4.1. *Let M be a spacelike surface immersed in L^3 by a mapping $X=(X^1, X^2, X^3): M\to L^3$. Let H and Ψ_i $(i=1,2)$ denote the mean curvature function of M and the Gauss map of M into C defined in §2, respectively. Then the following hold.*

(1) *On $\Psi_1^{-1}(C)$, we have*

$$\begin{aligned} H\frac{\partial X^1}{\partial z}&=\frac{1+\Psi_1^2}{(1-|\Psi_1|^2)^2}\frac{\partial\bar{\Psi}_1}{\partial z}\,,\\ H\frac{\partial X^2}{\partial z}&=\sqrt{-1}\frac{1-\Psi_1^2}{(1-|\Psi_1|^2)^2}\frac{\partial\bar{\Psi}_1}{\partial z}\,,\\ H\frac{\partial X^3}{\partial z}&=-2\frac{\Psi_1}{(1-|\Psi_1|^2)^2}\frac{\partial\bar{\Psi}_1}{\partial z}\,. \end{aligned} \tag{4.1}$$

(2) *On $\Psi_2^{-1}(C)$, we have*

$$\begin{aligned} H\frac{\partial X^1}{\partial z}&=\frac{1+\Psi_2^2}{(1-|\Psi_2|^2)^2}\frac{\partial\bar{\Psi}_2}{\partial z}\,,\\ H\frac{\partial X^2}{\partial z}&=-\sqrt{-1}\frac{1-\Psi_2^2}{(1-|\Psi_2|^2)^2}\frac{\partial\bar{\Psi}_2}{\partial z}\,, \end{aligned} \tag{4.2}$$

$$H\frac{\partial X^3}{\partial z}=2\frac{\Psi_2}{(1-|\Psi_2|^2)^2}\frac{\partial\bar{\Psi}_2}{\partial z}.$$

PROOF. (1) Recall that by (3.8) we have

$$\frac{\partial\bar{\Psi}_1}{\partial z}=\frac{H}{2}(1-|\Psi_1|^2)^2\frac{\partial(X^1-\sqrt{-1}X^2)}{\partial z} \tag{4.3}$$

on $\Psi_1^{-1}(C)$. From (3.1) and (3.2) it then follows that

$$\Psi_1^2\frac{\partial\bar{\Psi}_1}{\partial z}=\frac{H}{2}(1-|\Psi_1|^2)^2\frac{\partial(X^1+\sqrt{-1}X^2)}{\partial z}. \tag{4.4}$$

Hence, by adding (4.4) to (4.3), we get

$$(1+\Psi_1^2)\frac{\partial\bar{\Psi}_1}{\partial z}=H(1-|\Psi_1|^2)^2\frac{\partial X^1}{\partial z},$$

and, by subtracting (4.4) from (4.3),

$$(1-\Psi_1^2)\frac{\partial\bar{\Psi}_1}{\partial z}=-\sqrt{-1}H(1-|\Psi_1|^2)^2\frac{\partial X^2}{\partial z}.$$

Since $1-|\Psi_1|^2\neq 0$, it follows from these that on $\Psi_1^{-1}(C)$

$$H\frac{\partial X^1}{\partial z}=\frac{1+\Psi_1^2}{(1-|\Psi_1|^2)^2}\frac{\partial\bar{\Psi}_1}{\partial z}, \tag{4.5}$$

$$H\frac{\partial X^2}{\partial z}=\sqrt{-1}\frac{1-\Psi_1^2}{(1-|\Psi_1|^2)^2}\frac{\partial\bar{\Psi}_1}{\partial z}. \tag{4.6}$$

Now note that from (3.2) we also have

$$H\frac{\partial X^3}{\partial z}=-\Psi_1 H\frac{\partial(X^1-\sqrt{-1}X^2)}{\partial z}. \tag{4.7}$$

It then follows from (4.3) and (4.7) that on $\Psi_1^{-1}(C)$

$$H\frac{\partial X^3}{\partial z}=-2\frac{\Psi_1}{(1-|\Psi_1|^2)^2}\frac{\partial\bar{\Psi}_1}{\partial z}, \tag{4.8}$$

for $1-|\Psi_1|^2\neq 0$.

(2) can be proved in a similar fashion, or one can derive it from (1) by means of the relation $\Psi_1\cdot\Psi_2=-1$ valid on $\Psi_1^{-1}(C)\cap\Psi_2^{-1}(C)$.

REMARK 4.1. The Euclidean counterpart of Theorem 4.1, namely, the corresponding representation formula for surfaces in Euclidean 3-space has been proved in Kenmotsu [3].

REMARK 4.2. If we carry out the same argument, utilizing the equations (3.9), (3.11) instead of (3.8), (3.10), then we obtain the following representation formula in terms of Ψ and ϕ: On $\Psi_1^{-1}(C)$,

$$(4.9)\qquad \begin{aligned} \bar{\phi}\frac{\partial X^1}{\partial z} &= \frac{1+\Psi_1^2}{(1-|\Psi_1|^2)^2}\overline{\frac{\partial \Psi_1}{\partial z}}, \\ \bar{\phi}\frac{\partial X^2}{\partial z} &= \sqrt{-1}\frac{1-\Psi_1^2}{(1-|\Psi_1|^2)^2}\overline{\frac{\partial \Psi_1}{\partial z}}, \\ \bar{\phi}\frac{\partial X^3}{\partial z} &= -2\frac{\Psi_1}{(1-|\Psi_1|^2)^2}\overline{\frac{\partial \Psi_1}{\partial z}}. \end{aligned}$$

(The corresponding formula also holds on $\Psi_2^{-1}(C)$.)

Now, let M be a spacelike surface immersed in L^3 by $X=(X^1, X^2, X^3)$: $M \to L^3$, and assume that $\phi \neq 0$. If we set $F=[\bar{\phi}(1-|\Psi_1|^2)^2]^{-1}(\overline{\partial \Psi_1/\partial z})$, then it follows from (4.9) that

$$(4.10)\qquad \left(\frac{\partial X^1}{\partial z}, \frac{\partial X^2}{\partial z}, \frac{\partial X^3}{\partial z}\right) = (F(1+\Psi_1^2), \sqrt{-1}F(1-\Psi_1^2), -2F\Psi_1),$$

and, in consequence,

$$(4.11)\qquad F=\frac{1}{2}\left(\frac{\partial X^1}{\partial z}-\sqrt{-1}\frac{\partial X^2}{\partial z}\right).$$

Recall that if M is assumed to be maximal in L^3, then each component function of the immersion X is harmonic on M. It then follows from (4.11) that F is holomorphic in this case. This fact implies that (4.10) gives a Lorentzian counterpart of the classical Weierstrass-Enneper formula for minimal surfaces in Euclidean 3-space (cf. [8]). To be more precise, the following has been proved.

PROPOSITION 4.2 (Kobayashi [4], McNertney [10]). *Any simply connected maximal spacelike surface M in L^3 can be represented in the form*

$$(4.12)\qquad X(z)=2\,\mathrm{Re}\int^z (F(1+\Psi_1^2), \sqrt{-1}F(1-\Psi_1^2), -2F\Psi_1)\,dz+c,$$

where $z \in M$ and $c \in L^3$, the integral being taken along an arbitrary path from a fixed point to the point z.

PROOF. Here we remark only on the following matters. For more details, see [4], [10]. First, F is defined by (4.11), which is a holomorphic function on M. Ψ_1 is given as

$\Psi_1 = -(1/2F)(\partial X^3/\partial z)$ by virtue of (4.10), which defines a meromorphic function on M such that $F\Psi_1^2$ is holomorphic on M. (The exceptional case where $F\equiv 0$ corresponds to the (x^1, x^2)-plane in L^3, but it can be obtained by setting $F\equiv 1$ and $\Psi_1\equiv 0$ in (4.12).)

5. Integrability condition. In this section we shall show that the Gauss map Ψ of an arbitrary spacelike surface M in L^3 satisfies a nonlinear second order partial differential equation in Ψ and H. The equation we obtain will then turn out to be the complete integrability condition of the first order PDE system in Theorem 4.1 with given data H and Ψ. First we prove:

PROPOSITION 5.1. *Let M be a spacelike surface in L^3. Then the mean curvature function H of M and the Gauss map Ψ of M into $\mathbf{C}$ satisfy the following second order partial differential equation*

$$H\left(\frac{\partial^2\Psi}{\partial z\partial\bar{z}}+\frac{2\bar{\Psi}}{1-|\Psi|^2}\frac{\partial\Psi}{\partial z}\frac{\partial\Psi}{\partial\bar{z}}\right)=\frac{\partial H}{\partial z}\frac{\partial\Psi}{\partial\bar{z}}. \tag{5.1}$$

PROOF. We shall prove (5.1) for Ψ_1. To do this, we may consider only the case where $H\neq 0$. Indeed, if $H(p)=0$ at $p\in\Psi_1^{-1}(\mathbf{C})$, then $\partial\Psi_1/\partial\bar{z}(p)=0$ by (3.13), and hence (5.1) holds trivially there.

This being remarked, recall that from (3.7) and (3.8) we have

$$\frac{\partial\Psi_1}{\partial\bar{z}}=2H\frac{1}{(1-e_3^3)^2}\frac{\partial(X^1+\sqrt{-1}X^2)}{\partial\bar{z}}. \tag{5.2}$$

On the other hand, a simple calculation using (1.3), (3.6) and (3.7) yields

$$\frac{\partial(X^1+\sqrt{-1}X^2)}{\partial z\partial\bar{z}}=\lambda^2H\frac{\Psi_1}{1-|\Psi_1|^2}. \tag{5.3}$$

Hence, differentiating (5.2) with respect to z and applying (1.5) and (5.3), we get

$$\frac{\partial^2\Psi_1}{\partial z\partial\bar{z}}=\frac{1}{H}\frac{\partial H}{\partial z}\frac{\partial\Psi_1}{\partial\bar{z}}+(1-|\Psi_1|^2)\left[H\frac{\partial X^3}{\partial z}+\phi\frac{\partial X^3}{\partial\bar{z}}\right]\frac{\partial\Psi_1}{\partial\bar{z}}+\frac{\lambda^2H^2}{2}(1-|\Psi_1|^2)\Psi_1. \tag{5.4}$$

Substituting (4.1) and (4.9) into (5.4) and applying (3.16), we then obtain

$$\frac{\partial^2\Psi_1}{\partial z\partial\bar{z}}+\frac{2\bar{\Psi}_1}{1-|\Psi_1|^2}\frac{\partial\Psi_1}{\partial z}\frac{\partial\Psi_1}{\partial\bar{z}}=\frac{1}{H}\frac{\partial H}{\partial z}\frac{\partial\Psi_1}{\partial\bar{z}}, \tag{5.5}$$

thus proving (5.1) for Ψ_1. We also get the same equation for Ψ_2 by the same argument, or from (5.5) by using the relation $\Psi_1\cdot\Psi_2=-1$.

COROLLARY 5.2 (Milnor [6]). *The mean curvature of a spacelike surface M in L^3 is constant if and only if the Gauss map G of M is a harmonic mapping into $\boldsymbol{H}$.*

PROOF. It is not difficult to observe from (3.13) as well as (5.1), which is in fact a nonlinear elliptic system in Ψ, that H is constant if and only if Ψ satisfies

$$\frac{\partial^2\Psi}{\partial z\partial\bar{z}}+\frac{2\bar{\Psi}}{1-|\Psi|^2}\frac{\partial\Psi}{\partial z}\frac{\partial\Psi}{\partial\bar{z}}=0\,,$$

which shows that G, whose coordinates expression is Ψ, is a harmonic mapping into $\boldsymbol{H}$ (cf. [2]).

REMARK 5.1. (1) Equation (5.1) does not depend on the metric on M, but depends only on the complex structure on M.

(2) It should be noted that geometrically (5.5) means the following: The tension field $\tau(G)$ (see [2] for definition) of the Gauss map G coincides, up to translations in L^3, with the gradient ∇H of the mean curvature function H (cf. [9]).

REMARK 5.2. Corollary 5.2 gives a Lorentzian counterpart of a theorem of Ruh and Vilms [9] that the mean curvature of a hypersurface in Euclidean n-space is constant if and only if its Gauss map is harmonic.

In what follows, let M be a Riemann surface, and $\boldsymbol{H}$ denote, as before, the unit pseudosphere in L^3 with the induced metric of constant negative Gaussian curvature and natural complex structure defined in §2. Given a *nonvanishing* smooth function $H: M\to\boldsymbol{R}$ and a smooth mapping $G: M\to\boldsymbol{H}$, let us now look at the following system of first order partial differential equations:

$$\text{(5.6)}\qquad \begin{aligned}\frac{\partial X^1}{\partial z}&=\frac{1}{H}\frac{1+\Psi_i^2}{(1-|\Psi_i|^2)^2}\frac{\partial\bar{\Psi}_i}{\partial z}\\ \frac{\partial X^2}{\partial z}&=(-1)^{i-1}\frac{\sqrt{-1}}{H}\frac{1-\Psi_i^2}{(1-|\Psi_i|^2)^2}\frac{\partial\bar{\Psi}_i}{\partial z}\qquad \text{on}\quad \Psi_i^{-1}(C)\\ \frac{\partial X^3}{\partial z}&=(-1)^i\frac{2}{H}\frac{\Psi_i}{(1-|\Psi_i|^2)^2}\frac{\partial\bar{\Psi}_i}{\partial z}\,.\end{aligned}$$

Here Ψ_i denotes the composition $\Psi_i=\psi_i\circ G$ of G and the stereographic mapping ψ_i defined by (2.1), and $i=1, 2$. It should be noted that owing to the relation $\Psi_1\cdot\Psi_2=-1$, the right sides of (5.6) for $i=1, 2$ are compatible on $\Psi_1^{-1}(C)\cap\Psi_2^{-1}(C)$, and hence (5.6) defines a system defined globally on M.

With these prepared, we now prove the following

PROPOSITION 5.3. *Equation* (5.1) *is the complete integrability condition of the system* (5.6).

PROOF. Let P denote the right side of (5.6), that is,

$$\text{(5.7)}\qquad P=(f_i(1+\Psi_i^2),\,(-1)^{i-1}\sqrt{-1}f_i(1-\Psi_i^2),\,(-1)^i 2f_i\Psi_i)\,,$$

where $f_i=[H(1-|\Psi_i|^2)^2]^{-1}(\partial\bar{\Psi}_i/\partial z)$. Assuming that H and Ψ_i satisfy (5.1), we shall show that (5.6) is a completely integrable system. To do this, it suffices to see that $\partial P/\partial\bar{z}\in\boldsymbol{R}^3$. But this is immediate; in fact, by a direct calculation we can easily see that if (5.1) is satisfied,

$$\frac{\partial P}{\partial\bar{z}}=\frac{H}{2}\lambda^2\left(\frac{2\operatorname{Re}\Psi_i}{1-|\Psi_i|^2},(-1)^{i-1}\frac{2\operatorname{Im}\Psi_i}{1-|\Psi_i|^2},(-1)^i\frac{1+|\Psi_i|^2}{1-|\Psi_i|^2}\right), \tag{5.8}$$

where $\lambda=2[H(1-|\Psi_i|^2)]^{-1}|\partial\Psi_i/\partial\bar{z}|$.

6. Spacelike surfaces with prescribed mean curvature. We shall now prove a converse of Theorem 4.1. Namely, by solving the PDE system (5.6), we shall construct a spacelike surface M in L^3 with prescribed nonvanishing mean curvature H and Gauss map G. To be precise, we are going to prove the following

THEOREM 6.1. *Let M be a simply connected Riemann surface, $H: M\to\boldsymbol{R}$ be a nonvanishing real smooth function on M, and $G: M\to\boldsymbol{H}$ be a nowhere holomorphic smooth mapping of M into the unit pseudosphere $\boldsymbol{H}$ in L^3. For $i=1, 2$, let Ψ_i denote the composition $\Psi_i=\psi_i\circ G$ of G and the stereographic mapping ψ_i defined by* (2.1). *Suppose that H and Ψ_i satisfy the differential equation* (5.1). *Then there exists a spacelike immersion X: $M\to L^3$ with the following properties*:

(1) *The mean curvature of M is H, and the Gauss map of M is given by G.*

(2) *$X=(X^1, X^2, X^3)$ is given explicitly as*

$$\begin{aligned} X^1(z)&=2\operatorname{Re}\int^z\frac{1}{H}\frac{1+\Psi_i^2}{(1-|\Psi_i|^2)^2}\frac{\partial\bar{\Psi}_i}{\partial z}dz+c^1\,,\\ X^2(z)&=2\operatorname{Re}\int^z(-1)^{i-1}\frac{\sqrt{-1}}{H}\frac{1-\Psi_i^2}{(1-|\Psi_i|^2)^2}\frac{\partial\bar{\Psi}_i}{\partial z}dz+c^2\,,\\ X^3(z)&=2\operatorname{Re}\int^z(-1)^i\frac{2}{H}\frac{\Psi_i}{(1-|\Psi_i|^2)^2}\frac{\partial\bar{\Psi}_i}{\partial z}dz+c^3\,, \end{aligned} \tag{6.1}$$

where $z\in\Psi_i^{-1}(\boldsymbol{C})$ and $c=(c^1, c^2, c^3)\in L^3$, the integral being taken along an arbitrary path from a fixed point to the point z.

PROOF. For given function H and given mapping G, we shall look at the complex PDE system (5.6) defined on M. Note that, on account of Proposition 5.3, the system (5.6) is completely integrable, since H and Ψ_i satisfy (5.1). Moreover, any real solution $X=(X^1, X^2, X^3)$ of the system (5.6) can be represented as

$$X(z)=2\operatorname{Re}\int^z Pdz+c\,, \tag{6.2}$$

where P is defined by (5.7) and $c\in\boldsymbol{R}^3$. Indeed, since M is simply connected and $\partial P/\partial\bar{z}\in\boldsymbol{R}^3$ by (5.8), the right side of (6.2), where the integral is taken along an arbitrary path in

M from a fixed point to a variable point z, defines a single-valued mapping, and satisfies (5.6) with given H and Ψ_i. Thus we define a mapping $X\colon M\to L^3$ by (6.2), and shall prove that X has the desired properties.

It is easy to see from (5.6) that X satisfies

$$\left\langle\frac{\partial X}{\partial z},\frac{\partial X}{\partial \bar z}\right\rangle=\frac{\lambda^2}{2}\,,\qquad \left\langle\frac{\partial X}{\partial z},\frac{\partial X}{\partial z}\right\rangle=\left\langle\frac{\partial X}{\partial \bar z},\frac{\partial X}{\partial \bar z}\right\rangle=0\,, \tag{6.3}$$

where $\lambda=2[H(1-|\Psi_i|^2)]^{-1}|\partial\Psi_i/\partial\bar z|$. Note that, since G is nowhere holomorphic, $\partial\Psi_i/\partial\bar z\neq 0$ everywhere. Then it follows from (6.3) that X defines a spacelike immersion with induced metric $g=\lambda^2|dz|^2$, and by setting $z=\xi^1+\sqrt{-1}\,\xi^2$, we get an isothermal coordinates on M with respect to g, On the other hand, from (5.6) together with (3.5) and (2.2), it is immediate to verify that the Gauss map of M coincides with G and the mean curvature of M is given by H.

Remark 6.1. In Theorem 6.1, if we merely assume $G\colon M\to \boldsymbol{H}$ to be a smooth mapping which satisfies the complete integrability condition (5.1) with given H, then the mapping $X\colon M\to L^3$ given by (6.2) is, in general, not a spacelike immersion but have singularities which occur where $\partial\Psi_i/\partial\bar z=0$.

Corollary 6.2. *Let $X\colon M\to L^3$ be a spacelike immersion in Theorem* 6.1. *Then the following hold.*

(1) *The induced metric g on M is given by*

$$g=\left[\frac{2}{H(1-|\Psi|^2)}\left|\frac{\partial\Psi}{\partial\bar z}\right|\right]^2|dz|^2\,.$$

(2) *The Gaussian curvature K of M is given by*

$$K=H^2\left[\left|\frac{\Psi_z}{\Psi_{\bar z}}\right|^2-1\right].$$

Proof. (1) is already proved. (2) can be obtained by substituting (3.12) into (1.6).

As in the case of minimal surfaces in Euclidean 3-space, it is not difficult to see from Proposition 4.2 that two noncongruent maximal spacelike surfaces may have the same Gauss map (cf. [4]). However, for spacelike surfaces with nonvanishing mean curvature in Theorem 6.1, we have the uniqueness in the following sense.

Proposition 6.3. *Let X (resp. $\tilde X$) be a spacelike immersion in Theorem* 6.1 *of a simply connected Riemann surface M into L^3 with nonvanishing mean curvature function H (resp. $\tilde H$) and Gauss map G (resp. $\tilde G$) into $\boldsymbol{H}$. Then the following statements are equivalent*:

(1) *There exist a holomorphic diffeomorphism φ on M and an orientation preserving isometry τ of L^3 such that for $z\in M$*

$$\tau\circ X(z)=\tilde X\circ\varphi(z). \tag{6.5}$$

(2) *There exist a holomorphic diffeomorphism φ on M and an orientation preserving isometry σ of $\boldsymbol{H}$ such that for $z \in M$*

$$\sigma \circ G(z) = \tilde{G} \circ \varphi(z)\,, \qquad H(z) = \tilde{H} \circ \varphi(z)\,. \tag{6.6}$$

PROOF. [(1) ⇒ (2)] Putting $w = \varphi(z)$ and differentiating (6.5), we have $\tau_*(\partial X/\partial z)(z) = (\partial \tilde{X}/\partial w)(\varphi(z)) \cdot \varphi'(z)$ and $\tau_*(\partial X/\partial \bar{z})(z) = (\partial \tilde{X}/\partial \bar{w})(\varphi(z)) \cdot \overline{\varphi'(z)}$ for $z \in M$, τ_* being extended $\boldsymbol{C}$-linearly. Denoting by (e_A) (resp. $(\tilde{e}_A)$), $A = 1, 2, 3$, a Lorentzian frame field adapted to X (resp. $\tilde{X}$) in L^3, we then get

$$(\tilde{e}_1 + \sqrt{-1}\,\tilde{e}_2)(\varphi(z)) = |\varphi'(z)|\,\overline{\varphi'(z)}^{-1} \tau_*(e_1 + \sqrt{-1}\,e_2)(z)\,,$$

and hence

$$\begin{aligned} 2\tilde{e}_3(\varphi(z)) &= \sqrt{-1}(\tilde{e}_1 + \sqrt{-1}\,\tilde{e}_2)(\varphi(z)) \times (\tilde{e}_1 - \sqrt{-1}\,\tilde{e}_2)(\varphi(z)) \\ &= \tau_*(\sqrt{-1}(e_1 + \sqrt{-1}\,e_2)(z) \times (e_1 - \sqrt{-1}\,e_2)(z)) = 2\tau_*(e_3(z))\,, \end{aligned}$$

since τ is orientation preserving. Therefore, by setting $\sigma = \tau_*$, we obtain an orientation preserving isometry σ of $\boldsymbol{H}$ such that $\tilde{G} \circ \varphi(z) = \sigma \circ G(z)$ for $z \in M$. Now differentiating $\tilde{e}_3(\varphi(z)) = \tau_*(e_3(z))$ and substituting (1.5), it can be checked without difficulty that $\tilde{H}(\varphi(z)) = H(z)$ for $z \in M$, thus proving (6.6).

[(2)⇒(1)] Denote also by σ the extension of σ to an orientation preserving isometry of L^3. To show (6.5), we may assume σ = identity, considering $\sigma \circ X$ instead of X if necessary. Then we have $G(z) = \tilde{G}(\varphi(z))$, that is, $\Psi(z) = \tilde{\Psi}(\varphi(z))$, since σ is orientation preserving. It then follows from (6.1) that

$$\partial(X^A(z) - \tilde{X}^A(\varphi(z)))/\partial z = 0\,, \qquad A = 1, 2, 3\,.$$

Therefore, $X(z) = \tilde{X}(\varphi(z)) + c$ for some $c \in \boldsymbol{R}^3$. This means that $\sigma \circ X(z) = \tilde{X}(\varphi(z)) + c$, and hence there exists an orientation preserving isometry τ of L^3 such that $\tau \circ X(z) = \tilde{X} \circ \varphi(z)$ for $z \in M$.

In the case where a given H in Theorem 6.1 is constant, the complete integrability condition (5.1) requires simply that a given G should be a harmonic mapping. Consequently, given a nonzero real constant H and nowhere holomorphic harmonic mapping G of a simply connected Riemann surface M into $\boldsymbol{H}$, we can construct, by (6.1), a spacelike immersion $X: M \to L^3$ with constant mean curvature H and prescribed Gauss map G.

REMARK 6.2. More generally, given a nonzero real constant H and a non-holomorphic harmonic mapping $G: M \to \boldsymbol{H}$, the mapping $X: M \to L^3$ given by (6.1) defines a spacelike immersion except for possible isolated singular points, which has, away from these singular points, constant mean curvature H and prescribed Gauss map G. Indeed, it follows from a standard result in the theory of harmonic mappings (cf.

[2, (10.5)]) that if $G: M \to \boldsymbol{H}$ is a nonholomorphic harmonic mapping, then $\partial\Psi_i/\partial\bar{z}$ has at most isolated zeros where singularities of X occur.

From this point of view, we shall next exhibit some examples of spacelike surfaces of constant mean curvature in L^3.

EXAMPLE 6.1. Let $D=\{z\in \boldsymbol{C} \mid |z|<1\}$ be the unit disk in $\boldsymbol{C}$. Take $H=-1$, and define $\Psi_1: D\to \boldsymbol{C}$ by $\Psi_1(z)=-\bar{z}$. Then Ψ_1 satisfies (5.1), and the spacelike immersion X defined by (6.1) is written as

$$X(z)=\left(\frac{2\operatorname{Re} z}{1-|z|^2}, -\frac{2\operatorname{Im} z}{1-|z|^2}, \frac{1+|z|^2}{1-|z|^2}\right).$$

This is the standard immersion of the hyperboloid or the upper sheet of $\boldsymbol{H}$ in L^3.

EXAMPLE 6.2. Take $H=-1/2$, and define $\Psi_1: \boldsymbol{C}\to \boldsymbol{C}$ by $\Psi_1(z)=(e^{z+\bar{z}}-1)/(e^{z+\bar{z}}+1)$. Then Ψ_1 satisfies (5.1); indeed $\Psi_1(\boldsymbol{C})$ is a geodesic in D. The spacelike immersion X defined by (6.1) is written as

$$X(z)=\left(-\frac{1}{2}(e^{z+\bar{z}}-e^{-(z+\bar{z})}), \sqrt{-1}(\bar{z}-z), \frac{1}{2}(e^{z+\bar{z}}+e^{-(z+\bar{z})})\right).$$

This is the standard immersion of the hyperbolic cylinder, the surface defined by $(x^3)^2-(x^1)^2=1$ with $x^3>0$, in L^3.

EXAMPLE 6.3. Let M be a closed Riemann surface of genus $\geqq 2$. Then each homotopy class of mappings $M\to M$ contains a harmonic mapping, with respect to the hyperbolic metric of constant negative Gaussian curvature (cf. [2, (6.11)]). Lifting these to the universal covering $\tilde{M}$ of M, we get harmonic mappings $G: \tilde{M}\to \boldsymbol{H}$. With each of these and a nonzero real constant H, there is associated by (6.1) a spacelike immersion with possible isolated singular points $X: \tilde{M}\to L^3$, which has, away from singular points, constant mean curvature H and the Gauss map G. (Take the conjugate mapping $\bar{G}$ of G, if G is holomorphic.)

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